Conditions LMI de synthèse de commandes adaptatives directes
pour les systèmes linéaires ‘presque stables’

Dimitri PEAUCELLE
LAAS-CNRS - Université de Toulouse - FRANCE

Cooperation program between CNRS, RAS and RFBR:
A. Fradkov, B. Andrievsky
Application to Demeter satellite with CNES: A. Drouot, Ch. Pittet, J. Mignot
Application to ‘helicopter’ benchmark: B. Andrievsky, V. Mahout
Simple adaptive control (SAC)

For a system $y(t) = \sum u (t)$ to follow reference $y_r$

$$u(t) = K(t)e(t) \ , \ \dot{K}(t) = -Gy(t)e^T(t)\Gamma \ , \ e(t) = y(t) - y_r(t)$$

- $K$ is driven to minimize the square of the error $J(t) = e^T(t)e(t)$
- In the scalar case

$$\dot{k} = -\gamma \frac{\partial (y - y_r)}{\partial k} (y - y_r) \simeq -\gamma gy e$$

- $Gy$: approximation of the gradient of $J$ with respect to $K$ (for the closed-loop)
- $\Gamma > 0$: weight on the adaptation speed
- [Fradkov, Kaufman et al, Ioannou, Barkana]

$G$ is chosen with respect to closed-loop passivity conditions

- $G$ is chosen with respect to closed-loop passivity conditions

△ Choice of $G$ depends on the systems model

△ Adaptive control intended for uncertain systems: robust
Outline

1. Passivity conditions for Simple Adaptive Control (SAC)
   - LMI formulas for SAC - robustness issues

2. Robustness to noise on measurements & to parametric uncertainty
   - Barrier type corrective term
   - Passivity with respect to an output with feedthrough gain

3. Examples and some other features of SAC
   - $L_2$ performance
   - Stable neighborhood of the origin in case of time-varying systems
Passivity conditions for SAC

SAC for LTI systems

- Let a linear system \( \dot{x} = Ax + Bu \), \( y = Cx \) \( (u \in \mathbb{R}^m, y \in \mathbb{R}^p, m \leq p) \)
- and SAC \( u = Ky \), \( \dot{K} = -Gyy^T \Gamma \)
- Closed-loop stability is guaranteed if

\[ \exists F : \dot{x} = (A + BFC)x + Bw, \quad z = GCx \text{ strictly passive} \]

or equivalently if

\[ \exists F, P > 0 : (A + BFC)^T P + P(A + BFC) < 0, \quad PB = C^T G^T \]

- Proof using Lyapunov function

\[ V(x, K) = x^T Px + \text{Tr}((K - F)\Gamma^{-1}(K - F)^T). \]
\[ \dot{V} = x^T \Upsilon x + 2x^T PB(K - F)y + 2\text{Tr}(\dot{K}\Gamma^{-1}(K - F)^T) = x^T \Upsilon x \]
Passivity conditions for SAC

SAC versus SOF

- Closed-loop stability with SAC is guaranteed if system is ‘almost passive’

\[
\exists F, P : \begin{bmatrix}
(A + BFC)^T P + P(A + BFC) < 0 \\
\end{bmatrix} \,
\]

\[
 PB = C^T G^T
\]

\[\Upsilon\]

\[\Upsilon\]

- Stability with SAC proved by existence of some stabilizing SOF \((u = F y)\)
  - Why complicating the control ?

- The condition happens to be LMI+E (for given \(G\)):

\[
\exists F, P : A^T P + PA + C^T (G^T F + F^T G) C < 0 
\]

\[
 PB = C^T G^T
\]

- Any \(F = -kG\) with \(k\) large enough stabilizes the system (high gain)

\[\Upsilon\]

- Not all SOF stabilizable systems will satisfy such constraints

\[\Upsilon\]

- The SAC design problem is to find \(G\): non convex problem.
Passivity conditions for SAC

Robustness of SAC

- Let an uncertain LTI system $\dot{x} = A(\Delta)x + B(\Delta)u$, $y = C(\Delta)x$
- and SAC $u = Ky$, $\dot{K} = -Gyy^T \Gamma$
- Closed-loop robust stability with SAC is guaranteed if $\exists F(\Delta), P(\Delta)$:
  \[
  \]
  \[
  P(\Delta)B(\Delta) = C^T(\Delta)G^T
  \]
- Robustness techniques may be applied to the LMI (given $G$)
- Equality constraint almost impossible to guarantee robustly
  \[
  P(\Delta)B(\Delta) = C^T(\Delta)G^T, \quad \forall \Delta \in \Delta
  \]
Passivity conditions for SAC

Divergence of SAC due to noise

Assume noisy measurements $y(t) = Cx(t) + d(t)$

$$\dot{K} = -Gyy^T \Gamma = -G(Cx + d)(x^T C^T + d^T) \Gamma$$

$\Delta$ $K(t)$ will diverge even if $x \to 0$ (if $d$ does not go to zero).

$\Delta$ Not acceptable in practice

Most often corrective terms are added such as

$$\dot{K} = -Gyy^T \Gamma - \mu(K - F_0)$$

$\Delta$ But then $K(t) \to F_0$:

the closed-loop characteristics tend to those with SOF $u = F_0 y$

- Why complicating the control?
Passivity conditions for Simple Adaptive Control (SAC)
- LMI formulas for SAC - robustness issues

Robustness to noise on measurements & to parametric uncertainty
- Barrier type corrective term
- Passivity with respect to an output with feedthrough gain

Examples and some other features of SAC
- $L_2$ performance
- Stable neighborhood of the origin in case of time-varying systems
Robustness to noise on measurements & to parametric uncertainty

Dead-zone + barrier corrective term

- Usual corrective term
  \[ \dot{K} = -Gyy^T \Gamma - \mu(K - F_0) \]

- Corrective term always active even if \( K \) does not diverge

- Corrective term does not guarantee \( K \) to be bounded in given set

- Proposed corrective term
  \[ \dot{K} = -Gyy^T \Gamma - \phi(K - F_0) \Gamma \]

    \[ \phi(\hat{K}) = \psi(\|\hat{K}\|^2)\hat{K} \]

    \[ \psi(0 \leq k \leq \nu) = 0 , \quad \frac{d\psi}{dk}(\nu \leq k \leq \beta \nu) > 0 , \quad \psi(\nu \beta) = +\infty \]

- Example: weighted Frobenius norm
  \[ \|\hat{K}\|^2 = \text{Tr}(\hat{K} \hat{D} \hat{K}^T) \]

    \[ \psi(\nu \leq k \leq \beta \nu) = \exp(\mu k - \log(\beta \nu - k)) \]
Robustness to noise on measurements & to parametric uncertainty

**Dead-zone + barrier corrective term**

- **Proposed corrective term**
  \[
  \dot{K} = -Gyy^T \Gamma - \phi(K - F_0) \Gamma
  \]

\[
\phi(\hat{K}) = \psi(||\hat{K}||^2) \hat{K}
\]

\[
\psi(0 \leq k \leq \nu) = 0 \quad , \quad \frac{d\psi}{dk}(\nu \leq k \leq \beta \nu) > 0 \quad , \quad \psi(\nu \beta) = +\infty
\]

- **Corrective term active only when**
  \[\|K - F_0\| > \nu\]

- **Guarantees that** \(K(t)\) **is bounded around** \(F_0\): \[\|K - F_0\| < \nu \beta\]

\[\beta > 1\] can be chosen arbitrarily based on practical considerations

- **\(\hat{D}\)** defines the geometry of the set \[\|\hat{K}\| = \text{Tr}(\hat{K} \hat{D} \hat{K}^T) \leq \nu\]

- **\(\nu\)** defines the dead-zone and barrier levels

\[\beta > 1\] can be chosen arbitrarily based on practical considerations

- **Best to maximize the set** \[\|\hat{K}\| \leq \nu\], *i.e.* maximize \(\nu\) and minimize \(\text{Tr} \hat{D}\)
Robustness to noise on measurements & to parametric uncertainty

Feedthrough gain for robust passivity

\[
\dot{x} = Ax + Bu, \quad y = Cx \quad \text{with SAC} \quad u = Ky, \quad \dot{K} = -Gyy^T \Gamma
\]

Closed-loop stability is guaranteed if

\[
\exists F : \dot{x} = (A + BFC)x + Bw, \quad z = GCx \quad \text{strictly passive}
\]

or equivalently if

\[
\exists F, P : (A + BFC)^T P + P(A + BFC) < 0, \quad PB = C^T G^T
\]

Need for conditions without equality constraints

\[
\Rightarrow \text{need for a feedthrough gain} (z = GCx + Dw)
\]
Robustness to noise on measurements & to parametric uncertainty

Feedthrough gain for robust passivity

- Passivity conditions without equality constraints

\[
\dot{x} = (A + BFC)x + Bw, \quad z = GCx + Dw \text{ strictly passive}
\]

if and only if

\[
\exists P : \begin{bmatrix}
(A + BFC)^T P + P(A + BFC) & PB - C^T G^T \\
B^T P - GC & -D - D^T
\end{bmatrix} < 0
\]

- Feedthrough gain $D$ always exists if system is SOF stabilizable
- If $D$ is small, then conditions are close to original ones
- Choice of $F = -kG$ with $k \gg 1$ no more valid
- Gains should be bounded
- Gains are bounded thanks to corrective term $\phi$
Robustness to noise on measurements & to parametric uncertainty

Main result - part 1

- Let $F_0$ be a stabilizing SOF for $\dot{x} = Ax + Bu$, $y = Cx$
- There exists $(P > 0, G, \hat{D})$ solution to

$$
\begin{bmatrix}
(A + BF_0C)^T P + P(A + BF_0C) & PB - C^T G^T \\
B^T P - GC & -\hat{D}
\end{bmatrix} < 0
$$

- minimize $\text{Tr} \hat{D}$ and choose
  - $G$ for the adaptation gain $\dot{K} = -Gyy^T \Gamma - \phi(K - F_0) \Gamma$
  - $\|\hat{K}\|^2 = \text{Tr}(\hat{K} \hat{D} \hat{K}^T)$ for the norm in the corrective term $\phi$
  - $F_0$ as the center of the set around which the adaptation is performed
Robustness to noise on measurements & to parametric uncertainty

Main result - part 2

\( (F_0, G, \hat{D}) \) being chosen, there exist \((Q > 0, R, F, T, \nu)\) solution to

\[
\begin{bmatrix}
R & QB - C^T G^T \\
B^T Q - GC & \hat{D}
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
T & (F - F_0)^T \\
(F - F_0) & \hat{D}^{-1}
\end{bmatrix} \geq 0, \quad \text{Tr} T \leq \nu
\]

\( (A + BF_0C)^T Q + Q(A + BF_0C) + \nu \beta C^T C + R 
+ C^T (G^T (F - F_0) + (F - F_0)^T G) C < 0 \)

maximize \( \nu \) and take it for the levels in the corrective term \( \phi \)

SAC defined by \((G, F_0, \hat{D}, \nu)\) stabilizes the system. Proof with

\[
V(x, K) = x^T Q x + \text{Tr}((K - F) \Gamma^{-1} (K - F)^T).
\]
Robustness to noise on measurements & to parametric uncertainty

Characteristic of the results

- ‘Almost passive’ conditions extended to ‘almost stable’
  SAC can be applied to all SOF stabilizable systems

- Stability is proved for SAC with the corrective barrier function
  Moreover, \( K(t) \) is strictly bounded, even w.r.t. perturbations and noise

- The gain \( K(t) \) remains ‘close’ to initial SOF guess \( F_0 \)
  Interesting feature for practitioners: keep close to a ‘safe’ situation
  Benefit of adaptation expected to be improved if domain is large

[Submitted to IFAC World Congress 2011, Milano]
Robustness to noise on measurements & to parametric uncertainty

Guaranteed robustness

- Results formulated as LMIs:
  
  Can be extended to uncertain models $A(\Delta), B(\Delta), C(\Delta)$

- Procedure for robust SAC design
  
  1. Choose an SOF $F_0$ (stabilizes nominal system $A(O), B(O), C(O)$)
  2. Solve first LMI problem (robust version) to get $G, \hat{D}$
  3. Solve second LMI problem (robust version) to get $\nu$

- Stability is proved with a parameter-dependent Lypunov function

\[
V(x, K) = x^T Q(\Delta) x + \text{Tr}((K - F(\Delta))\Gamma^{-1}(K - F(\Delta))^T).
\]

- SAC and parameter-dependent $u = F(\Delta)y$ stabilize the system
- SAC does it without measure/estimation of $\Delta$
Outline

1. Passivity conditions for Simple Adaptive Control (SAC)
   - LMI formulas for SAC - robustness issues

2. Robustness to noise on measurements & to parametric uncertainty
   - Barrier type corrective term
   - Passivity with respect to an output with feedthrough gain

3. Examples and some other features of SAC
   - $L_2$ performance
   - Stable neighborhood of the origin in case of time-varying systems
Examples and some other features of SAC

Demeter satellite

- Given stabilizing PD gains ($F_0$) replaced by adaptive gains
- LMIs solved on LTI model
design of $G$
and corrective term
Examples and some other features of SAC

Outputs of closed-loop system with $F_0$ (dotted) and SAC
Examples and some other features of SAC

Input of closed-loop system with $F_0$ (dotted) and SAC
Examples and some other features of SAC

Control gains of SAC

- \( K_p \): 0.15, 0.2, 0.25, 0.3, 0.35
- \( K_d \): 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55

Time [s]: 0, 50, 100, 150
Examples and some other features of SAC

From LTI to non-linear systems

- Parametric uncertain LTI systems $\simeq$ slowly TV systems

- Will properties be lost for non-linear or rapidly time-varying systems?

- LMI-based results for:
  - $L_2$ norm minimization: robustness to norm bounded non-linearities
  - LTV systems with bounded rates of variations (classical LPV hyp)
Examples and some other features of SAC

$L_2$-gain performance

- Lur’e type modeling of a close-to-linear non-linear system

\[
\dot{x}(t) = A(\Delta)x(t) + B_\Phi(\Delta)w_\Phi(t) + B(\Delta)u(t)\\
z_\Phi(t) = C_\Phi(\Delta)x(t) + D_\Phi(\Delta)w_\Phi(t)\\
y(t) = C(\Delta)x(t)
\]

- Small gain theorem: guarantee input/output performance

\[
\dot{x}(t) = A(\Delta)x(t) + B_\Phi(\Delta)w_\Phi(t) + B(\Delta)u(t)\\
z_\Phi(t) = C_\Phi(\Delta)x(t) + D_\Phi(\Delta)w_\Phi(t)\\
y(t) = C(\Delta)x(t)
\]

\[
\|w_\Phi\|_2 \leq \gamma \|z_\Phi\|_2
\]

\[
\|z_\Phi\|_2 \leq \frac{1}{\gamma} \|w_\Phi\|_2 \quad , \quad \|\Sigma(\Delta)\|_2 \leq \frac{1}{\gamma}
\]
Examples and some other features of SAC

$L_2$-gain performance

- LMI results that give a PD-SOF $F(\Delta)$ used to prove $L_2$ performance of SAC:

$$\| \Sigma(\Delta) \ast K(t) \|_2 \leq \| \Sigma(\Delta) \ast F(\Delta) \|_2$$

- Guaranteed $L_2$ performance of SAC
- Not worse than the PD-SOF
- Result explained by the fact that SAC is conceived to minimize square of error
- No similar result expected for other criteria (convergence time, etc.)

Examples and some other features of SAC

UAV Example

4 states, 2 scalar uncertainties, $\delta_2 \in [0, 2.5]

Tests on large intervals of $\delta_1$

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\min \gamma$</th>
<th>$\delta_1$</th>
<th>$\min \gamma$</th>
<th>$\delta_1$</th>
<th>$\min \gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-1, 0]$</td>
<td>0.2</td>
<td>$[0.7, 0.72]$</td>
<td>141</td>
<td>$[0.72, 0.722]$</td>
<td>1001</td>
</tr>
<tr>
<td>$[-1, 0.7]$</td>
<td>24</td>
<td>$[0.7, 0.73]$</td>
<td>infeas.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[-1, 0.72]$</td>
<td>infeas.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

infeas.
Examples and some other features of SAC

UAV Example

Tests on small intervals of $\delta_1$
Examples and some other features of SAC

SAC simulations with impulse disturbances $w_L$ (every 20s) and slowly varying $\delta_1$ (beyond proved stable values).
Examples and some other features of SAC

UAV Example

Zoom on the output responses.
Examples and some other features of SAC

UAV Example

Time histories of the SAC gains
Examples and some other features of SAC

UAV Example

\( \nu = 10, \beta = 1.2 \) : the gains are bounded \( \text{Tr}(K^T K) \leq \nu \beta \).
Robust stability in case of time varying uncertainties

Uncertain time-varying linear system

\[\dot{x}(t) = A(\Delta(t))x(t) + B(\Delta(t))u(t), \quad y = C(\Delta(t))x(t)\]

Stability proof based on the Lyapunov function \(V(x, K, \Delta) = \)

\[x^T(t)P(\Delta(t))x(t) + \text{Tr}(K(t) - F(\Delta(t))\Gamma^{-1}(K(t) - F(\Delta(t))))^T\]

\(\Delta\) If \(\dot{\Delta}\) is unbounded, then \(\dot{V}(x, K, \Delta)\) exists only if:

\(P(\Delta) = P, F(\Delta) = F,\) are constant

i.e. the robust stabilisation is solved with constant SOF \(F\).

\(\Delta\) If \(\dot{\Delta}\) is bounded, then [Auto.R.Ctr’09], LMI conditions for

\(\dot{V}(x, K, \Delta) < 0\) whatever \(x\) s.t. \(x^TQx \geq 1,\)

i.e. Lasalle’s principle \(x^TQx \leq 1\) attractive set.

\(\bullet\) Attractive domain can be made arbitrarily small if \(\dot{\Delta} \to 0\) or \(\Gamma \to \infty\)

\[u(t) = K(t)y(t) + w(t), \quad \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\]
Robust stability in case of time varying uncertainties

Example: State of the UAV for input impulses every 20s and

\[ \delta_1(t) = 0.75 \sin(0.125t + 3\pi/2) + 0.1 \sin(49t + 3\pi/2) - 0.15 \leq 0.7 \]
Robust stability in case of time varying uncertainties

Example Gains of SAC:
Conclusions

- SAC revisited in LMI-based Robust Control framework
- Guaranteed robustness
- Form ‘almost passive’ to ‘almost stable’ systems
- Bounded control gains in given regions

- SAC is intended for non-linear systems
  - Implementation not trivial: choosing $\Gamma$, $\psi$...
  - Need to validate on real applications
  - Can other features such as rapidity, damping, noise rejection performance etc. be treated?