CNRS - RAS cooperation

Seminar - 18th June, 2004 - IPME, St. Petersburg

Quadratic Separation for Robustness and Design

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What is LAAS-CNRS?

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE French National Center for Scientific Research.

- ➥ Public basic-research org. producing knowledge and making it available to society.
- ➥ 26,000 employees (11,600 researchers).
- ➥ 1,260 units, spread throughout the country, cover all fields of research.

LAAS Analysis and Architecture of Systems

- ➥ Part of CNRS - STIC department

(Science and Technology for Information and Communication)

- ➥ 500 employees (200 researchers)
- ➥ 12 research groups
- ➥ Control Theory, Robotics, Micro and Nano-Systems, Computer science
- ➥ In Toulouse, France.

Methods and Algorithms in Control group

MAC group http://www.laas.fr/MAC

- ❑ Research fields : Robust control & Non-linear control
- ❑ Application fields : Aeronautics & Space industry & Environment

Research in robust control

- ❑ MIMO LTI systems with parametric uncertainty
- ❑ State-space modeling and Lyapunov theory
- ❑ Stability and performance (H_∞, H_2, pole location, impulse to peak)
- ❑ Analysis & Control design (state-feedback, full-order and static output-feedback)
- ❑ LMI based results & Semi-definite programming
- ❑ Robust MULti-Objective Control toolbox (V1 in September)

  http://www.laas.fr/OLOCEP

Outline

Quadratic separation for LTI systems

Examples of results for robustness and design

- Preliminaries and notations
- Robust analysis and losslessness of quadratic separators
- Quadratic separation and control design

Methodology and notations

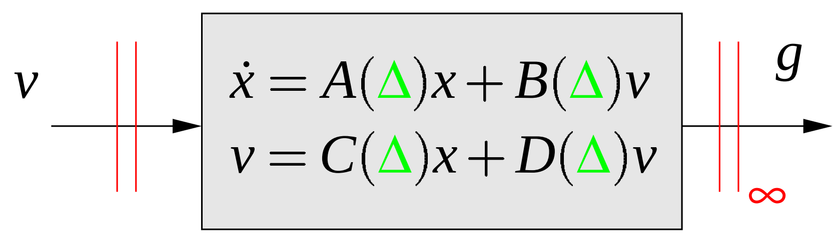
Uncertain model

- ❏ Engineering problem modeled as uncertain differential equations with constraints
- ❏ State-space LTI systems / parametric uncertainty / pole and induced norm constraints

$\Downarrow$

Optimization problem

- ❏ At best: lossless formulation with a global polynomial-time algorithm
- ❏ Conveniently: Conservative formulation with a global polynomial-time algorithm
- ❏ Usually: Conservative with sub-optimal heuristic algorithm
- ❏ LMI formulated results ⋗ convex SDP & $O(n^{6.5})$ algorithms
- ❏ YALMIP interface in Matlab & Solvers: SeDuMi, SDPT3, CSDP,...



$$\dot{x} = A(\Delta)x + B(\Delta)v$$
$$v = C(\Delta)x + D(\Delta)v$$

$$\Rightarrow \quad \min \gamma \quad : \quad \mathcal{L}(P,\Theta,\gamma) < 0$$

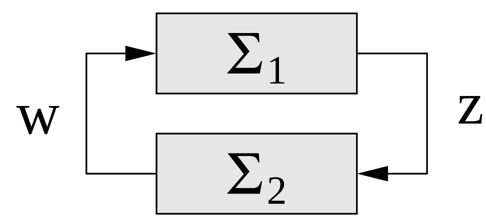
Topological Separation

Graph of Σ_1 and inverse graph of Σ_2:

$$\mathcal{G}(\Sigma_1) = \left\{ x = \begin{pmatrix} z \\ w \end{pmatrix} \ : \ \Sigma_1(z,w) = 0 \right\} \qquad \mathcal{G}^I(\Sigma_2) = \left\{ x = \begin{pmatrix} z \\ w \end{pmatrix} \ : \ \Sigma_2(w,z) = 0 \right\}$$

Stability result [Safonov]:

The interconnected system



is stable

- iff $\mathcal{G}(\Sigma_1) \bigcap \mathcal{G}^I(\Sigma_2) = \{0\}$
- iff $\exists\, d$:
$$\begin{cases} d(x) > 0 \ , & \forall\, x \in \mathcal{G}(\Sigma_1) \\ d(x) \leq 0 \ , & \forall\, x \in \mathcal{G}^I(\Sigma_2) \end{cases}$$

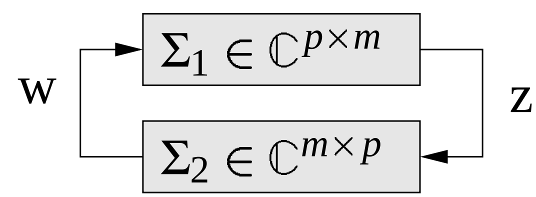
d : topological separator (see also "supply rate" in dissipative theory [Willems])

Quadratic function for the topological separator:

$d(x) = x^* \Theta x \qquad \Theta = \Theta^* \in \mathbb{C}^{(m+p)\times(m+p)}$ Lossless for linear systems

E.g. for matrix gains

The interconnected system



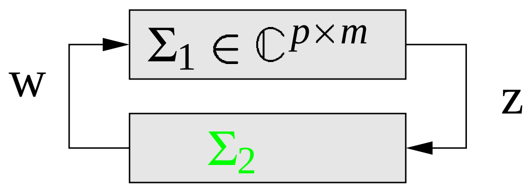
is well-posed

- iff $\det(\mathbb{1} - \Sigma_1 \Sigma_2) \neq 0$
- iff $\exists \Theta$:

$$\begin{cases} w^* \begin{bmatrix} \Sigma_1^* & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma_1 \\ \mathbb{1} \end{bmatrix} w > 0 \ , \ \forall\, w \neq 0 \\ z^* \begin{bmatrix} \mathbb{1} & \Sigma_2^* \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ \Sigma_2 \end{bmatrix} z \leq 0 \ , \ \forall\, z \neq 0 \end{cases}$$

One interconnected operator is uncertain

Robust analysis



is robustly well-posed for all $\Sigma_2 \in \mathbb{U} \subset \mathbb{C}^{m\times p}$

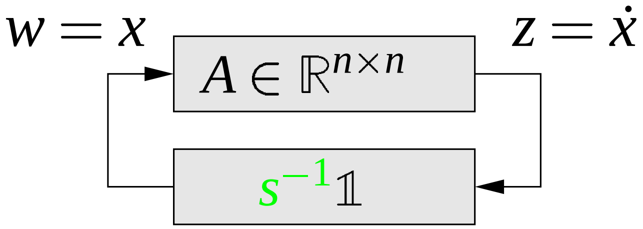
- iff $\det(\mathbb{1} - \Sigma_1\Sigma_2) \neq 0 \;,\; \forall\, \Sigma_2 \in \mathbb{U}$
- iff $\exists\, \Theta \;:\; \begin{cases} \begin{bmatrix} \Sigma_1^* & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma_1 \\ \mathbb{1} \end{bmatrix} > \mathbb{0} \\ \begin{bmatrix} \mathbb{1} & \Sigma_2{}^* \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ \Sigma_2 \end{bmatrix} \leq \mathbb{0} \;,\; \forall\, \Sigma_2 \in \mathbb{U} \end{cases}$

➥ Quadratic separation results are LMI-based.

➥ Is it possible to handle the infinite-dimensional constraint without conservatism?

One interconnected operator is uncertain

Example: Lyapunov matrix



is stable (interconnection well-posed for all $s^{-1} \in \mathbb{C}^+$)

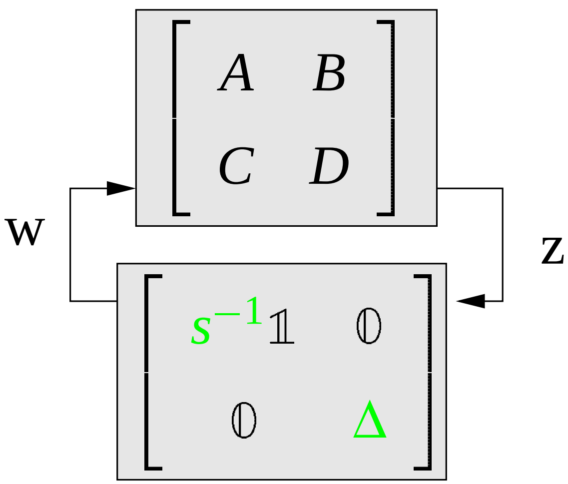
- iff $\exists \Theta$: $\left\{ \begin{array}{l} \begin{bmatrix} A^T & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} A \\ \mathbb{1} \end{bmatrix} > 0 \\ \begin{bmatrix} \mathbb{1} & s^{-*}\mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ s^{-1}\mathbb{1} \end{bmatrix} \leq 0 \ , \ \forall s^{-1} \in \mathbb{C}^+ \end{array} \right.$
- iff $\exists \Theta = \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix}$: $P > 0$, $\begin{bmatrix} A^T & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} A \\ \mathbb{1} \end{bmatrix} = -A^T P - P A > 0$

➥ Lossless quadratic separator.

➥ Other sets than $\mathbb{C}^+$ ⋗ pole location.

One interconnected operator is uncertain

Example: bounded real lemma



is robustly stable ($s^{-1} \in \mathbb{C}^+$, $\Delta^T \Delta \leq \mathbb{1}$)

❑ iff there exists a separator such as:

$$\Theta = \begin{bmatrix} \mathbb{0} & \mathbb{0} & -P & \mathbb{0} \\ \mathbb{0} & -\tau\mathbb{1} & \mathbb{0} & \mathbb{0} \\ -P & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \tau\mathbb{1} \end{bmatrix}, \quad P > \mathbb{0}, \quad \tau > 0$$

that satisfies the LMI constraint:

$$\begin{bmatrix} \Sigma_1^* & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma_1 \\ \mathbb{1} \end{bmatrix} = - \begin{bmatrix} A^T P + PA + \tau C^T C & PB + \tau C^T D \\ B^T P + \tau D^T C & -\tau\mathbb{1} + \tau D^T D \end{bmatrix} > \mathbb{0}$$

➥ Lossless quadratic separator.

Lossless quadratic separators

❑ Full-block dissipative (𝒮-procedure)

$$\begin{bmatrix} \mathbb{1} & \Delta_D^* \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \Delta_D \end{bmatrix} \leq \mathbb{0} \quad \Leftrightarrow \quad \Theta = \tau \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} , \quad \tau > 0$$

❑ Disk located, repeated, complex valued scalar

$$\begin{array}{l} \Delta = \delta_c \mathbb{1} \\ \alpha + \delta_c \beta + \delta_c^* \beta^* + \delta_c \delta_c^* \gamma \leq 0 \end{array} \quad \Leftrightarrow \quad \Theta = \begin{bmatrix} \alpha P & \beta P \\ \beta^* P & \gamma P \end{bmatrix} , \quad P > \mathbb{0}$$

❑ Bounded, repeated, real valued scalar

$$\begin{array}{l} \Delta = \delta_r \mathbb{1} \\ \alpha + 2\delta_r \beta + \delta_r^2 \gamma \leq 0 \end{array} \quad \Leftrightarrow \quad \Theta = \begin{bmatrix} \alpha P & \beta P + Q \\ \beta P + Q^* & \gamma P \end{bmatrix} , \quad \begin{array}{l} P > \mathbb{0} \\ Q = -Q^* \end{array}$$

Conservative and lossless separators

Conservative quadratic separators for block diagonal uncertainty

❏ Repeated full-block dissipative

$$\Delta = \mathbb{1}_r \otimes \Delta_D$$

$$\begin{bmatrix} \mathbb{1} & \Delta_D^* \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \Delta_D \end{bmatrix} \leq 0 \quad \Rightarrow \quad \Theta = \begin{bmatrix} T \otimes X & T \otimes Y \\ T \otimes Y^* & T \otimes Z \end{bmatrix}, \quad T > 0$$

❏ Block diagonal polytopic

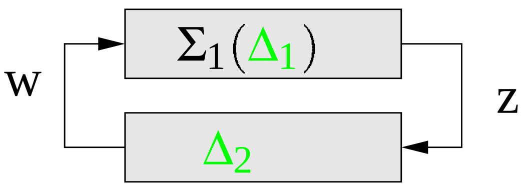
$$\Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_r)$$

$$\Delta = \sum \zeta_i \Delta^{[i]} \quad : \quad \sum \zeta_i = 1 \quad , \quad \zeta_i \geq 0 \quad \Rightarrow \quad \begin{bmatrix} \mathbb{1} & \Delta^{[i]} \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ \Delta^{[i]} \end{bmatrix} \leq 0, \quad \Theta_{ii}^{22} \geq 0$$

❏ Any block diagonal structure of previous types (lossless if $2(m_r + m_c) + m_D \leq 3$)

$$\Delta = \mathrm{diag}(\Delta_1, \dots) \quad \Rightarrow \quad \Theta = \begin{bmatrix} \mathrm{diag}(\Theta_1^{11}, \dots) & \mathrm{diag}(\Theta_1^{12}, \dots) \\ \mathrm{diag}(\Theta_1^{21}, \dots) & \mathrm{diag}(\Theta_1^{22}, \dots) \end{bmatrix}$$

Robust analysis: parameter-dependent separators



is robustly well-posed for all $\forall \Delta_1 \in \mathbb{U}_1 \;,\; \forall \Delta_2 \in \mathbb{U}_2$

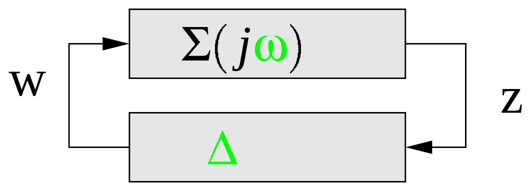
❑ iff $\forall \Delta_1 \in \mathbb{U}_1 \;\exists\, \Theta(\Delta_1)$:

$$\left\{ \begin{array}{l} \begin{bmatrix} \Sigma_1^*(\Delta_1) & \mathbb{1} \end{bmatrix} \Theta(\Delta_1) \begin{bmatrix} \Sigma_1(\Delta_1) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} \\ \begin{bmatrix} \mathbb{1} & \Delta_2{}^* \end{bmatrix} \Theta(\Delta_1) \begin{bmatrix} \mathbb{1} \\ \Delta_2 \end{bmatrix} \leq \mathbb{0} \;,\; \forall \Delta_2 \in \mathbb{U}_2 \end{array} \right.$$

➥ Infinite number of LMI variables & infinite number of constraints

Both interconnected operators are uncertain

Example: μ-analysis



is robustly stable ($\omega \in \mathbb{R}^+$, $\Delta \in \mathbb{U}$)

❑ iff $\forall\, \omega \in \mathbb{R}^+ \; \exists\, \Theta(j\omega)$:

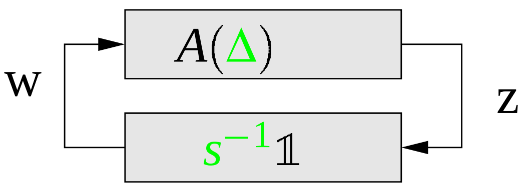
$$\left\{ \begin{array}{l} \begin{bmatrix} \Sigma^*(j\omega) & \mathbb{1} \end{bmatrix} \Theta(j\omega) \begin{bmatrix} \Sigma(j\omega) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} \\ \begin{bmatrix} \mathbb{1} & \Delta^* \end{bmatrix} \Theta(j\omega) \begin{bmatrix} \mathbb{1} \\ \Delta \end{bmatrix} \leq \mathbb{0} \;,\; \forall\, \Delta \in \mathbb{U} \end{array} \right.$$

➥ An optimistic bound on μ can then be obtained by gridding $\{\omega_1, \ldots, \omega_N\} \subset \mathbb{R}^+$.

➥ For each ω_i, build finite dimensional LMIs.

Both interconnected operators are uncertain

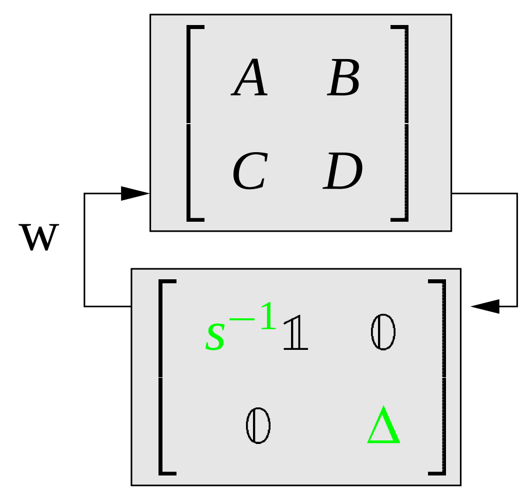
Example: Parameter-dependent Lyapunov Function



is robustly stable ($s^{-1} \in \mathbb{C}^+$, $\Delta \in \mathbb{U}$)

- iff $\forall \Delta \in \mathbb{U} \;\; \exists P(\Delta) \;\; : \;\; \begin{cases} A^T(\Delta)P(\Delta) + P(\Delta)A(\Delta) < \mathbb{0} \\ P(\Delta) > \mathbb{0} \end{cases}$

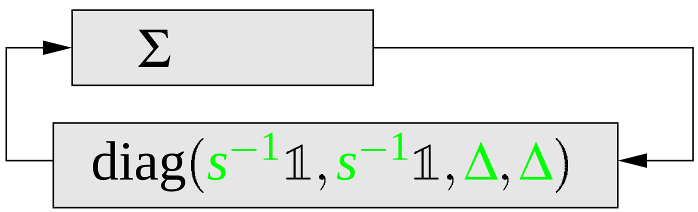
Question: When $A(\Delta) = A + B\Delta(\mathbb{1} - D\Delta)^{-1}C$ how is this result related to the robust stability of



?

Towards lossless robust analysis

Answer: Consider the implicit augmented system



$$\Sigma \; : \; \begin{cases} \dot{x} = Ax + Bw & \\ \dot{w} = & \dot{w} \\ z = Cx + Dw & \\ \dot{z} = CAx + CBw & + D\dot{w} \\ 0 = \quad -w & + w \end{cases}$$

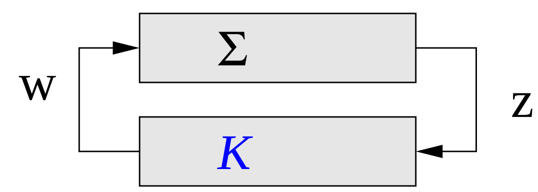
Quadratic separation ⇒ parameter-dependent Lyapunov matrix

$$P(\Delta) = \begin{bmatrix} \mathbb{1} & C^T(\mathbb{1} - D\Delta)^{-T}\Delta^T \end{bmatrix} P \begin{bmatrix} \mathbb{1} \\ \Delta(\mathbb{1} - D\Delta)^{-1}C \end{bmatrix}$$

Prospective work:

- ❑ In relation with [Bliman], build asymptotically lossless P-D Lyapunov functions
- ❑ Take into account information on the derivatives of Δ.

Quadratic separation v.s. control law

There exists a matrix K such that  is stable

- iff $\exists K$, $\exists \Theta$: $\begin{bmatrix} \Sigma^* & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma \\ \mathbb{1} \end{bmatrix} > \mathbb{0}$, $\begin{bmatrix} \mathbb{1} & K^* \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0}$

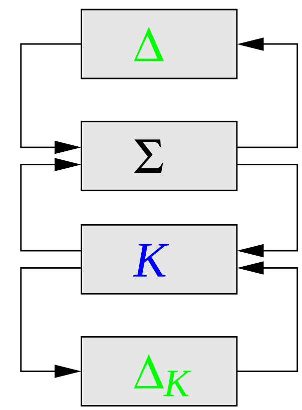
- iff $\exists \Theta = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$: $\left\{ \begin{array}{ll} \begin{bmatrix} \Sigma^* & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma \\ \mathbb{1} \end{bmatrix} > \mathbb{0} & \\ X \leq YZ^{-1}Y^* & Z > \mathbb{0} \end{array} \right.$

> All matrices K such that $\begin{bmatrix} \mathbb{1} & K^* \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0}$, stabilize the interconnection.

➥ The non-linear constraint $X \leq YZ^{-1}Y^*$ is not convex

Improvements and challenge

Quadratic separation for design, features:



- ❏ Design of sets of controllers: handles fragility
- ❏ LMI formulations for state-feedback and full-order output-feedback
- ❏ All robust multi-objective problems can be recast as LMIs + $X \leq YZ^{-1}Y^*$

Challenge:

- ❏ Algorithms to handle the non-linear matrix inequality.
- ❏ Successful results of a gradient-type algorithm - to be improved.

Some general prospectives

- Towards lossless robust analysis - PDLF and beyond
- Topological separation for control design
- Algorithms for NLMIs
- Free academic software to test and compare results.
  `http://www.laas.fr/OLOCEP`
- Control theory and optimization - Duality, separation...