Robust stability analysis of discrete-time systems with parametric and switching uncertainties

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Study of LMIs for stability analysis of discrete-time polytopic systems

\[ x_{k+1} = A(\theta_k)x_k, \quad A(\theta) = \sum_{v=1}^{\bar{v}} \theta_v A^{[v]} : \quad \theta \in \Xi_{\bar{v}} = \left\{ \theta_{v=1} \ldots \bar{v} \geq 0, \sum_{v=1}^{\bar{v}} \theta_v = 1 \right\} \]

Classical “quadratic stability” result [Bar85]

\[ \exists P \succ 0 : A^{[v]^T} PA^{[v]} - P \prec 0 \quad \forall v = 1 \ldots \bar{v} \]

PDLF result for “switching” uncertainties \( \theta_k \neq \theta_{k+1}, \forall k \geq 0 \) [DB01, DRI02]

\[ \exists P^{[v]} \succ 0 : A^{[v]^T} P^{[w]} A^{[v]} - P^{[v]} \prec 0, \quad \forall v = 1 \ldots \bar{v}, \forall w = 1 \ldots \bar{v} \]

PDLF result for “parametric” uncertainties \( \theta_k = \phi, \forall k \geq 0 \) [PABB00]

\[ \exists P^{[v]} \succ 0 : \begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^S, \quad \forall v = 1 \ldots \bar{v} \]

Difference and links between the two PDLF results?

The PDLF in both cases is \( P(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P^{[v]} \).
Outline

- PDLF result for "switching" descriptor systems
- Non-conservative reduction of the numerical burden
- Robustness w.r.t. parametric and switching uncertainties
- Numerical example
- Conclusions
PDLF result for "switching" descriptor systems

General descriptor models, affine in the uncertainties \([CTF02\ MAS03]\)

\[
E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k
\]

\[
\begin{bmatrix}
E_x(\theta) & E_\pi(\theta) & -F(\theta)
\end{bmatrix}
= \sum_{v=1}^{\bar{v}} \theta_v
\begin{bmatrix}
E_x^{[v]} & E_\pi^{[v]} & -F^{[v]}
\end{bmatrix} : \theta \in \Xi_{\bar{v}}
\]

In this paper \(\begin{bmatrix}
E_x(\theta) & E_\pi(\theta)
\end{bmatrix}\) is assumed square invertible \(\forall \theta \in \Xi_{\bar{v}}\)

This modeling is an alternative to LFTs:

Any rationally dependent non descriptor state-space model can be reformulated as such.

Example: \(x_{k+1} = \begin{bmatrix}
-b_k^2/a_k & -b_k \\
1 & 0
\end{bmatrix} x_k\) writes also as

\[
\begin{bmatrix}
a_k & 0 \\
0 & 1
\end{bmatrix} x_{k+1}
+ \begin{bmatrix}
b_k \\
0
\end{bmatrix} \pi_k =
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} x_k
\]
Stability of the descriptor system with “switching” uncertainties $\theta_k \neq \theta_{k+1}$, $\forall k \geq 0$ if

$$\exists P[v] > 0 : \begin{bmatrix} P[w] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P[v] \end{bmatrix} \prec \left\{ G[v] \begin{bmatrix} E_x[v] & E_\pi[v] & -F[v] \end{bmatrix} \right\}^S \forall v = 1 \ldots \bar{v} \forall w = 1 \ldots \bar{w}$$

- The proof combines characteristics of the both previously cited PDLF methods
- $G[v]$ are S-variables with many interesting properties, see

  *The S-Variable Approach to LMI-Based Robust Control*


- Major drawback: many large decision variables and many large LMI constraints
Assume there exists a basis in which the descriptor matrix has $\theta$ independent columns

$$\exists T : \begin{bmatrix} E^{[v]}_x & E^{[v]}_{\pi} \end{bmatrix} T = \begin{bmatrix} E_1 & E_2^{[v]} \end{bmatrix}, \quad \forall v = 1 \ldots \bar{v}$$

then the LMIs can be replaced losslessly by an LMI of the type (formulas given in the paper)

$$\exists P^{[v]} \succ 0 : N^{[v]}_1 T \hat{M}(P^{[w]}, P^{[v]}) N^{[v]}_1 < \left\{ \hat{G}^{[w]} N^{[v]}_2 \right\}^S, \quad \forall v = 1 \ldots \bar{v}$$

$$\exists \hat{G}^{[v]}$$

Let $n$ be the order of the system,
$q$ the size of the exogenous $\pi$ vector
and $p$ the number of $\theta$ independent columns $E_1$
then :

- the number of decision variables is reduced by $\bar{v}(3n + 2q - p)p$
- the number of rows of the LMI problem is reduced by $\bar{v}^2 p$
In the case of non-descriptor systems \( x_{k+1} = A(\theta_k) x_k \) the two equivalent LMIs read as

\[
\begin{bmatrix}
P[w] & 0 \\
0 & -P[v]
\end{bmatrix} \prec \left\{ G[w] \begin{bmatrix} I & -A[v] \end{bmatrix} \right\}^S
\]


In such cases, the S-variables are useless (known result [DRI02]).
In the paper we also provide a reduced lossless LMI condition for the case when there are vertex independent rows in the system representation:

\[ \exists S : S \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix}, \quad \forall v = 1 \ldots \bar{v} \]

The two results can be combined for further reducing the numerical burden.
Robustness w.r.t. parametric and switching uncertainties

- General descriptor models, affine in both “switching” and “parametric” uncertainties

\[ E_x(\theta_k, \phi)x_{k+1} + E_\pi(\theta_k, \phi)\pi_k = F(\theta_k, \phi)x_k, \quad \theta \in \Xi_v, \quad \phi \in \Xi_{\mu} \]

\[
\begin{bmatrix}
E_x(\theta, \phi) & E_\pi(\theta, \phi) & -F(\theta, \phi)
\end{bmatrix}
= \sum_v \sum_{\mu} \theta_v \phi_\mu \begin{bmatrix}
E_x^{[v, \mu]} & E_\pi^{[v, \mu]} & -F^{[v, \mu]}
\end{bmatrix}
\]

- Stability assessed by:

\[
\exists P^{[v, \mu]} > 0 : \begin{bmatrix}
P^{[w, \mu]} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -P^{[v, \mu]}
\end{bmatrix} \prec \begin{bmatrix}
G^{[w]} & E_x^{[v, \mu]} & E_\pi^{[v, \mu]} & -F^{[v, \mu]}
\end{bmatrix} \quad \forall v = 1 \ldots \bar{v}, \forall w = 1 \ldots \bar{v}, \forall \mu = 1 \ldots \bar{\mu}
\]

- Similar size reduction methods apply for these LMIs

- The two LMI conditions expressed in the introduction are special cases of this general result.
Numerical example

Considered system: \(a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0\) with affine descriptor model

\[
\begin{bmatrix}
    a_k & 0 \\
    0 & 1 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_{k+1}
\end{bmatrix}
+ \begin{bmatrix}
    b_k \\
    0 \\
    1
\end{bmatrix}
\begin{bmatrix}
    \pi_k
\end{bmatrix}
= \begin{bmatrix}
    0 & 0 \\
    1 & 0 \\
    b_k & a_k
\end{bmatrix}
\begin{bmatrix}
    x_k
\end{bmatrix}
\]

Uncertainties bounded by \(a \in [1, 2]\) and \(b \in [-0.5, \beta]\)

Aim: find maximal \(\beta\) that preserves robust stability in the four cases

\(\Delta\) \(a_k\) and \(b_k\) are both time-varying ("switching")

\(\Delta\) \(a_k\) is switching and \(b\) is constant ("parametric")

\(\Delta\) \(a\) is parametric and \(b\) is switching

\(\Delta\) \(a\) and \(b\) are parametric
By adding one step ahead information, the system also reads as
\[
\begin{bmatrix}
0 \\
 a_{k+1}
\end{bmatrix} y_{k+3} + \begin{bmatrix}
 a_k \\
 b_{k+1}^2
\end{bmatrix} y_{k+2} + \begin{bmatrix}
 b_k^2 \\
 a_{k+1} b_{k+1}
\end{bmatrix} y_{k+1} + \begin{bmatrix}
 a_k b_k \\
 0
\end{bmatrix} y_k = 0
\]
and admits an affine descriptor representation to which the LMI conditions can be applied.

The LMI conditions for the augmented system are less conservative (see \[EPAH05, PAHG07\]), but with increased numerical burden.
## Numerical example

<table>
<thead>
<tr>
<th>$\beta$ (nb vars/nb rows)</th>
<th>original syst.</th>
<th>augmented syst.</th>
<th>true bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_k, b_k$</td>
<td>0.81094 (44/64)</td>
<td>0.84677 (480/1536)</td>
<td>?</td>
</tr>
<tr>
<td>$a, b_k$</td>
<td>0.89027 (28/32)</td>
<td>0.90293 (144/192)</td>
<td>?</td>
</tr>
<tr>
<td>$a_k, b$</td>
<td>0.82658 (28/32)</td>
<td>0.85375 (144/192)</td>
<td>?</td>
</tr>
<tr>
<td>$a, b$</td>
<td>0.98059 (20/16)</td>
<td>0.99519 (48/24)</td>
<td>1</td>
</tr>
</tbody>
</table>

### Conclusions

- New general result for both time varying and parametric uncertainties
- Methodology that allows systematic reduction of numerical burden
- Conservatism reduction achieved by system augmentation
References


