

Robust LFR-based technique
for stability analysis of periodic solutions

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Robustness and Non-Linear problems

- ★ Robustness: prove stability whatever uncertainties in bounded sets
- ★ Lyapunov theory: prove stability whatever initial conditions in bounded sets
- ➔ Model non-linearities on the states as uncertainties on an LTI model

Stability of periodic solutions and periodic systems

- ★ Dynamics of relative motion: stability of the origin of periodic system
- ★ Numerical methods for periodic systems: sampling
- ➔ Modeling of sampled uncertainties

Outline

- ① Discrete-time Periodic Uncertain models for stability of periodic solutions
- ② Linear Matrix Inequality (LMI) results
- ③ Illustrative example

Linear-Fractional Representation (LFR)

Non-linear, uncertain, system $\dot{\eta}(t) = f(\eta(t), t, \delta)$

with periodic solution:

$$\dot{\eta}_s(t) = f(\eta_s(t), t, \delta) \quad , \quad \forall \delta \in [\underline{\delta}, \bar{\delta}] \quad : \quad \eta_s(t + T) = \eta_s(t)$$

Quasi-linearization around periodic solution ($x^c = \eta_s - \eta$)

$$\dot{x}^c(t) = A^c(t)x^c(t) + B_\delta^c(t)w_\delta(t) + B_\Omega^c(t)w_\Omega(t)$$

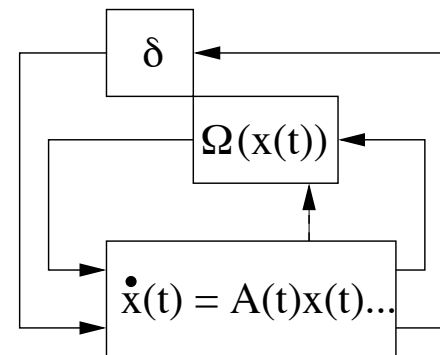
$$z_\delta^c(t) = C_\delta^c(t)x^c(t) + D_{\delta\delta}^c(t)w_\delta(t) + D_{\delta\Omega}^c(t)w_\Omega(t)$$

$$z_\Omega^c(t) = C_\Omega^c(t)x^c(t) + D_{\Omega\delta}^c(t)w_\delta(t) + D_{\Omega\Omega}^c(t)w_\Omega(t)$$

where uncertainties and non-linearities are feedback connected

$$w_\delta^c(t) = \delta z_\delta^c(t)$$

$$w_\Omega^c(t) = \Omega(x^c(t))z_\Omega^c(t)$$



LMI results for robust stability need quadratic separators

$$\exists \Theta \quad : \quad \Upsilon_{\Theta}(\delta \mathbf{1}) = \begin{bmatrix} 1 \\ \delta \mathbf{1} \end{bmatrix}^T \Theta \begin{bmatrix} 1 \\ \delta \mathbf{1} \end{bmatrix} \leq 0 \quad , \quad \forall \delta \in [\underline{\delta}, \bar{\delta}]$$

→ Let Θ the set of all Θ such that this holds.

Bounded $\Omega(x)$ for x bounded

★ HYP 0: Exists a positive definite matrix Q and a matrix Ξ such that

$$x^T Q x \leq \gamma \quad \Rightarrow \quad \Upsilon_{\Xi}(\Omega(x)) = \begin{bmatrix} 1 \\ \Omega(x) \end{bmatrix}^T \Xi \begin{bmatrix} 1 \\ \Omega(x) \end{bmatrix} \leq 0$$

→ For given Q and γ , let Ξ_{γ} the set of all Ξ such that HYP 0 holds.

→ Based on HYP 0, non-linearities may be treated as if uncertainties.

Non-linear system

$$\ddot{\eta}_1 + (\eta_1^2 + \dot{\eta}_1^2 - 1)\dot{\eta}_1 + \eta_1 = u$$

with uncertain control input $u(t) = (\kappa + \delta)(\cos t - \eta_1(t)) \Rightarrow \eta_{s1}(t) = \cos t$

LFR around periodic solution

$$x_1^c = \eta_{s1} - \eta_1, \quad x_2^c = \dot{x}_1^c$$

$$A^c(t) = \begin{bmatrix} 0 & 1 \\ -1 - \kappa + \sin 2t & \cos 2t - 1 \end{bmatrix} \quad B_\delta^c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$B_\Omega^c(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sin t & 2 \cos t & -3 \sin t & -1 \end{bmatrix}$$

$$C_\delta^c = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D_{\delta\delta}^c = 0 \quad D_{\delta\Omega}^c = 0$$

$$C_\Omega^c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{\Omega\delta}^c = 0 \quad D_{\Omega\Omega}^c = 0$$

$$\Omega(x^c) = \begin{bmatrix} x_1^c & 0 \\ x_2^c & 0 \\ 0 & x_2^c \\ 0 & x_1^{c2} + x_2^{c2} \end{bmatrix}$$

Periodic sampling strategy

★ Sampling sequence $\{T_s(k)\}_{k \geq 0}$: $T_s(N) = T$, $T_s(k + N) = T_s(k)$

$$x(k) = x^c(T_s(k)) , w_\delta(k) = w_\delta^c(T_s(k)) \dots \Omega(k) = \Omega(x^c(T_s(k)))$$

★ HYP 1: median approximation of LTI model

$$\forall t \in [T_s(k), T_s(k+1)] : A^c(t) \simeq \tilde{A}(k) = A^c\left(\frac{1}{2}(T_s(k) + T_s(k+1))\right) \dots$$

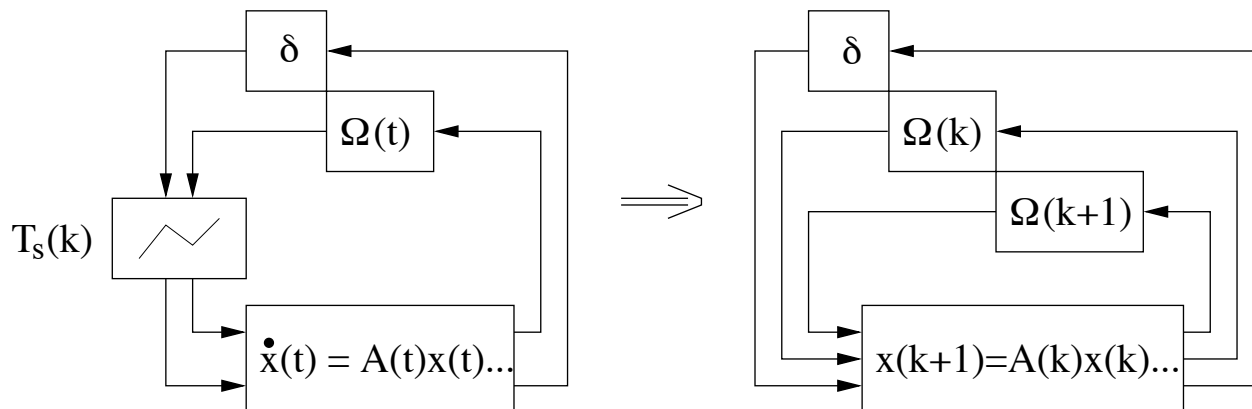
★ HYP 2: first-order hold approximation of exogenous signals [Imbert 2001]

$$w_\delta^c(t) \simeq w_\delta(k) + \frac{t - T_s(k)}{T_s(k+1) - T_s(k)} (w_\delta(k+1) - w_\delta(k)) \dots$$

★ HYP 3: β -bounded growth of the state x over sampling period ($\beta \geq 1$)

$$x^T(k)Qx(k) \leq q \Rightarrow x^T(k+1)Qx(k+1) \leq \beta q$$

HYP 0-3 : T -periodic continuous $\Rightarrow N$ -periodic discrete



$$x(k+1) = A(k)x(k) + B_\delta(k)\tilde{w}_\delta(k) + B_\Omega(k)\tilde{w}_\Omega(k)$$

$$\tilde{z}_\delta(k) = C_\delta(k)x(k) + D_{\delta\delta}\tilde{w}_\delta(k) + D_{\delta\Omega}(k)\tilde{w}_\Omega(k)$$

$$\tilde{z}_\Omega(k) = C_\Omega(k)x(k) + D_{\Omega\delta}\tilde{w}_\delta(k) + D_{\Omega\Omega}(k)\tilde{w}_\Omega(k)$$

$$\tilde{w}_\delta(k) = \delta\tilde{z}_\delta(k)$$

$$\tilde{w}_\omega(k) = \tilde{\Omega}(k)\tilde{z}_\Omega(k)$$

$$\tilde{\Omega} = \begin{bmatrix} \Omega(k) & 0 \\ 0 & \Omega(k+1) \end{bmatrix} \text{ with } \Upsilon_{\Xi \in \Xi_q}(\Omega(k)) \leq 0 \quad \text{if } x^T(k)Qx(k) \leq q.$$

$$\Upsilon_{\hat{\Xi} \in \hat{\Xi}_{\beta q}}(\Omega(k+1)) \leq 0$$

Invariant sets $x^T Q x \leq q$:

If there exist $\Xi(k) \in \Xi_q$, $\hat{\Xi}(k) \in \Xi_{\beta q}$, $\Theta(k) \in \Theta$
that satisfy for all $k = 1 \dots N$ the LMIs

$$N_x^T(k) \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} N_x(k) < \begin{aligned} & N_\delta^T(k) \Theta(k) N_\delta(k) \\ & + N_\Omega^T(k) \Xi(k) N_\Omega(k) \\ & + \hat{N}_\Omega^T(k) \hat{\Xi}(k) \hat{N}_\Omega(k) \end{aligned}$$

where N_x , N_δ , N_Ω and \hat{N}_Ω matrices are functions of model matrices A , $B_\delta \dots$
then for any bounded initial conditions such that $x^T(0) Q x(0) \leq q$
the trajectory remains bounded such that $x^T(k) Q x(k) \leq q$.

Convergence to periodic solution for initial conditions $x^T \hat{Q} x \leq q$:

If there exist $P(k) \geq 0$, $\Xi(k) \in \Xi_q$, $\hat{\Xi}(k) \in \Xi_{\beta q}$, $\Theta(k) \in \Theta$

that satisfy for all $k = 1 \dots N$ the LMIs

$$\hat{Q}(k) = Q + P(k) > 0$$

$$N_x^T(k) \begin{bmatrix} \hat{Q}(k) & 0 \\ 0 & -\hat{Q}(k) \end{bmatrix} N_x(k) < \begin{aligned} & N_\delta^T(k) \Theta(k) N_\delta(k) \\ & + N_\Omega^T(k) \Xi(k) N_\Omega(k) \\ & + \hat{N}_\Omega^T(k) \hat{\Xi}(k) \hat{N}_\Omega(k) \end{aligned}$$

then for any bounded initial conditions such that $x^T(0) \hat{Q}(0) x(0) \leq q$

the trajectory satisfies invariance $x^T(k) Q x(k) \leq x^T(k) \hat{Q}(k) x(k) \leq q$

decreasing Lyapunov function $x^T(k+1) \hat{Q}(k+1) x(k+1) \leq x^T(k) \hat{Q}(k) x(k)$

and convergence to the periodic solution $x(k) = \eta_s(k) - \eta(k) \rightarrow 0$.

Need for tools to choose $\Theta \in \Theta$ and $\Xi \in \Xi_\gamma$

Vertex separator: If Θ satisfies the LMIs

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \Theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq 0, \Upsilon_\Theta(\underline{\delta}\mathbf{1}) \leq 0, \Upsilon_\Theta(\bar{\delta}\mathbf{1}) \leq 0$$

then $\Theta \in \Theta$ (i.e. $\Upsilon_\Theta(\delta\mathbf{1}) \leq 0 \quad \forall \delta \in [\underline{\delta}, \bar{\delta}]$)

LMIs conditions of Ξ for $\Upsilon_\Xi(\Omega(x)) \leq 0$ whatever $x^T Q x \leq \gamma$?

Example: $\Omega(x) = \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \\ 0 & x_2 \\ 0 & x_1^2 + x_2^2 \end{bmatrix}$ with $x^T Q x = x^T x = x_1^2 + x_2^2$.

Take Ξ structured as

$$\Xi = \left[\begin{array}{cc|cccc} -\gamma(\alpha_1 + \alpha_4) & -\gamma(\alpha_2 + \alpha_5) & 0 & 0 & 0 & \alpha_2 \\ -\gamma(\alpha_2 + \alpha_5) & -\gamma(2\alpha_3 + \alpha_6 + \gamma\alpha_7) & 0 & 0 & 0 & \alpha_3 \\ \hline 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 + \alpha_4 & \alpha_5 & 0 \\ 0 & 0 & 0 & \alpha_5 & \alpha_6 & 0 \\ \alpha_2 & \alpha_3 & 0 & 0 & 0 & \alpha_7 \end{array} \right]$$

$$\Xi_1 = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 2\alpha_3 \end{bmatrix} \geq 0, \quad \Xi_2 = \begin{bmatrix} \alpha_4 & \alpha_5 \\ \alpha_5 & \alpha_6 \end{bmatrix} \geq 0, \quad \Xi_3 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_7 \end{bmatrix} \geq 0$$

then

$$\Upsilon_{\Xi}(\Omega(x)) = (x_1^2 + x_2^2 - \gamma)\Xi_1 + (x_2^2 - \gamma)\Xi_2 + ((x_1^2 + x_2^2)^2 - \gamma^2)\Xi_3 \leq 0$$

for all $x^T Q x = x^T x = x_1^2 + x_2^2 \leq \gamma$.

Non-linear system $\ddot{\eta}_1 + (\eta_1^2 + \dot{\eta}_1^2 - 1)\dot{\eta}_1 + \eta_1 = u$

with uncertain control input $u(t) = (\kappa + \delta)(\cos t - \eta_1(t)) \Rightarrow \eta_{s1}(t) = \cos t$

Tests made for control gain $\kappa = 4.5$ and $\bar{\delta} = -\underline{\delta} = 0.5$ uncertainty.

LMI tests

LMIs solved for $N = 20$ samples per period (uniformly spaced),

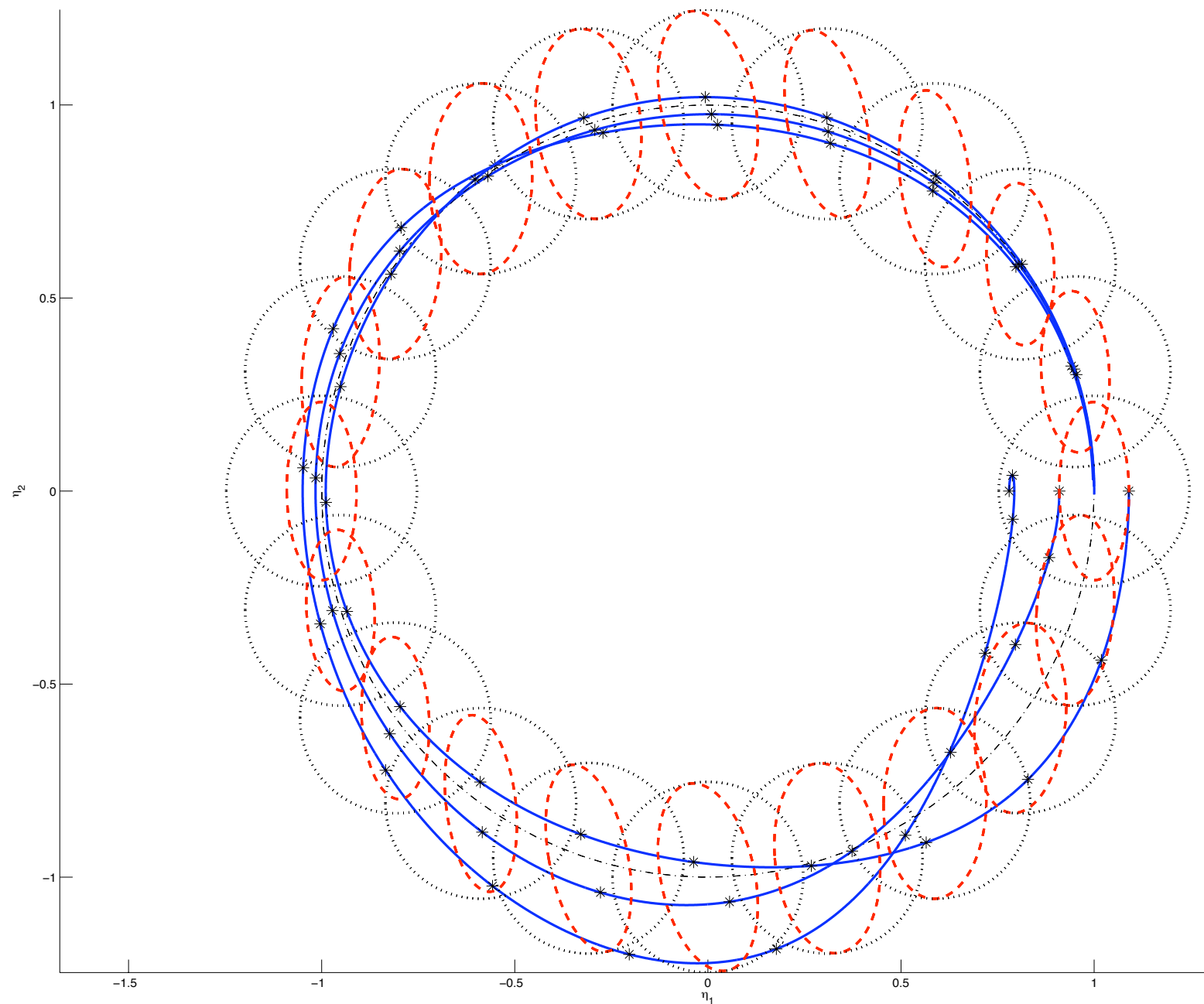
$\beta = 1.5$ (assumed bound on state growth over one sample)

and various values of q : feasible for all $q \leq q_{max} = 0.056$.

When taking $\beta = 1.1$ then $q_{max} = 0.061$.

Computation time (YALMIP+SeDuMi, 3GHz+1GB) $\simeq 1.4$ sec.

Simulations for $\eta_1(0) = 0.78, 0.91 and $1.09$$



Conclusions

- ➔ Methodology to analyze robust stability of periodic trajectories
- ➔ HYP 3 on bounded growth of the state embeds errors due to sampling
- LMI type results provide access to efficient numerical tools
- Results are conservative

- State-feedback design and others may be considered as well
- Difficulties to describe the sets Ξ via LMIs.