

# Robust Multi-Objective Control for Linear Systems

Elements of theory and RoMuIOC toolbox

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The logo for LAAS-CNRS, featuring the text "LAAS-CNRS" in a bold, blue, sans-serif font. The text is centered between two horizontal lines: a red line above and a yellow line below.

also with Guilherme Chevarria, Alberto Bortott, Maud Sevin

## ■ Robust control theory

- Robustness properties of the feedback loop
- Aim for guaranteed properties (stability and performances)
- Uncertain modeling of systems: tradeoff between complexity of systems & simplicity of models

## ■ Optimization based tools

- Linear Matrix Inequalities (LMI) framework [1990's]
- Efficient fast solvers and nice parser for Matlab [2000's]
- Possibility of a tool gathering established results : RoMulOC

## ① Uncertain LTI systems and performances

- Control objectives: stability, transient response, perturbation rejection...
- Structured parametric uncertainties: extremal values, bounded sets...

## ② LMIs and convex polynomial-time optimization

- Semi-Definite Programming and LMIs
- SDP solvers and parsers

## ③ Conservative LMI results

- Methods: Lyapunov, S-procedure, Finsler lemma, Topologic Separation...
- The ROMULOC toolbox

## ■ Linear Time-Invariant State-Space Multi-Input Multi-Output models

$$\begin{aligned} \vartheta[x](t) &= Ax(t) + B_u u(t) & \vartheta[\eta](t) &= K_A \eta(t) + K_B y(t) \\ y(t) &= C_y x(t) + D_{yu} u(t) & u(t) &= K_C \eta(t) + K_D y(t) \end{aligned},$$
$$x \in \mathbb{C}^n \quad u \in \mathbb{C}^{q_u} \quad y \in \mathbb{C}^{p_y} \quad \eta \in \mathbb{C}^{n_K}$$

● Continuous ( $\vartheta[x](t) = \dot{x}(t)$ ) and discrete-time ( $\vartheta[x](t) = x(t + T)$ )

■ analysis problem: For given  $(K_A, K_B, K_C, K_D)$  prove closed-loop properties of

$$\vartheta \begin{bmatrix} x \\ \eta \end{bmatrix} (t) = A(K) \begin{pmatrix} x \\ \eta \end{pmatrix} (t)$$

■ Design problem: Find  $(K_A, K_B, K_C, K_D)$  providing closed loop properties

- For  $n_k = 0$ : Static output-feedback (SOF)
- For  $n_k = 0$  and  $y = x$ : State feedback problem
- For  $n_k = n$ : full order output-feedback problem

## Control objectives

● Stability of  $\vartheta[x](t) = A(K)x(t)$

▲ For continuous-time: poles are all in left-hand half of complex plane

▲ For discrete-time: poles are all in unit circle of complex plane

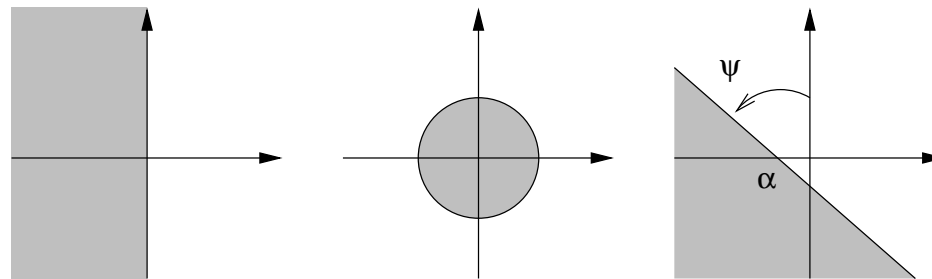
●  $D_R$ -Stability of  $\vartheta[x](t) = A(K)x(t)$ :

▲ Poles are in region defined by

$$D_R = \{ s \in \mathbb{C} : r_{11} + sr_{12} + s^*r_{12} + ss^*r_{22} \leq 0 \} , \quad R = (r_{ij})$$

Such regions are half-planes and discs

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} -2\alpha \cos \psi & \cos \psi - i \sin \psi \\ \cos \psi + i \sin \psi & 0 \end{bmatrix}$$



## Input/output objectives

$$\begin{aligned} \vartheta[x](t) &= A(K)x(t) + B_w(K)w(t) \\ z(t) &= C_z(K)x(t) + D_{zw}(K)w(t) \end{aligned} \quad \begin{array}{l} w \in \mathbb{C}^{q_w} \\ z \in \mathbb{C}^{p_z} \end{array}$$

- Induced  $L_2$  gain:  $\|z\| \leq \gamma_\infty \|w\|$  ,  $(\|w\|^2 = \int_0^\infty w^* w dt)$   
Also known as:  $H_\infty$  performance (max singular value  $H(j\omega)$  ,  $\omega \in \mathbb{R}$ ),  
Robustness to unmodeled dynamics  $w = \Delta z$  ,  $\|\Delta\| \leq 1/\gamma_\infty$  (bounded-real lemma)
- Impulse-to-norm performance:  $\|z\| \leq \gamma_2$  if  $w(t) = \delta(t)\mathbf{1}$   
Also known as:  $H_2$  performance (mean value  $H(j\omega)$  ,  $\omega \in \mathbb{R}$ ),  
Energy of output in response to Gaussian white noise  
Norm-to-peak performance ( $\max |z| \leq \gamma_2 \|w\|$ ,  $z \in \mathbb{R}$ )
- Impulse-to-peak performance:  $\max |z| \leq \gamma_{i2p}$  if  $w(t) = \delta(t)\alpha$ ,  $\|\alpha\| \leq 1$ .  
Also known as: Invariant ellipsoids  
Non saturating initial conditions

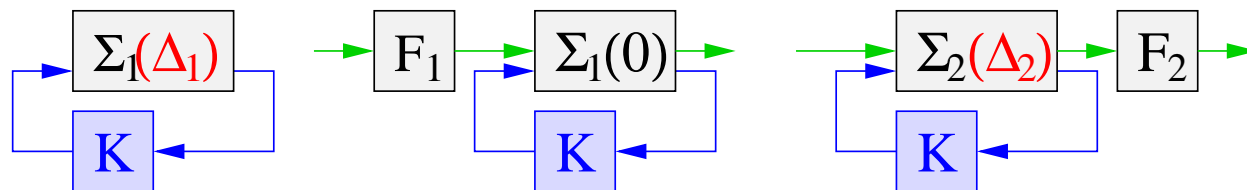
# 1 Uncertain LTI systems and performances

## ■ Robust Multi-Objective Control

- $\Delta$ : errors in modeling, operating conditions, mass-production...
- $\Delta$ : parametric uncertainty, assumed constant, belongs to a set  $\Delta$ .

$$\begin{aligned}\vartheta[x](t) &= A(\Delta, K)x(t) + B_w(\Delta, K)w(t) \\ z(t) &= C_z(\Delta, K)x(t) + D_{zw}(\Delta, K)w(t)\end{aligned}$$

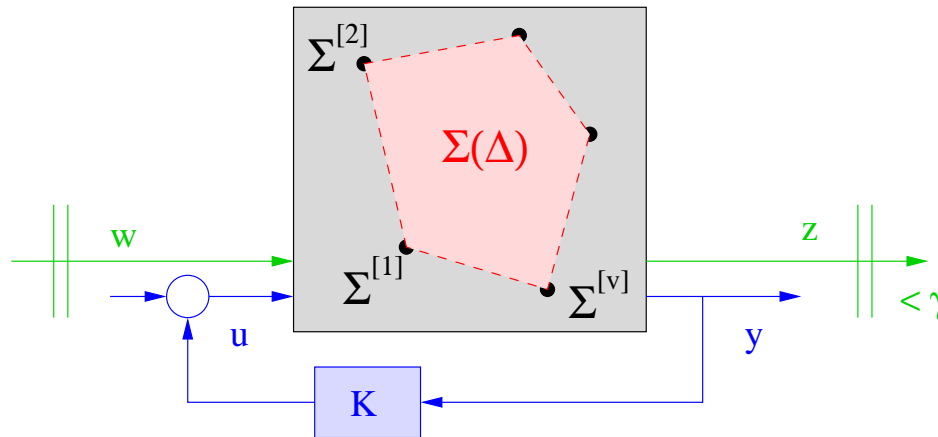
- Design: Find a controller  $K$  that fulfills all robust specifications  $\prod_{p=1 \dots \bar{p}}$  defined for models  $\Sigma_p(\Delta_p)$  subject to uncertainties  $\Delta_p \in \Delta_p$ .



- Analysis: For given  $K$  prove for each  $\Sigma_{p=1 \dots \bar{p}}(\Delta_p)$  that the specification  $\prod_p$  holds for all uncertainties  $\Delta_p \in \Delta_p$ .

## ■ Uncertain LTI systems: Affine with scalar parametric uncertainty

### ● Polytopic models



Convex hull of  $\bar{v}$  vertices

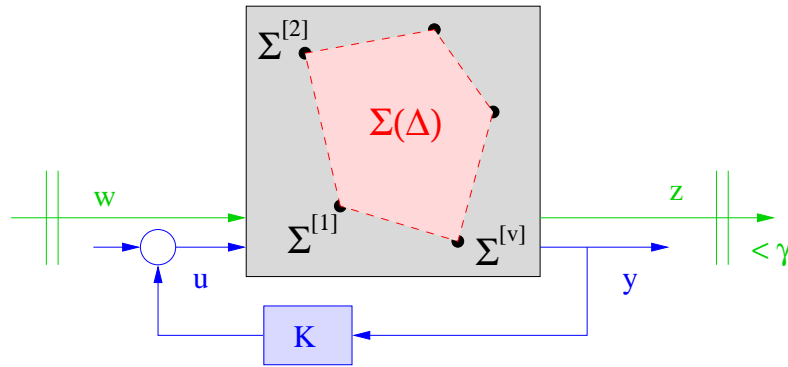
$$A(\Delta) = \sum_{v=1}^{\bar{v}} \xi_v A^{[v]} , \quad B_w(\Delta) = \sum_{v=1}^{\bar{v}} \xi_v B_w^{[v]} \quad \dots \quad : \quad \xi_v \geq 0 , \quad \sum_{v=1}^{\bar{v}} \xi_v = 1$$

- ▲ Example: Linear combination of linear models identified on different operating points.
- ▲ The  $\xi_v$  parameters may not have physical meaning
- ▲  $\bar{v}$  vertices can define a volume in  $\bar{v} - 1$  space of parameters  
(possible to divide space in polytopes with low number of vertices)

# 1 Uncertain LTI systems and performances

## ■ Uncertain LTI systems: Affine with scalar parametric uncertainty

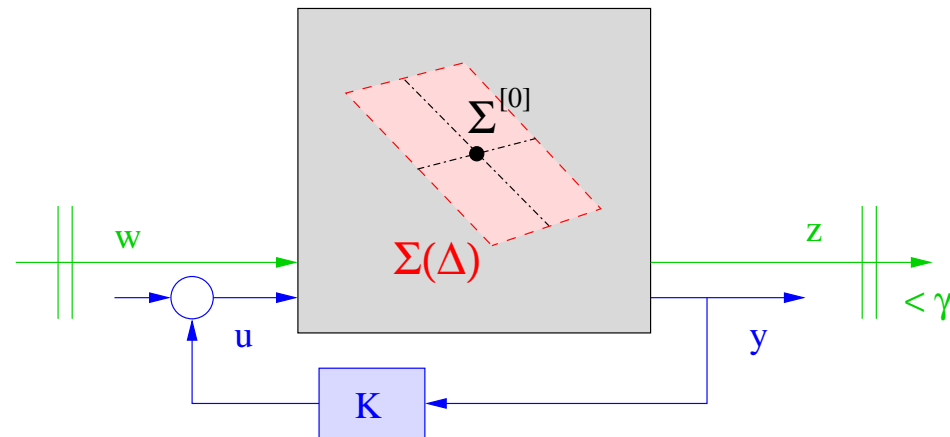
### ● Polytopic models



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$$A(\Delta) = \sum_{v=1}^{\bar{v}} \xi_v A^{[v]}, \quad \dots \quad : \quad \xi_v \geq 0, \quad \sum_{v=1}^{\bar{v}} \xi_v = 1$$

### ● Parallelotopic models with $\bar{\varsigma}$ axes



$$A(\Delta) = A^{[0]} + \sum_{\varsigma=1}^{\bar{\varsigma}} \delta_{\varsigma} A^{|\varsigma|}, \quad \dots \quad : \quad |\delta_{\varsigma}| \leq 1$$

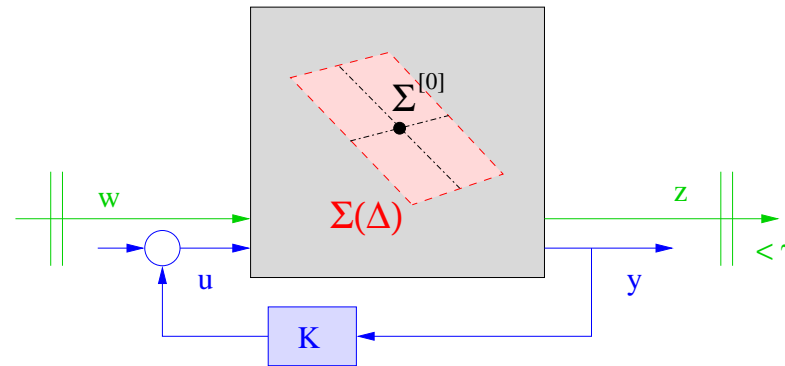
▲  $\bar{\varsigma}$  independent parameters  $\delta_{\varsigma}$  identified in intervals

▲ polytope with  $\bar{v} = 2^{\bar{\varsigma}}$  vertices

# 1 Uncertain LTI systems and performances

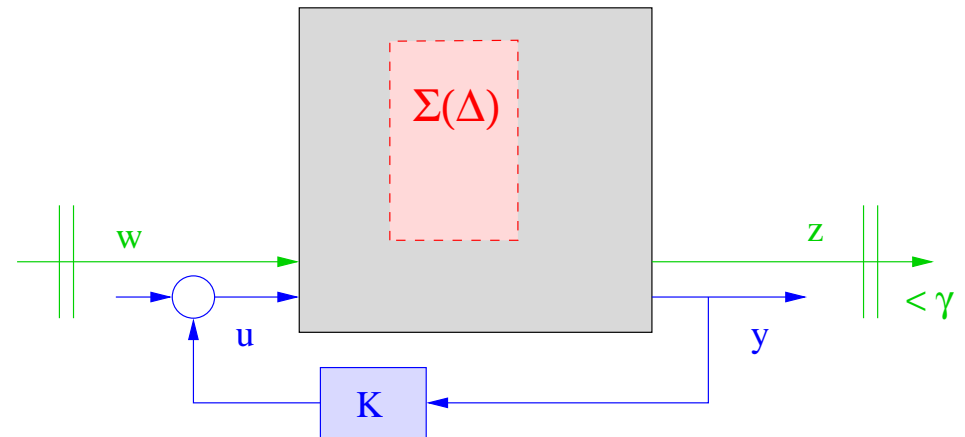
## ■ Uncertain LTI systems: Affine with scalar parametric uncertainty

### ● Parallelotopic models with $\bar{\zeta}$ axes



$$A(\Delta) = A^{[0]} + \sum_{\zeta=1}^{\bar{\zeta}} \xi_{\zeta} A^{[\zeta]}, \quad \dots \quad : \quad |\xi_{\zeta}| \leq 1$$

### ● Interval models with $\bar{\zeta}$ non-equal coefficients



$$\underline{A} \preceq A(\Delta) \preceq \bar{A} \quad : \quad \underline{a}_{ij} \leq a_{ij}(\Delta) \leq \bar{a}_{ij} \quad \dots$$

- ▲ Parallelotope with axes in the euclidian basis of matrices
- ▲ All coefficients independent
- ▲ Change of basis does not preserve the structure

# 1 Uncertain LTI systems and performances

- Polytopic models can also be written as

$$A(\Delta) = A + \underbrace{\begin{bmatrix} B_{\Delta}^{[1]} & \dots & B_{\Delta}^{[\bar{v}]} \end{bmatrix}}_{B_{\Delta}} \underbrace{\begin{bmatrix} \xi_1 \mathbf{1}_{q_1} & & 0 \\ & \ddots & \\ 0 & & \xi_{\bar{v}} \mathbf{1}_{q_{\bar{v}}} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} C_{\Delta}^{[1]} \\ \vdots \\ C_{\Delta}^{[\bar{v}]} \end{bmatrix}}_{C_{\Delta}}$$

where for each vertex  $A^{[v]} = A + B_{\Delta}^{[v]} C_{\Delta}^{[v]}$  with  $B_{\Delta}^{[v]} \in \mathbb{C}^{n \times q_v}$ .

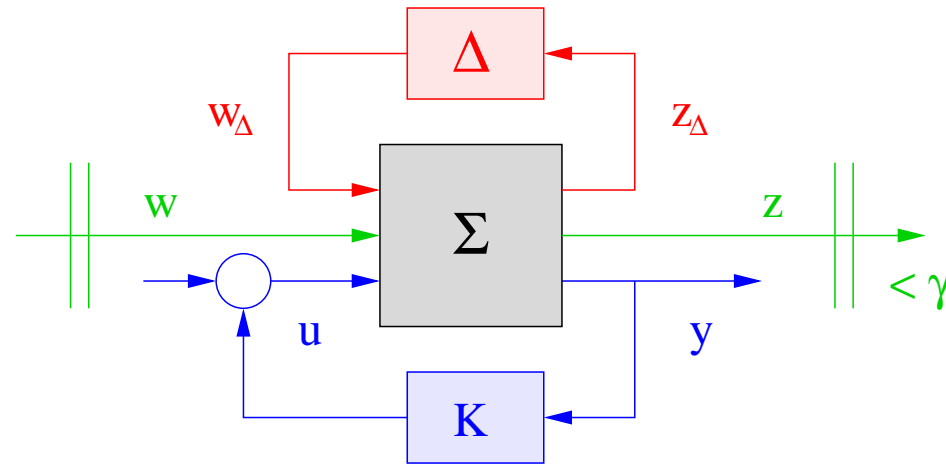
- Parallelotopic models can also be written as

$$A(\Delta) = \underbrace{A^{[0]}}_A + \underbrace{\begin{bmatrix} B_{\Delta}^{[1]} & \dots & B_{\Delta}^{[\bar{\varsigma}]} \end{bmatrix}}_{B_{\Delta}} \underbrace{\begin{bmatrix} \delta_1 \mathbf{1}_{p_1} & & 0 \\ & \ddots & \\ 0 & & \delta_{\bar{\varsigma}} \mathbf{1}_{p_{\bar{\varsigma}}} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} C_{\Delta}^{[1]} \\ \vdots \\ C_{\Delta}^{[\bar{\varsigma}]} \end{bmatrix}}_{C_{\Delta}}$$

where for each axis  $A^{[\varsigma]} = B_{\Delta}^{[\varsigma]} C_{\Delta}^{[\varsigma]}$  with  $B_{\Delta}^{[\varsigma]} \in \mathbb{C}^{n \times p_{\varsigma}}$ .

- ▲ Factorisation as  $A(\Delta) = A + B_{\Delta} \Delta C_{\Delta}$  is not unique.

## ■ Uncertain LTI systems: Linear Fractional Representation (LFR)



$$\begin{aligned}
 \vartheta[x](t) &= Ax(t) + B_{\Delta} w_{\Delta}(t) + B_w w(t) + B_u u(t) \\
 z_{\Delta}(t) &= C_{\Delta} x(t) + D_{\Delta\Delta} w_{\Delta}(t) + D_{\Delta w} w(t) + D_{\Delta u} u(t) \\
 z(t) &= C_z x(t) + D_{z\Delta} w_{\Delta}(t) + D_{zw} w(t) + D_{zu} u(t) \\
 y(t) &= C_y x(t) + D_{y\Delta} w_{\Delta}(t) + D_{yw} w(t) + D_{yu} u(t)
 \end{aligned}
 \quad \begin{array}{l}
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot
 \end{array}
 \quad \begin{array}{l}
 w_{\Delta} \in \mathbb{C}^{q_{\Delta}} \\
 z_{\Delta} \in \mathbb{C}^{p_{\Delta}}
 \end{array}$$

Linear - Fractional Transformation (LFT):

$$\begin{aligned}
 A(\Delta) &= A + B_{\Delta} \Delta (1 - D_{\Delta\Delta} \Delta)^{-1} C_{\Delta} = A + B_{\Delta} (1 - \Delta D_{\Delta\Delta})^{-1} \Delta C_{\Delta}, \\
 B_w(\Delta) &= B_w + B_{\Delta} \Delta (1 - D_{\Delta\Delta} \Delta)^{-1} D_{\Delta w} \quad \dots
 \end{aligned}$$



## ■ Sets of uncertainties in LFRs

### ● $\{X, Y, Z\}$ –dissipative uncertain matrices

$$\{ \Delta_j : X + Y \Delta_j + \Delta_j^* Y^* + \Delta_j^* Z \Delta_j \leq 0, X \leq 0, Z \geq 0 \}$$

▲ Norm-bounded :  $\|\Delta_j\| \leq \rho 1$  (gain limited operators)  $\Rightarrow \{-\rho^2 1, 0, 1\}$ –dissipative

▲ Positive real :  $\Delta_j + \Delta_j^* \geq 0$  (passive operators)  $\Rightarrow \{0, -1, 0\}$ –dissipative

## ■ Sets of uncertainties in LFRs

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$$\left\{ \Delta_j : X + Y \Delta_j + \Delta_j^* Y^* + \Delta_j^* Z \Delta_j \leq 0, X \leq 0, Z \geq 0 \right\}$$

▲ Norm-bounded :  $\|\Delta_j\| \leq \rho \mathbf{1}$  (gain limited operators)  $\Rightarrow \{-\rho^2 \mathbf{1}, 0, \mathbf{1}\}$ –dissipative

▲ Positive real :  $\Delta_j + \Delta_j^* \geq 0$  (passive operators)  $\Rightarrow \{0, -1, 0\}$ –dissipative

### ● Polytopic uncertainties

$$\left\{ \Delta_j = \sum \xi_{j,v} \Delta_j^{[v]} : \xi_{j,v} \geq 0, \sum \xi_{j,v} = 1 \right\}$$

### ● Parallelotopic uncertainties

$$\left\{ \Delta_j = \Delta_j^{[0]} + \sum \delta_{j,i} \Delta_j^{[i]} : |\delta_{j,i}| \leq 1 \right\}$$

### ● Interval uncertainties

$$\left\{ \underline{\Delta}_j \preceq \Delta_j \preceq \overline{\Delta}_j \right\}$$

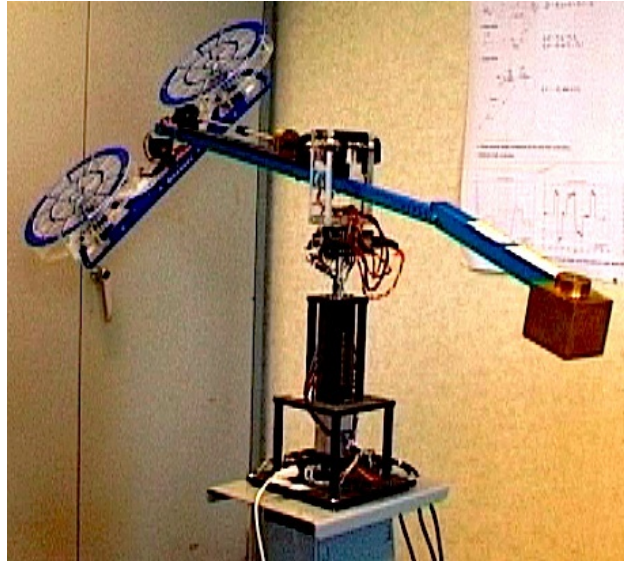
## ■ Example

$$\dot{x} = \frac{\delta_1^2}{1+\delta_2} x \Rightarrow \left\{ \begin{array}{l} \dot{x} = w_{\Delta 1} \\ z_{\Delta 1} = \frac{\delta_1}{1+\delta_2} x \end{array} \right. \quad \begin{array}{l} w_{\Delta 1} = \delta_1 z_{\Delta 1} \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} \dot{x} = w_{\Delta 1} \\ z_{\Delta 1} = w_{\Delta 2} \\ z_{\Delta 2} = \frac{1}{1+\delta_2} x \end{array} \right. \quad \begin{array}{l} w_{\Delta 1} = \delta_1 z_{\Delta 1} \\ w_{\Delta 2} = \delta_1 z_{\Delta 2} \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} \dot{x} = w_{\Delta 1} \\ z_{\Delta 1} = w_{\Delta 2} \\ z_{\Delta 2} = x - w_{\Delta 3} \\ z_{\Delta 3} = x - w_{\Delta 3} \end{array} \right. \quad \begin{array}{l} w_{\Delta 1} = \delta_1 z_{\Delta 1} \\ w_{\Delta 2} = \delta_1 z_{\Delta 2} \\ w_{\Delta 3} = \delta_2 z_{\Delta 3} \end{array}$$

- ▲  $z_{\Delta 3} = z_{\Delta 2}$  added to have  $\Delta$  diagonal
- ▲  $\delta_1$  repeated twice
- ▲  $\delta_1, \delta_2$  if independent can be defined in two intervals, or as norm-bounded
- ▲  $\delta_1, \delta_2$  if dependent can be defined in polytope



## ■ "Helicopter" example

### ● System defined at maximal value of parameters

```
>> sysmax = ssmode1( 'Helicopter' );  
>> sysmax.A = [0 1 0 ; 0 0 1; 0 -2.8 -0.14];  
>> sysmax.Bw = [0; 0; -14];  
>> sysmax.Bu = [0; 0; 8];  
>> sysmax.Dzu = 1  
name: Helicopter  
  
          n=3      mw=1      mu=1  
n=3      dx  =  A*x +  Bw*w +  Bu*u  
pz=1      z  =                Dzu*u  
continuous time ( dx : derivative operator )
```

## ● System defined at maximal value of parameters

```
>> sysmax = ssmodel( 'Helicopter' );  
>> sysmax.A = [0 1 0 ; 0 0 1; 0 -2.8 -0.14];  
>> sysmax.Bw = [0;0;-14];  
>> sysmax.Bu = [0;0;8];  
>> sysmax.Dzu = 1;
```

## ● System defined at minimal value of parameters

```
>> sysmin = ssmodel( 'Helicopter' );  
>> sysmin.A = [0 1 0 ; 0 0 1; 0 -3 -0.2];  
>> sysmin.Bw = [0;0;-14];  
>> sysmin.Bu = [0;0;8];  
>> sysmin.Dzu = 1  
name: Helicopter  
  
          n=3    mw=1    mu=1  
n=3    dx =  A*x +  Bw*w +  Bu*u  
pz=1    z =                Dzu*u  
continuous time ( dx : derivative operator )
```

## ● Uncertain system defined as interval of max and min

```
>> usys = uinter( sysmin, sysmax )
Uncertain model : interval 2 param
----- WITH -----
name: Helicopter
           n=3      mw=1      mu=1
n=3      dx  =  A*x +  Bw*w +  Bu*u
pz=1      z  =                      Dzu*u
continuous time ( dx : derivative operator )
```

## ● Interval model converted to polytopic model

```
>> usys = u2poly( usys )
Uncertain model : polytope 4 vertices
----- WITH -----
name: Helicopter
           n=3      mw=1      mu=1
n=3      dx  =  A*x +  Bw*w +  Bu*u
pz=1      z  =                      Dzu*u
continuous time ( dx : derivative operator )
```

- Declare a state-feedback design problem

```
>> quiz = ctrpb( 'state-feedback', 'Lyap-unique' )
control problem: STATE-FEEDBACK design
Lyapunov function: UNIQUE (quadratic stability)
No specified performance
```

- Add an  $H_\infty$  performance objective

```
>> quiz = quiz + hinfty( usys, 4 );
```

- Add a pole location performance objective

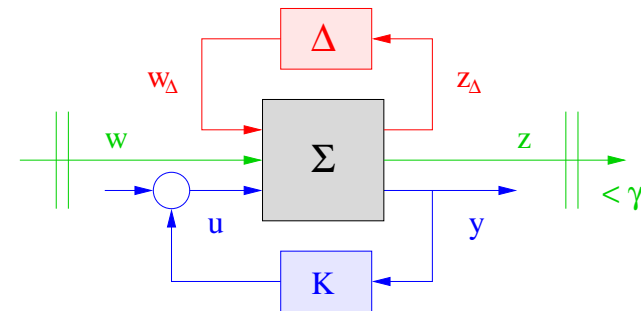
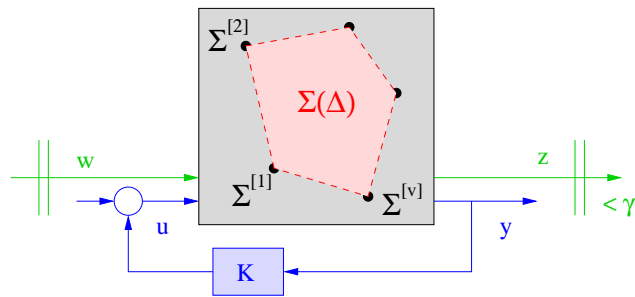
```
>> r = region( 'plane', -0.1 )
Half-plane such that: Re(z)<-0.1
>> quiz = quiz + dstability( usys, r )
```

- Add an impulse-to-peak performance minimization objective

```
>> quiz = quiz + i2p( usys )
control problem: STATE-FEEDBACK design
Lyapunov function: UNIQUE (quadratic stability)
Specified performances / systems:
# Hinfty < 4 / Helicopter
# D-stability / Helicopter
# minimize I2P / Helicopter
```

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- Control objectives: stability, transient response, perturbation rejection...
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## ② LMIs and convex polynomial-time optimization

- Semi-Definite Programming and LMIs
- SDP solvers and parsers

## ③ Conservative LMI results

- Methods: Lyapunov, S-procedure, Finsler lemma, Topologic Separation...
- The ROMULOC toolbox

### ■ Semi-Definite Programming and LMIs

- Extension of LP to semi-definite matrices

$$\min c^T x \quad : \quad Ax = b \quad , \quad x_i \geq 0 \quad (LP) \quad | \quad \text{mat}(x) \geq 0 \quad (SDP)$$

- Convexity, duality, polynomial-time algorithms ( $\mathcal{O}(n^{6.5} \log(1/\epsilon))$ ).

$$\max b^T y \quad : \quad A^T y - c^T = z \quad , \quad \text{mat}(z) \geq 0$$

- 1st developments and 1st results : LMI formalism & Control Theory

$$\min \sum g_i y_i \quad : \quad F_0 + \sum F_i y_i \geq 0$$

- ▲ The  $H_\infty$  norm computation example for  $G(s) \sim (A, B, C, D)$  :

$$\|G(s)\|_\infty^2 = \min \gamma \quad : \quad P > 0 \quad , \quad \begin{bmatrix} A^T P + P A + C_z^T C_z & B_w P + C_z^T D_{zw} \\ P B_w^T + D_{zw}^T C_z & -\gamma \mathbf{1} + D_{zw}^T D_{zw} \end{bmatrix} \leq 0$$

### ■ SDP solvers and parsers

- LMI Control Toolbox  $\Rightarrow$  Control Toolbox

1st solver, dedicated to LMIs issued from Control Theory, Matlab, owner.

- SDP solvers: SP, SeDuMi, SDPT3, CSDP, DSDP, SDPA...

Active field, mathematical programming, C/C++, free.

- Parsers: tklimitool, sdpsol, SeDuMiInterface, **YALMIP**

Convert LMIs to SDP solver format, Matlab (Scilab), free.

### ■ SDP-LMI issues and perspectives

● Any SDP representable problem is "solved" (numerical problems due to size and structure)

▲ Find "SDP-ables" problems

(linear systems, performances, robustness, LPV, saturations, delays, singular systems...)

▲ Equivalent SDP formulations  $\Rightarrow$  distinguish which are numerically efficient

▲ New SDP solvers: faster, precise, robust (need for benchmark examples)

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- Any "SDP-able" problem has a dual interpretation
- ▲ New theoretical results (worst case)
- ▲ New proofs (Lyapunov functions = Lagrange multipliers; related to SOS)
- ▲ SDP formulas numerically stable (KYP-lemma)

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- ▲ SDP formulas numerically stable (KYP-lemma)
- Non "SDP-able" : Robustesse & Multi-objective & Relaxation of NP-hard problems
- ▲ Optimistic / Pessimistic (conservative) results
- ▲ Reduce the gap (upper/lower bounds) while handling numerical complexity growth.

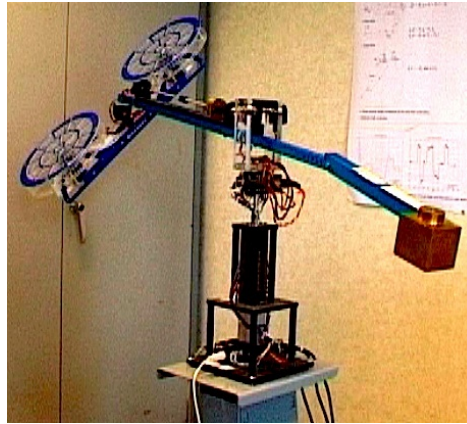
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- ▲ Reduce the gap (upper/lower bounds) while handling numerical complexity growth.
- Develop software for "industrial" application / adapted to the application field

$\Rightarrow$  ROMULOC toolbox



### ■ "Helicopter" example

### ● The quiz object

```
>> quiz
control problem: STATE-FEEDBACK design
Lyapunov function: UNIQUE (quadratic stability)
Specified performances / systems:
# HinfTy < 4 / Helicopter
# D-stability / Helicopter
# minimize I2P / Helicopter
```

### ● Contains decision variables

```
>> quiz.vars
[3x3 sdpvar]      'Lyapunov matrix'
[1x3 sdpvar]      'S=-K*P'
[1x1 sdpvar]      'S-procedure scaling'
[1x1 sdpvar]      'g > (I2P cost)^2'
```

## ② LMIs and convex polynomial-time optimization

### ● Constrained by LMIs

```
>> quiz.lmi
```

```
+++++
|   ID |   Constraint |                               Type |   Tag |
+++++
|   #1 | Numeric value | Matrix inequality 3x3 |   Lyap >0 |
|   #2 | Numeric value | Matrix inequality 4x4 | Var Lyap <0 |
|   #3 | Numeric value | Matrix inequality 4x4 | Var Lyap <0 |
|   #4 | Numeric value | Matrix inequality 4x4 | Var Lyap <0 |
|   #5 | Numeric value | Matrix inequality 4x4 | Var Lyap <0 |
|   #6 | Numeric value | Matrix inequality 3x3 | Var Lyap <0 |
|   #7 | Numeric value | Matrix inequality 3x3 | Var Lyap <0 |
|   #8 | Numeric value | Matrix inequality 3x3 | Var Lyap <0 |
|   #9 | Numeric value | Matrix inequality 3x3 | Var Lyap <0 |
|  #10 | Numeric value | Matrix inequality 3x3 | Constraint 1 |
|  #11 | Numeric value | Matrix inequality 4x4 | Constraint 2 |
|  #12 | Numeric value | Matrix inequality 3x3 | Constraint 3 |
|  #13 | Numeric value | Element-wise 1x1 | Constraint 4 |
|  #14 | Numeric value | Matrix inequality 3x3 | Constraint 1 |
|  #15 | Numeric value | Matrix inequality 4x4 | Constraint 2 |
|  #16 | Numeric value | Matrix inequality 3x3 | Constraint 3 |
|  #17 | Numeric value | Element-wise 1x1 | Constraint 4 |
```

## ② LMIs and convex polynomial-time optimization

---

#18	Numeric value	Matrix inequality 3x3	Constraint 1
#19	Numeric value	Matrix inequality 4x4	Constraint 2
#20	Numeric value	Matrix inequality 3x3	Constraint 3
#21	Numeric value	Element-wise 1x1	Constraint 4
#22	Numeric value	Matrix inequality 3x3	Constraint 1
#23	Numeric value	Matrix inequality 4x4	Constraint 2
#24	Numeric value	Matrix inequality 3x3	Constraint 3
#25	Numeric value	Element-wise 1x1	Constraint 4

+++++

## ② LMIs and convex polynomial-time optimization

### ● And can be solved (SeDuMi solver by default)

```
>> K = solvesdp( quiz )
```

```
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
```

```
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
```

```
eqs m = 11, order n = 76, dim = 250, blocks = 22
```

```
nnz(A) = 282 + 0, nnz(ADA) = 117, nnz(L) = 64
```

it	:	b*y	gap	delta	rate	t/tP*	t/tD*	feas	cg	cg	prec
0	:		6.96E+01	0.000							
1	:	-1.79E+02	1.85E+01	0.000	0.2657	0.9000	0.9000	-0.09	1	1	5.0E+02
2	:	-1.05E+02	5.96E+00	0.000	0.3223	0.9000	0.9000	1.55	1	1	1.1E+02
3	:	-2.56E+01	1.38E+00	0.000	0.2312	0.9000	0.9000	1.73	1	1	1.9E+01
4	:	-5.54E+00	2.62E-01	0.000	0.1902	0.9000	0.9000	1.21	1	1	3.2E+00
5	:	-1.84E+00	8.00E-02	0.000	0.3050	0.9000	0.9000	1.29	1	1	8.3E-01
6	:	-7.08E-01	2.90E-02	0.000	0.3621	0.9000	0.9000	1.35	1	1	2.6E-01
7	:	-2.95E-01	1.05E-02	0.000	0.3637	0.9000	0.9000	1.27	1	1	8.3E-02
8	:	-2.30E-01	3.57E-03	0.000	0.3393	0.9000	0.9000	1.12	1	1	2.7E-02
9	:	-1.97E-01	6.73E-04	0.000	0.1882	0.9000	0.9000	1.00	1	1	5.1E-03
10	:	-1.91E-01	2.02E-05	0.000	0.0300	0.9900	0.9900	0.98	1	1	1.6E-04
11	:	-1.91E-01	1.13E-06	0.000	0.0558	0.9900	0.9900	1.00	1	1	8.7E-06
12	:	-1.91E-01	3.08E-07	0.000	0.2737	0.9000	0.9000	1.00	1	1	2.4E-06
13	:	-1.91E-01	1.33E-08	0.000	0.0433	0.9900	0.9900	1.00	1	1	1.0E-07
14	:	-1.91E-01	3.01E-09	0.000	0.2261	0.9000	0.9000	1.00	2	2	2.3E-08

## ② LMIs and convex polynomial-time optimization

---

```
15 : -1.91E-01 7.53E-10 0.000 0.2498 0.9000 0.9000 1.00 2 2 5.8E-09
16 : -1.91E-01 4.50E-11 0.087 0.0598 0.9900 0.9900 1.00 2 2 3.5E-10
```

```
iter seconds digits          c*x          b*y
 16         0.4   Inf -1.9059654950e-01 -1.9059654919e-01
|Ax-b| = 3.6e-10, [Ay-c]_+ = 2.6E-11, |x|= 5.0e-01, |y|= 3.7e+02
```

Detailed timing (sec)

```
      Pre          IPM          Post
1.800E-01  4.000E-01  7.000E-02
Max-norms: ||b||=1, ||c|| = 196,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 42153.4.
```

Feasibility is not strictly determined

Worst constraint residual is  $-2.59066e-11 < 0$

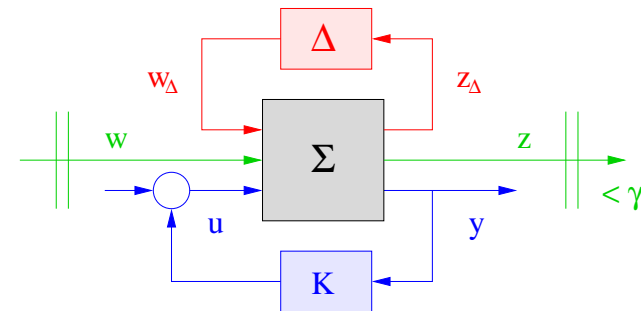
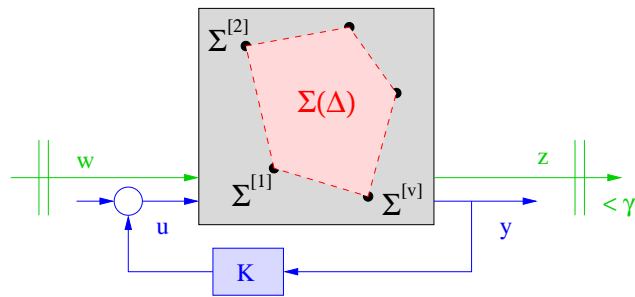
0.436574 (=sqrt(double(CTRPB.vars{4}))) may be a guaranteed I2P norm

K =

```
0.0442    0.0091    0.0305
```

## 1 Uncertain LTI systems and performances

- Control objectives: stability, transient response, perturbation rejection...
- Structured parametric uncertainties: extremal values, bounded sets...



## 2 LMIs and convex polynomial-time optimization

- Semi-Definite Programming and LMIs
- SDP solvers and parsers

⇒ YALMIP and all solvers

## 3 Conservative LMI results

- Methods: Lyapunov, S-procedure, Finsler lemma, Topologic Separation...
- The ROMULOC toolbox

### 3 Conservative LMI results

■ Nominal performance analysis:  $V(x) = x^T P x$  Lyapunov function ( $P > 0$ )

● Stability  $A^T P + P A < 0 \quad | \quad A^T P A - P < 0$

●  $D_R$ -Stability  $\begin{bmatrix} 1 & A^* \end{bmatrix} \begin{bmatrix} r_{11} P & r_{12} P \\ r_{12}^* P & r_{22} P \end{bmatrix} \begin{bmatrix} 1 \\ A \end{bmatrix} < 0$

●  $H_\infty$  norm  $\begin{bmatrix} A^T P + P A + C_z^T C_z & P B_w + C_z^T D_{zw} \\ B_w^T P + D_{zw}^T C_z & -\gamma^2 \mathbf{1} + D_{zw}^T D_{zw} \end{bmatrix} < 0$

●  $H_2$  norm  $A^T P + P A + C_z^T C_z < 0$

$$\text{trace}(B_w^T P B_w) < \gamma^2$$

● Impulse-to-peak  $A^T P + P A < 0 \quad B_w^T P B_w < \gamma^2 \mathbf{1}$

$$C_z^T C_z < P \quad D_{zw}^T D_{zw} < \gamma^2 \mathbf{1}$$

### 3 Conservative LMI results

---

■ Robust performance analysis:  $V(x, \Delta)$  parameter-dependent Lyapunov function.

▲ Nominal analysis (LMI)  $\rightarrow$  Robust analysis (NP-hard)

$$\exists P : \mathcal{L}_{\Sigma}(P) < 0 \quad \rightarrow \quad \forall \Delta \in \Delta, \exists P(\Delta) : \mathcal{L}_{\Sigma(\Delta)}(P(\Delta)) < 0$$

▲ Test over sample values  $\{\Delta_{1\dots N}\} \in \Delta$  gives optimistic results

(some results exist if  $\{\Delta_{1\dots N}\}$  is uniform distribution of  $\Delta$  and large  $N$ )

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(some results exist if  $\{\Delta_{1\dots N}\}$  is uniform distribution of  $\Delta$  and large  $N$ )

● Choice of  $P(\Delta)$  for having a finite number of decision variables :

$\rightarrow$  “Quadratic Stability”:  $P(\Delta) = P$

$\rightarrow$  Polytopic PDLF:  $P(\Delta) = \sum \zeta_i P^{[i]}$

$\rightarrow P(\Delta)$  polynomial w.r.t.  $\zeta_i$  (not coded in RoMulOC)

$\rightarrow$  Quadratic-LFT PDLF:  $P(\Delta) = \begin{bmatrix} 1 & \Delta_C^T \end{bmatrix} P \begin{bmatrix} 1 \\ \Delta_C \end{bmatrix} : \Delta_C = (1 - \Delta D_{\Delta\Delta})^{-1} \Delta C_{\Delta}$

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▲ LMIs over infinite number of variables

$$\forall \Delta \in \mathbb{\Delta}, \exists P(\Delta) : \mathcal{L}_{\Sigma(\Delta)}(P(\Delta)) < 0$$

$$\Leftrightarrow \exists P^{[i]} : \forall \Delta \in \mathbb{\Delta}, \mathcal{L}_{\Sigma(\Delta)}(P(\Delta)) < 0$$

■ Conservative LMIs for polytopic models (Example of stability analysis)

$$\dot{x} = A(\xi)x \text{ with } A(\xi) = \sum_{v=1}^{\bar{v}} \xi_v A^{[v]} : \xi \in \Xi = \{\xi_v \geq 0, \sum_{v=1}^{\bar{v}} \xi_v = 1\}$$

### 3 Conservative LMI results

---

■ Conservative LMIs for polytopic models (Example of stability analysis)

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● “Quadratic Stability”:  $P(\Delta) = P$

$$\dot{V}(x) < 0 \Leftrightarrow A^T(\Delta)P + PA(\Delta) < 0 \Leftrightarrow A^{[i]T}P + PA^{[i]} < 0$$

### 3 Conservative LMI results

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● Polytopic PDLF:  $P(\Delta) = \sum \zeta_i P^{[i]}$

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}^T \begin{bmatrix} 0 & P(\Delta) \\ P(\Delta) & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} < 0 : \begin{bmatrix} A(\Delta) & -1 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = 0$$

$\Leftrightarrow$  Finsler Lemma

$$\begin{bmatrix} 0 & P(\Delta) \\ P(\Delta) & 0 \end{bmatrix} + G(\Delta) \begin{bmatrix} A(\Delta) & -1 \end{bmatrix} + \begin{bmatrix} A^T(\Delta) \\ -1 \end{bmatrix} G^T(\Delta) < 0$$

$\Leftarrow G(\Delta) = G$  & convexity

$$\begin{bmatrix} 0 & P^{[i]} \\ P^{[i]} & 0 \end{bmatrix} + G \begin{bmatrix} A^{[i]} & -1 \end{bmatrix} + \begin{bmatrix} A^{[i]T} \\ -1 \end{bmatrix} G^T < 0$$

### ③ Conservative LMI results

---

■ Conservative LMIs for LFT models (Example of stability analysis)

$$\dot{x} = Ax + B_{\Delta}w_{\Delta} \quad \text{with} \quad w_{\Delta} = \Delta z_{\Delta} = \Delta C_{\Delta}x + \Delta D_{\Delta\Delta}w_{\Delta}$$

### 3 Conservative LMI results

■ Conservative LMIs for LFT models (Example of stability analysis)

$$\dot{x} = Ax + B_{\Delta}w_{\Delta} \text{ with } w_{\Delta} = \Delta z_{\Delta} = \Delta C_{\Delta}x + \Delta D_{\Delta\Delta}w_{\Delta}$$

● “Quadratic Stability”:  $P(\Delta) = P$

$$\begin{aligned} \dot{V}(x) &= \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix}^* \begin{bmatrix} 1 & 0 \\ A & B_{\Delta} \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ A & B_{\Delta} \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix} < 0 \\ &: \begin{bmatrix} \Delta & -1 \end{bmatrix} \begin{bmatrix} C_{\Delta} & D_{\Delta\Delta} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix} = 0 \end{aligned}$$

$\Leftrightarrow$  Finsler Lemma

$$M_A^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_A < \tau M_C^* \begin{bmatrix} \Delta^* \\ -1 \end{bmatrix} \begin{bmatrix} \Delta & -1 \end{bmatrix} M_C$$

### 3 Conservative LMI results

■ Conservative LMIs for LFT models (Example of stability analysis)

$$\dot{x} = Ax + B_{\Delta}w_{\Delta} \text{ with } w_{\Delta} = \Delta z_{\Delta} = \Delta C_{\Delta}x + \Delta D_{\Delta\Delta}w_{\Delta}$$

● “Quadratic Stability”:  $P(\Delta) = P$

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$$: \begin{bmatrix} \Delta & -1 \end{bmatrix} \begin{bmatrix} C_{\Delta} & D_{\Delta\Delta} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix} = 0$$

⇔ Finsler Lemma

$$M_A^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_A < M_C^* \Theta M_C \leq \tau M_C^* \begin{bmatrix} \Delta^* \\ -1 \end{bmatrix} \begin{bmatrix} \Delta & -1 \end{bmatrix} M_C$$

$$\text{with } \begin{bmatrix} 1 & \Delta^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0$$

### 3 Conservative LMI results

■ Conservative LMIs for LFT models (Example of stability analysis)

$$\dot{x} = Ax + B_{\Delta}w_{\Delta} \text{ with } w_{\Delta} = \Delta z_{\Delta} = \Delta C_{\Delta}x + \Delta D_{\Delta\Delta}w_{\Delta}$$

● “Quadratic Stability”:  $P(\Delta) = P$

$$\dot{V}(x) = \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix}^* \begin{bmatrix} 1 & 0 \\ A & B_{\Delta} \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ A & B_{\Delta} \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix} < 0$$

$$: \begin{bmatrix} \Delta & -1 \end{bmatrix} \begin{bmatrix} C_{\Delta} & D_{\Delta\Delta} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix} = 0$$

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$$\text{with } \begin{bmatrix} 1 & \Delta^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0$$

● Quadratic-LFT PDLF - same methodology (yet needs many matrix manipulations).

#### ■ Conservative LMIs for LFT models

#### ● LMI constraints on Quadratic Separators $\Theta$

▲  $\{X, Y, Z\}$ -dissipative matrices  $\mathbb{\Delta} = \{\Delta : X + Y\Delta + \Delta^*Y^* + \Delta^*Z\Delta \leq 0\}$

$$\begin{bmatrix} 1 & \Delta^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0 : \forall \Delta \in \mathbb{\Delta} \Leftrightarrow \Theta \leq \tau \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}, \tau \geq 0$$

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▲  $\{X, Y, Z\}$ -dissipative real repeated scalars  $\Delta = \{ \Delta = \delta 1 : x + 2y\delta + z\delta^2 \leq 0 \}$

$$\Theta \leq \begin{bmatrix} xD & yD + G \\ yD - G & zD \end{bmatrix}, \begin{matrix} Q \geq 0 \\ G = -G^* \end{matrix}$$

### 3 Conservative LMI results

#### ■ Conservative LMIs for LFT models

#### ● LMI constraints on Quadratic Separators $\Theta$

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$$\begin{bmatrix} 1 & \Delta^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0 : \forall \Delta \in \Delta \Leftrightarrow \Theta \leq \tau \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}, \tau \geq 0$$

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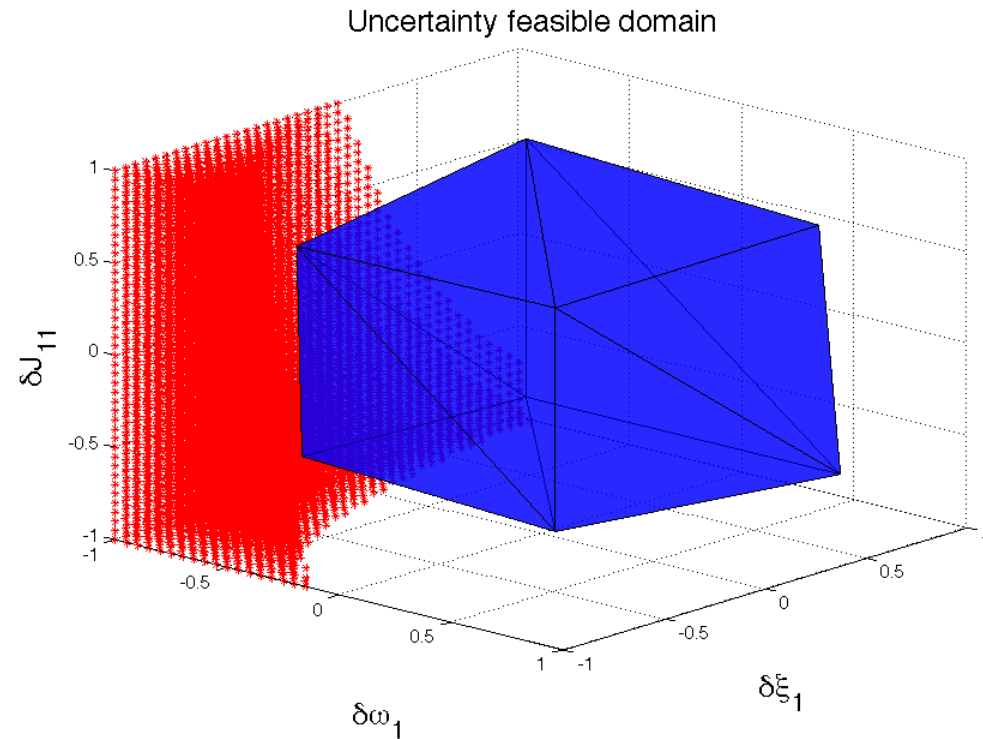
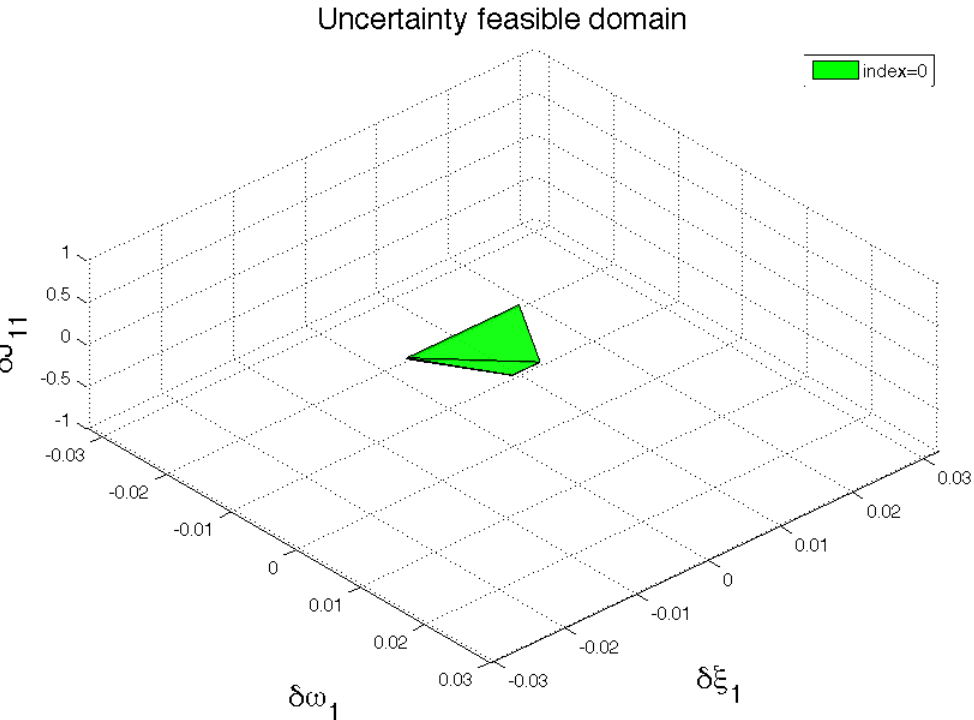
▲ Polytopic uncertainties  $\Delta = \{ \Delta = \sum \xi_i \Delta^{[i]} : \xi_i \geq 0, \sum \xi_i = 1 \}$

$$\begin{bmatrix} 1 & \Delta^{[i]*} \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta^{[i]} \end{bmatrix} \leq 0, \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \Theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq 0$$

# Conclusions

## ■ RoMulOC today

- Large variety of uncertain models associated with multiple performances
- Several associated LMI-based theoretical results coded
- Access to efficient LMI solvers (thanks to YALMIP)
- Testing being done on applications



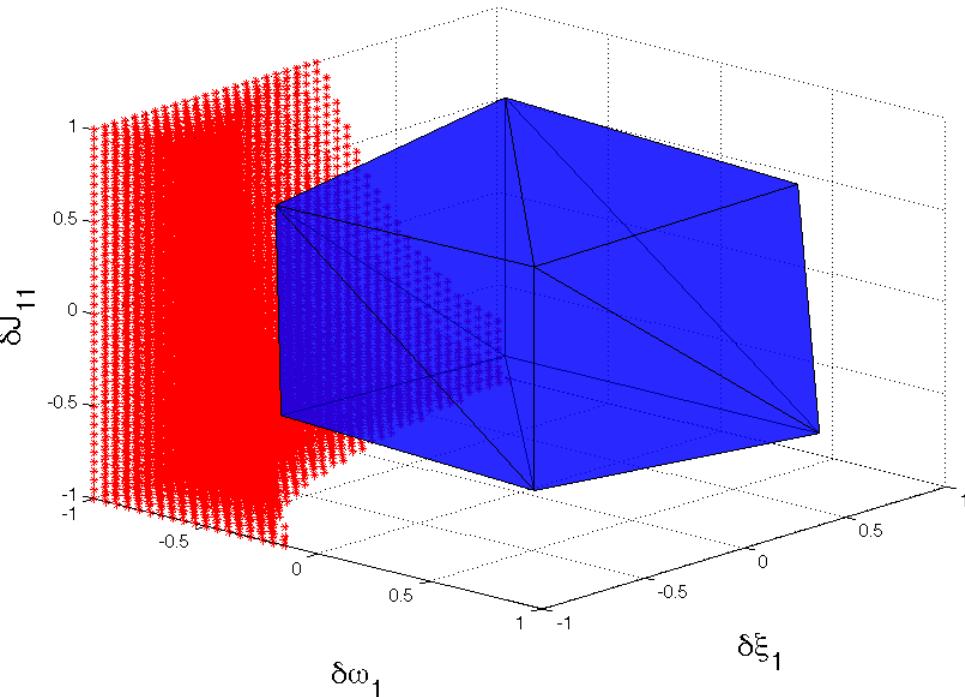
- Design: limited to state-feedback

# Conclusions

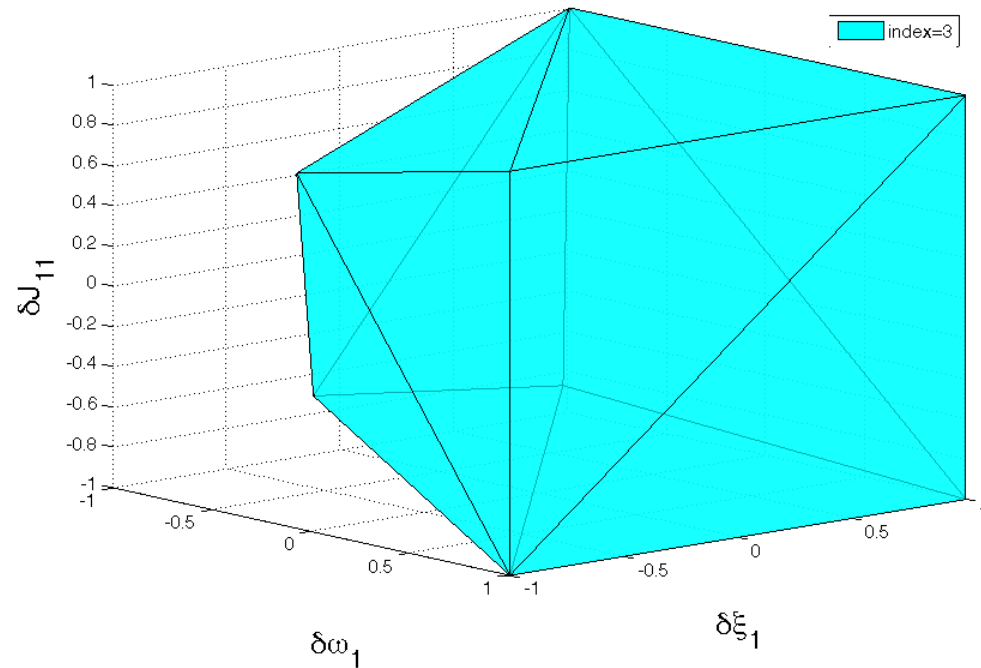
## ■ RoMulOC in the future

- Other design problems: Dynamic output-feedback (LMI) Static output-feedback (not LMI)
- Time-varying uncertainties (with bounded derivatives)
- Time-delay systems (constant or time-varying)
- Non-linearities ( Saturations, dead-zone ...)
- Other Lyapunov functions

Uncertainty feasible domain



Uncertainty feasible domain



■ Descriptor systems:  $E(\Delta)\dot{x} = A(\Delta)x \Rightarrow$  Romuald