S-variables for the positivity check of matrix polynomials with matrix indeterminates

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Introduction

Many mathematical tools in robust control for building LMI conditions

- Lyapunov, S-procedure, KYP, DG-scaling, IQC, Quadratic Separation,
  Finsler lemma, S-variables, Positivstellensatz, SOS, Polya...

- Developed separately for specific uncertainties and view points
- Conservative SDP relaxations
- Hierarchies of relaxations with decreasing conservatism

This presentation: Attempt to establish links between these tools

- Positivity of matrix polynomials with matrix indeterminates
- Continuation of work of S-variable approach [Ebihara]
- Strongly inspired by Quadratic Separation [Iwasaki]
- Connexions to be done with Generalized Frequency Variables [Hara]
- Technicalities linking SOS and S-variables [Sato]
Motivation - Lyapunov

- Stability of a linear system $\dot{x} = Ax$

- All eigenvalues of $A$ have negative real part

- $sI - A$ is non-singular for all $s \in \mathbb{C}_+$

- $I - As^{-1}$ is non-singular for all $s^{-1} \in \mathbb{C}_+$

- $\exists \epsilon : (I - As^{-1})^*(I - As^{-1}) \succeq \epsilon I > 0$ for all $s^{-1} + s^* \geq 0$

- Matrix valued polynomial inequality constrained by a polynomial inequality

- Indeterminate is complex-valued $s^{-1} \in \mathbb{C}$

- $\exists P \succeq 0, \exists \epsilon > 0$ such that $A^*P + PA \preceq -\epsilon I$

- **Equivalent** LMI formulation

- $P$ is a Lagrange-like multiplier such that $P(s^{-1} + s^*) \succeq 0$
Stability of a linear system \( \dot{x} = Ax \) with \( A \in CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} \)

\[
CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} = \{ A = \sum_{v=1}^{\bar{v}} \xi_v A^{[v]} : \xi_v \geq 0, \sum_{v=1}^{\bar{v}} \xi_v = 1 \}
\]

- \( \lambda(A) \) have negative real part \( \forall A \in CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} \)
- \( I - As^{-1} \) is non singular \( \forall s^{-1} \in \overline{C}_+, \forall A \in CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} \)
- \[
\begin{bmatrix}
I & -s^{-1}I \\
-A & I
\end{bmatrix}
\]
is non singular \( \forall s^{-1} \in \overline{C}_+, \forall A \in CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} \)

- \( \exists \epsilon : \left[ \begin{array}{cc}
I & -s^{-1}I \\
-A & I
\end{array} \right]^* \left[ \begin{array}{cc}
I & -s^{-1}I \\
-A & I
\end{array} \right] \geq \epsilon I > 0 \) for all \( s^{-1} + s^{-*} \geq 0 \)
  \( A \in CO\{A^{[1]}, \ldots, A^{[\bar{v}]}\} \)

▲ Matrix polynomial inequality with indeterminates constrained by polynomial inequalities & polytopes

▲ Indeterminates are in independent rows and columns
Motivation - Lyapunov & S-variables

- Stability of a linear system $\dot{x} = Ax$ with $A \in CO\{A^1, \ldots, A^{\bar{v}}\}$

  $\exists \epsilon : \begin{bmatrix} I & -s^{-1} \\ -A & I \end{bmatrix} * \begin{bmatrix} I & -s^{-1} \\ -A & I \end{bmatrix} \succeq \epsilon I \succ 0$ for all $s^{-1} + s^{-*} \geq 0$

  $A \in CO\{A^1, \ldots, A^{\bar{v}}\}$

- Matrix polynomial inequality with indeterminates constrained by polynomial inequalities & polytopes

  $\exists S : \forall v = 1 \ldots \bar{v}, \exists P[v] \succeq 0$ such that

  $\begin{bmatrix} \epsilon I & P[v] \\ P[v] & \epsilon I \end{bmatrix} \succeq S \begin{bmatrix} A[v] & -I \end{bmatrix} + (S \begin{bmatrix} A[v] & -I \end{bmatrix})^*$

- Conservative LMI formulation

  $P(A) = \sum_{v=1}^{\bar{v}} \xi_v P[v]$, parameter-dependent, s.t. $P(A)(s^{-1} + s^{-*}) \succeq 0$

- S-variable copes with the polytopic uncertainty
Well-posedness of $\Delta \star M$

- $\Delta = \begin{bmatrix} \delta_1 I_{r_1} & 0 \\ 0 & \Delta_k \end{bmatrix}$
- \( \star \) : feedback-loop

- $\tilde{w}$
- $w$
- $z$
- $\tilde{z}$

- Well-posedness: internal \((w, z)\) bounded for all bounded disturbances \((\tilde{w}, \tilde{z})\)
Motivation - DG-scalings

**Well-posedness of** $\Delta \star M$

\[ \Delta = \begin{bmatrix} \delta_1 I_{r_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Delta_k \end{bmatrix} \]

- independent uncertainties
- scalar repeated or matrix valued
- real or complex
- norm-bounded by 1: $|\delta_k| \leq 1$ or $\|\Delta_k\| \leq 1$

\[ \delta_k \in \mathbb{C}, \quad |\delta_k| \leq 1 \quad \Leftrightarrow \quad 1 \geq \delta_k^* \delta_k \]
\[ \Delta_k \in \mathbb{C}^{m_{1k}, m_{2k}}, \quad \|\Delta_k\| \leq 1 \quad \Leftrightarrow \quad I \succeq \Delta_k \star \Delta_k \]
\[ \delta_k \in \mathbb{R}, \quad |\delta_k| \leq 1 \quad \Leftrightarrow \quad -j\delta_k^* + j\delta_k = 0, \quad 1 \geq \delta_k^* \delta_k \]
\[ \delta_k \in \mathbb{R}, \quad |\delta_k| \leq 1 \quad \Leftrightarrow \quad \delta_k \in CO \{-1, 1\} \]

- Indeterminates constrained by polynomial inequalities/equalities & polytopes
- Uncertainties are repeated $I_{r_k} \otimes \Delta_k$ (generalization of $\delta_1 I_{r_1}$ to matrices)
Motivation - DG-scalings

- Well-posedness of $\Delta \star M$

- $\Delta = \ldots$

- $\star$: feedback-loop

- Well-posedness: internal $(w, z)$ bounded for all bounded disturbances $(\tilde{w}, \tilde{z})$

$$
\begin{bmatrix}
I_m & \Delta \\
M & I_m
\end{bmatrix}
\begin{bmatrix}
w \\
z
\end{bmatrix} =
\begin{bmatrix}
\tilde{w} \\
\tilde{z}
\end{bmatrix}
$$

- $\exists \epsilon > 0$ such that for all admissible $\Delta$

$$
(I_{m_2} - M \Delta)^* (I_{m_2} - M \Delta) \succeq \epsilon I_{m_2}
$$

- Matrix polynomial inequality with indeterminates constrained by polynomial inequalities/equalities & polytopes

- Indeterminates are in independent rows and columns ($\Delta$ block-diagonal)
Motivation - DG-scalings

- Well-posedness of $\Delta \star M$

\[ \Delta = \begin{bmatrix}
I_{r_1} \otimes \Delta_1 & 0 \\
\vdots & \ddots \\
0 & I_{r_k} \otimes \Delta_k
\end{bmatrix} \]

- polynomial inequality on $\Delta_k$

- polynomial equality on $\Delta_k$

- polytopic constraint $\Delta_k$

- $\exists \epsilon > 0$ such that for all admissible $\Delta$

\[ (I_{m_2} - M\Delta)^* (I_{m_2} - M\Delta) \succeq \epsilon I_{m_2} \]

- Conservative LMI condition

\[ \exists D_k \succeq 0, G_k : \begin{bmatrix}
I & M^* \\
M & \Theta(D_k, G_k)
\end{bmatrix} \begin{bmatrix}
I \\
M
\end{bmatrix} \succeq \epsilon I \]

- $\Theta(D_k, G_k)$ : linear in the decision variables

- $D_k, G_k$ : Lagrange-like multipliers w.r.t inequality and equality constraints
Motivation - Proving positivity under constraints

- Robustness analysis of linear time-invariant systems

- Most problems can be recast as proving positivity of polynomials
  - matrix valued (semi-definite constraints)
  - indeterminates are matrices (or scalars), complex valued
  - constrained by polynomial inequalities, equalities and in polytopes

- Many LMI results in the literature,
  - in general \textbf{conservative} (problems are NP-hard)
  - some results are proved to be less conservative
  - on examples conservatism may vanish
  - duality of SDPs can extract worst case indeterminates (prove conservatism vanishes)
  - Numerical issues: limit size of LMIs using the structure of the data
Introduction

- Many mathematical tools in robust control for building LMI conditions
  - Lyapunov, S-procedure, KYP, DG-scaling, IQC, Quadratic Separation, Finsler lemma, S-variables, Positivstellensatz, SOS, Pólya...
- Developed separately for specific uncertainties and view points
- Conservative SDP relaxations
- Hierarchies of relaxations with decreasing conservatism
- This presentation: Attempt to establish links between these tools
- Positivity of matrix polynomials with matrix indeterminates
- Continuation of work of S-variable approach [Ebihara]
- Strongly inspired by Quadratic Separation [Hara, Iwasaki]
- Technicalities linking SOS and S-variables [Sato]
Goal: proving positivity of a matrix valued polynomial $F(\delta) \succeq 0$

- with scalar indeterminates $\delta \in \mathbb{R}^k$
- constrained by scalar polynomial inequalities $f_i(\delta) \geq 0$.

Some key methods for solving the problem using SDPs

- Positivstellensatz
- Polynomials modeled as quadratic functions of monomials
- SDP relaxation
- Hierarchies
- Moment problem
\( F(\delta) \geq 0 \) for all \( \delta \in \mathbb{R}^k \) such that \( f_i(\delta) \geq 0 \)

\[ \exists D_i(\delta) \text{ SOS : } F_D(\delta) = d_0(\delta) F(\delta) + \sum D_i(\delta) f_i(\delta) \text{ SOS} \]

\( \uparrow \) Lossless when taking polynomials \( D_i(\delta) \) with sufficiently high order

\( \uparrow \) Variants of Positivstellensatz are: S-procedure, D-scalings, Lagrange multipliers...
Sum-Of-Squares

- $F(\delta) \succeq 0$ for all $\delta \in \mathbb{R}^k$ such that $f_i(\delta) \geq 0$

- Positivstellensatz: Polynomial positive multipliers for each constraint

  \[ \exists D_i(\delta) \text{ SOS } : \quad F_D(\delta) = d_0(\delta)F(\delta) + \sum D_i(\delta)f_i(\delta) \text{ SOS} \]

- Express the polynomials as a quadratic functions of monomials

  \[
  \delta^{\{0:p\}*} = \begin{bmatrix}
  I & \delta_1 I & \delta_2 I & \cdots & \delta_1\delta_2 I & \cdots & \delta_1^{p_1}\delta_2^{p_2} I & \cdots
  \end{bmatrix}
  \]

  \[
  D_i(\delta) = \delta^{\{0:p\}*}Q_i(d_i)\delta^{\{0:p\}}, \quad F_D(\delta) = \delta^{\{0:p\}*}Q(d)\delta^{\{0:p\}}
  \]

  $Q_i(d_i)$ and $Q(d)$ are linear in decision variables $d_i$ (coeff. of polynomials $D_i(\delta)$)

- Quadratic representations are not unique

  \[
  F_D(\delta) = \delta^{\{0:p\}*}(Q(d) + V)\delta^{\{0:p\}} \quad \text{whatever } V \text{ s.t. } \delta^{\{0:p\}*}V\delta^{\{0:p\}} = 0 \quad \forall \delta
  \]

  these are linear constraints on $V$ denoted $V \in \mathcal{N}(\delta^{\{0:p\}})$
$F(\delta) \geq 0$ for all $\delta \in \mathbb{R}^k$ such that $f_i(\delta) \geq 0$

- Positivstellensatz + monomials (for large enough order $p$)

\[ \exists \delta^{\{0:p\}} \ast (Q_i(d_i)) \delta^{\{0:p\}} \quad SOS : \quad \delta^{\{0:p\}} \ast (Q(d)) \delta^{\{0:p\}} \quad SOS \]

- SDP relaxation (for fixed order $p$)

\[ \exists V_i \in \mathcal{N}(\delta^{\{0:p\}}) : \quad Q_i(d_i) + V_i \succeq 0 \quad , \quad Q(d) + V \succeq 0 \]

- LMI : convex and exist efficient solvers (for not too large size problems)

- Conservative

- Parameterization of $\mathcal{N}(\delta^{\{0:p\}})$ is not easy
Sum-Of-Squares

\[ F(\delta) \succeq 0 \text{ for all } \delta \in \mathbb{R}^k \text{ such that } f_i(\delta) \geq 0 \]

- Positivstellensatz + monomials + SDP relaxation (for fixed order \( p \))

\[ \exists V_i \in \mathcal{N}(\delta^{\{0,p\}}) : Q_i(d_i) + V_i \succeq 0 \ , \ Q(d) + V \succeq 0 \]

- Hierarchies: increasing \( p \), the order of the polynomials (and scalings)
  - Conservatism decreases (rapidly)
  - Under mild assumption conservatism vanishes (for finite and low orders)
  - Numerical burden increases rapidly (need to exploit the structure)

- Dual of the LMIs is the relaxation of generalized moment problem
Motivation - Proving positivity under constraints

- Robustness analysis of linear time-invariant systems
- Most problems can be recast as proving positivity of polynomials
  - matrix valued (semi-definite constraints)
  - indeterminates are matrices (or scalars), complex valued
  - constrained by polynomial inequalities, equalities and in polytopes
- Many LMI results in the literature,
  - in general **conservative** (problems are NP-hard)
  - some results are proved to be less conservative
  - on examples conservatism may vanish
  - duality of SDPs can extract worst case indeterminates (prove conservatism vanishes)
  - Numerical issues: limit size of LMIs using the structure of the data
- Same characteristics in the SOS-Moment framework
  - Unification needs to manipulate polynomials with matrix indeterminates

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Nagoya, November 9, 2018
Goal: proving $F(\Delta) = F(I_{r_1} \otimes \Delta_1, \ldots, I_{r_{\bar{k}}} \otimes \Delta_{\bar{k}}) \succeq 0$

under polynomials + polytopic constraints of the type

$$F_{i_k}(\Delta_k) \succeq 0, \quad F_{e_k}(\Delta_k) = 0, \quad \Delta_k \in CO\{\Delta^{[1]}_k, \ldots, \Delta^{[\bar{v}_k]}_k\}$$
Polynomials with matrix indeterminates

- Monomials with matrix indeterminates \( \Delta_k \in \mathbb{C}^{m_1 \times m_2} \)
  \[
  \Delta_k \{0\} = I_{m_2}, \quad \Delta_k \{1\} = \Delta_k, \quad \Delta_k \{2\} = \Delta_k \Delta_k, \\
  \Delta_k \{3\} = \Delta_k \Delta_k \Delta_k, \quad \Delta_k \{4\} = \Delta_k \Delta_k \Delta_k \Delta_k, \ldots
  \]

- Matrix with monomials from degree 0 to degree \( p_k \):
  \[
  \Delta_k \{0:p_k\} = \begin{bmatrix}
    \Delta_k \{0\} \\
    \vdots \\
    \Delta_k \{p_k\}
  \end{bmatrix} = \begin{bmatrix}
    I_{m_2} \\
    \Delta_k \\
    \Delta_k \Delta_k \\
    \vdots
  \end{bmatrix}, \quad \left( I_{r_k} \otimes \Delta_k \right) \{0:p_k\} = \begin{bmatrix}
    I_{r_k} \otimes \Delta_k \{0\} \\
    \vdots \\
    I_{r_k} \otimes \Delta_k \{p_k\}
  \end{bmatrix}
  \]

- \( \Delta \{0:p\} = \begin{bmatrix}
    (I_{r_1} \otimes \Delta_1) \{0:p_1\} & 0 & \ldots \\
    0 & \ddots & \ddots \\
    0 & \ldots & (I_{r_{\bar{k}}} \otimes \Delta_{\bar{k}}) \{0:p_{\bar{k}}\}
  \end{bmatrix} \)
Polynomials with matrix indeterminates

With assumption that $\Delta_k$ enter in independent columns

$$F(\Delta) = \Delta^{\{0:p\}*}(F_0 + F_1^*F_1)\Delta^{\{0:p\}} \succeq 0$$

under constraints

$$F_{ik}(\Delta_k) = \Delta_k^{\{0:p_k\}*}\Phi_{ik} \Delta_k^{\{0:p_k\}} \succeq 0, \quad \Delta_k \in CO\{\Delta_k^{[1]} , \ldots , \Delta_k^{\bar{v_k}}\}$$

$$F_{ek}(\Delta_k) = \Delta_k^{\{0:p_k\}*}\Phi_{ek} \Delta_k^{\{0:p_k\}} = 0, \quad \Delta_k \in CO\{\Delta_k^{[1]} , \ldots , \Delta_k^{\bar{v_k}}\}$$

"Positivstellensatz" : Exist $D_k(\Delta) \succeq 0$, $G_k(\Delta) = G_k^*(\Delta)$ such that

$$\Delta^{\{0:p\}*}(F_0 + F_1^*F_1 + \text{diag}(\cdots D_k(\Delta) \boxtimes \Phi_{ik} + G_k(\Delta) \boxtimes \Phi_{ek} \cdots))\Delta^{\{0:p\}} \succeq 0$$

under polytopic constraints $\Delta_k \in CO\{\Delta_k^{[1]} , \ldots , \Delta_k^{\bar{v_k}}\}$ ($\bar{v}_k = 0$ if no constraint)

Conservative ?

How to build the SDP relaxation for fixed order $p$ ?

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Nagoya, November 9, 2018
Exists affine $H_k(\Delta_k) = J_0 + J_1 (I \otimes \Delta_k) J_2 + J_3 (I \otimes \Delta_k^*) J_4$

such that $\Delta_k \{0:p_k\}$ spans the null space of $H_k(\Delta_k)$
\( \Delta_k^{\{0:p_k\}} \) null space of \( H_k(\Delta_k) = J_0 + J_1(I \otimes \Delta_k)J_2 + J_3(I \otimes \Delta_k^*)J_4 \)

SDP relaxation for \( \Delta_k \) in polytope

assuming \( \Psi( X(\Delta) ) \) is affine of indeterminate-dependent \( X(\Delta) \)

\[ \exists S_k, X[v_k] : \Psi( X[v_k] ) + S_k H_k(\Delta_k[v_k]) + (S_k H_k(\Delta_k[v_k]))^* \succeq 0 \quad \forall v_k = 1 \ldots \bar{v}_k \]

\[ \Downarrow \]

\[ \exists X(\Delta) : \Delta_k^{\{0:p_k\}*} \Psi( X(\Delta) ) \Delta_k^{\{0:p_k\}} \succeq 0 \quad \forall \Delta_k \in CO\{\Delta_k^{[1]}, \ldots, \Delta_k^{[\bar{v}_k]} \} \]
S-variables and SDPs for SOS

Proof

\[ \text{Affine } \Psi \text{ and } M_k \text{ and } \Delta_k = \sum_{v_k=1}^{\bar{v}_k} \xi_v \Delta_k^{[v_k]} \]

\[ \exists S_k, X^{[v_k]} : \Psi(X^{[v_k]}) + S_k H_k(\Delta_k^{[v_k]}) + (S_k H_k(\Delta_k^{[v_k]}))^* \succeq 0 \quad \forall v_k = 1 \ldots \bar{v}_k \]

\[ \vdash \exists S_k, : \Psi(X(\Delta)) + S_k H_k(\Delta_k) + (S_k H_k(\Delta_k))^* \succeq 0 \quad \forall \xi_v \geq 0, \quad \sum_{v=1}^{\bar{v}} \xi_v = 1 \]

\[ X(\Delta) = \sum_{v_k=1}^{\bar{v}_k} \xi_v X^{[v_k]} \]

By congruence with the fact that \( H_k(\Delta_k^{[0:p_k]}) \Delta_k^{[0:p_k]} = 0 \)

\[ \Downarrow \]

\[ \exists X(\Delta) : \Delta_k^{[0:p_k]} \Psi(X(\Delta)) \Delta_k^{[0:p_k]} \succeq 0 \quad \forall \Delta_k \in CO\{\Delta_k^{[1]}, \ldots, \Delta_k^{[\bar{v}_k]} \} \]

Conservatism comes for the choice of indeterminate-independent \( S_k \)
S-variables and SDPs for SOS

- $\Delta_k\{0:p_k\}$ null space of $H_k(\Delta_k) = J_0 + J_1(I \otimes \Delta_k)J_2 + J_3(I \otimes \Delta_k^*)J_4$

- SDP relaxation for unbounded $\Delta_k$

  $\exists \hat{S}_k, X : \Psi(X) + \hat{S}_kH_k(0) + (\hat{S}_kH_k(0))^* \succeq 0 \ \forall v_k = 1 \ldots \bar{v}_k$

  $\hat{S}_k = \begin{bmatrix} J_T^4 & J_T^2 \\ J_T^4 & J_T^2 \end{bmatrix} \begin{bmatrix} T_k \otimes I & 0 \\ 0 & T_k \otimes I \end{bmatrix} \begin{bmatrix} J_T^4 \\ J_T^3 \end{bmatrix}^T$, $T_k = -T_k^*$

  $\Downarrow$

  $\exists X : \Delta_k\{0:p_k\}^*\Psi(X)\Delta_k\{0:p_k\} \succeq 0 \ \forall \Delta_k$

\[\text{\textbullet} \ \text{Structured S-variables provides a parameterization of } \mathcal{N}(\Delta_k\{0:p_k\})\]
\[\text{\textbullet} \ \text{Is } \mathcal{N}(\Delta_k\{0:p_k\}) \text{ fully parameterized in this way?}\]
\[\text{\textbullet} \ \text{SOS-Moment SDP relaxations are a special case of S-variable relaxation}\]
Main result (combining described techniques)

\[
\begin{aligned}
\hat{F}_1^{\perp*} \left( \hat{F}_0 - \text{diag} \begin{pmatrix} D_{k}^{[v_{K_2}]} \otimes \Phi_{ik} + G_{k}^{[v_{K_2}]} \otimes \Psi_{ek} \end{pmatrix} \right) \hat{F}_1^{\perp} \\
+ \sum_{k \in K_1} \left\{ \hat{F}_1^{\perp*} \hat{S}_{k}^{[v_{K_2}]} \hat{H}_k(0) \hat{F}_1^{\perp} \right\}^\mathcal{H} + \sum_{k \in K_2} \left\{ S_{k}^{[v_{K_2}\setminus k]} \hat{H}_k(\Delta_k^{[v_k]}) \hat{F}_1^{\perp} \right\}^\mathcal{H} \succeq 0
\end{aligned}
\]

\( K_1, K_2 \): uncertainties without and with polytopic constraints respectively

\( \mathcal{K}_1, \mathcal{K}_2 \): uncertainties without and with polytopic constraints respectively

\( \mathcal{K}_1 \): uncertainties without polytopic constraints

\( \mathcal{K}_2 \): uncertainties with polytopic constraints

Large number of constraints and decision variables (when several polytopes)

Are all the decision variables needed?

Can we build hierarchies as in usual SOS methods?

When applied to special cases we get exactly the same LMI conditions
Conclusion

▲ Ongoing work to make links between many existing results in Robust Control
▲ Used tool is inspired by SOS-Moments relaxations
▲ Motivation for dealing with polynomial matrix inequalities of matrix indeterminates
▼ Sub-case of all possible polynomial matrix inequalities of matrix indeterminates
▼ At this stage does not allow to build non-existing results
▼ Numerical experiments to be done