LMI Results for Resilient State-Feedback with H_∞ Performance

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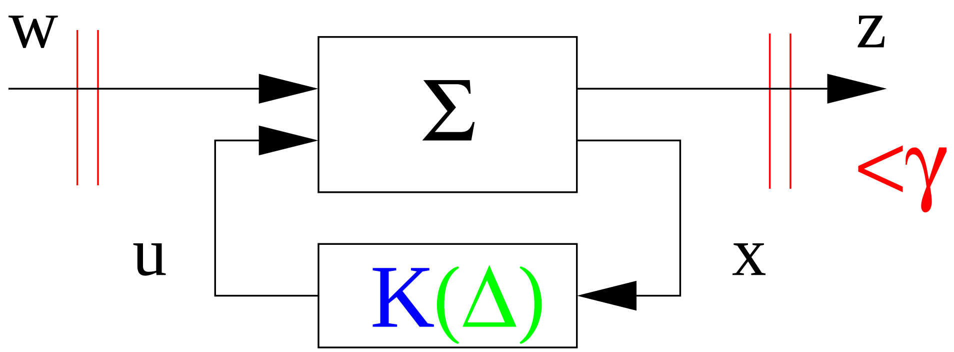
Considered problem: Resilient state-feedback design with H_∞ performance.



Fragile: Uncertainties on the control destroy expected closed-loop properties.

Uncertainties on the control:

→ Tuning margins for practical implementation

→ Implementation errors

→ Finite-precision elements and computation.

Resilient: Not fragile

State-space LTI systems in continuous-time with performance output/input signals:

$$\Sigma \sim \quad \begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & D_{zw} & D_{zu} \\ C & D_{yw} & D \end{bmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix}$$

Alternative representation:

$$Q_x \begin{pmatrix} x \\ -\dot{x} \end{pmatrix} + Q_w \begin{pmatrix} w \\ -z \end{pmatrix} + Q_u \begin{pmatrix} u \\ -y \end{pmatrix} = 0 \quad , \quad Q_x = \begin{bmatrix} A & \mathsf{I} \\ C_z & \mathsf{O} \\ C_y & \mathsf{O} \end{bmatrix} \quad \cdots$$

Define $\Sigma \star \mathbf{K}$ the closed-loop system whith $u = \mathbf{K}y$.

Preliminary result: Ellipsoidal sets of controllers

Derived from results published at IFAC'02 Barcelona, ECC'03 Cambridge:

Theorem 1 *There exists a solution to the following matrix inequality problem*

$$Q_x \begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{P} & 0 \end{bmatrix} Q_x^T < Q_w \begin{bmatrix} -\mathrm{I} & 0 \\ 0 & \gamma^2 \mathrm{I} \end{bmatrix} Q_w^T + Q_u \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^T & \mathbf{Z} \end{bmatrix} Q_u^T$$

$$\mathbf{P} > 0 \quad , \quad \mathbf{Z} > 0 \quad , \quad \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^T$$

if and only if all controls $u = \mathbf{K}y$ *that satisfy the quadratic constraint*

$$\mathbf{X} + \mathbf{K}\mathbf{Y}^T + \mathbf{Y}\mathbf{K}^T + \mathbf{K}\mathbf{Z}\mathbf{K}^T \leq 0$$

perform the closed-loop H_∞ *performance* $\|\Sigma \star \mathbf{K}\|_\infty \leq \gamma$

Remarks:

→ Non-linear matrix inequalities, NP-hard in general

→ Design of sets of controllers (matrix $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}-$ellipsoids)

Main result: The state-feedback case

Theorem 2 *Let the following matrices*

$$\hat{Q}_x = \begin{bmatrix} A & \mathsf{I} \\ C_z & \mathsf{0} \end{bmatrix} \ , \ \hat{Q}_w = \begin{bmatrix} B_w & \mathsf{0} \\ D_{zw} & \mathsf{I} \end{bmatrix} \ , \ \hat{Q}_u = \begin{bmatrix} B & \mathsf{I} \\ D_{zu} & \mathsf{0} \end{bmatrix}$$

There exists a solution to the following LMI problem

$$\hat{Q}_x \begin{bmatrix} \mathsf{0} & \mathbf{P} \\ \mathbf{P} & \mathsf{0} \end{bmatrix} \hat{Q}_x^T < \hat{Q}_w \begin{bmatrix} -\mathsf{I} & \mathsf{0} \\ \mathsf{0} & \gamma^2 \mathsf{I} \end{bmatrix} \hat{Q}_w^T + \hat{Q}_u \begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} \\ \hat{\mathbf{Y}}^T & -\hat{\mathbf{Z}} \end{bmatrix} \hat{Q}_u^T$$

$$\mathbf{P} > \mathsf{0} \quad , \quad \hat{\mathbf{Z}} > \mathsf{0} \quad , \quad \hat{\mathbf{X}} \leq \mathsf{0}$$

if and only if all state-feedback gains $u = \mathbf{K}x$ *of the* $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}-$*ellipsoid where*

$$\mathbf{X} = \hat{\mathbf{X}} + \hat{\mathbf{Y}}\hat{\mathbf{Z}}^{-1}\hat{\mathbf{Y}}^T \quad , \quad \mathbf{Y} = \hat{\mathbf{Y}}\hat{\mathbf{Z}}^{-1}\mathbf{P} \quad , \quad \mathbf{Z} = \mathbf{P}\hat{\mathbf{Z}}\mathbf{P}$$

perform the closed-loop H_∞ *performance* $\|\Sigma \star \mathbf{K}\|_\infty \leq \gamma$

Remarks:

→ Finding ellipsoidal set of state-feedback controllers is purely LMI.

→ $\mathbf{K_o} = -\mathbf{Y}\mathbf{Z}^{-1} = -\hat{\mathbf{Y}}\mathbf{P}^{-1}$ is the center of the ellipsoidal set.

→ The classical LMI for the design of a unique control gain is a sub-case where

$$\hat{\mathbf{X}} = 0 \quad , \quad \mathbf{K} = \mathbf{K_o}$$

and $\hat{\mathbf{Z}}$ is removed because redundant in the constraints.

Can the LMI design of ellipsoidal sets solve the resilience problem?

→ Avoid sets close to singletons.

→ Specify the set geometry for additive or multiplicative uncertainty on $\mathbf{K}$.

→ Find a set that includes a known feedback gain K_n.

Corollary 1

If the solution to theorem 2 satisfies simultaneously the LMI constraints

$$\begin{bmatrix} \tau_a \mathsf{I} & \mathbf{P} \\ \mathbf{P} & \hat{\mathbf{Z}} \end{bmatrix} \geq 0 \quad , \quad \hat{\mathbf{X}} \leq -\tau_a \delta_a^2 \mathsf{I}$$

then $\|\Sigma \star K(\Delta)\|_\infty \leq \gamma$ *for any additive norm-bounded uncertainty on* $\mathbf{K_o}$ *such that:*

$$K(\Delta) = \mathbf{K_o} + \Delta \quad : \quad \Delta\Delta^T \leq \delta_a^2 \mathsf{I}$$

Remarks:

➞ Generate ellipsoidal sets that contain a ”sphere” of radius δ_a.

➞ If $\delta_a = 1/2$ it implies that the gain build out of the rounded coefficients of $\mathbf{K_o}$ still guarantees the closed-loop stability and H_∞ performance.

Corollary 2

If the solution to theorem 2 satisfies simultaneously the LMI constraints

$$\begin{bmatrix} \tau_m \mathsf{I} & S_2 \hat{\mathbf{Y}} \\ \hat{\mathbf{Y}}^T S_2^T & \hat{\mathbf{Z}} \end{bmatrix} \geq \mathsf{O} \quad , \quad \hat{\mathbf{X}} \leq -\tau_m \delta_m^2 S_1 S_1^T$$

then $\|\Sigma \star K(\Delta)\|_\infty \leq \gamma$ *for any multiplicative norm-bounded uncertainty on* $\mathbf{K_o}$ *such that:*

$$K(\Delta) = (\mathsf{I} + S_1 \Delta S_2)\mathbf{K_o} \quad : \quad \Delta\Delta^T \leq \delta_m^2 \mathsf{I}$$

Remarks:

➞ In this case the uncertainty is not totally known *a priori* but related to the designed value $\mathbf{K_o}$.

➞ Uncertainty is known to have influence only on some elements of $\mathbf{K_o}$.

➞ Modeling found in [Yee *et al.*, IJSS 2001]

Corollary 3

If the solution to theorem 2 satisfies simultaneously the LMI constraints

$$\begin{bmatrix} \hat{\mathbf{X}} & \bar{\delta}\hat{\mathbf{Y}} \\ \bar{\delta}\hat{\mathbf{Y}}^T & -\hat{\mathbf{Z}} \end{bmatrix} \leq 0$$

then $\|\Sigma \star K(\Delta)\|_\infty \leq \gamma$ *for any scalar multiplicative uncertainty on* $\mathbf{K_o}$ *such that:*

$$K(\Delta) = (1+\delta)\mathbf{K_o} \quad : \quad |\delta| \leq \bar{\delta}$$

Remarks:

➞ Allows a set of proportional state-feedback gains.

➞ δ is a design parameter for fine tuning.

➞ Modeling found in [Corrado and Haddad, ACC 1999]

$$A = \begin{bmatrix} 1 & 1 \\ -10 & 1 \end{bmatrix} \;,\; B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;,\; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \gamma = 1$$

$$C_z = \begin{bmatrix} 1 & 1 \end{bmatrix} \;,\; D_{zw} = 0.5 \;,\; D_{zu} = 0.1$$

→ Classical design (single state-feedback):

$$K_1 = \begin{bmatrix} -10.2357 & 12.2276 \end{bmatrix} \quad : \quad \|\Sigma \star K_1\|_\infty = 0.8307$$

Fragile to $\bar{\delta} = 0.5$ and $\delta_a = 10$:

$$\|\Sigma \star (1-0.5)K_1\|_\infty = 1.1160 \quad , \quad \|\Sigma \star \begin{bmatrix} -2 & 12 \end{bmatrix}\|_\infty = \infty$$

→ New ellipsoidal set design without constraints on the set. Center:

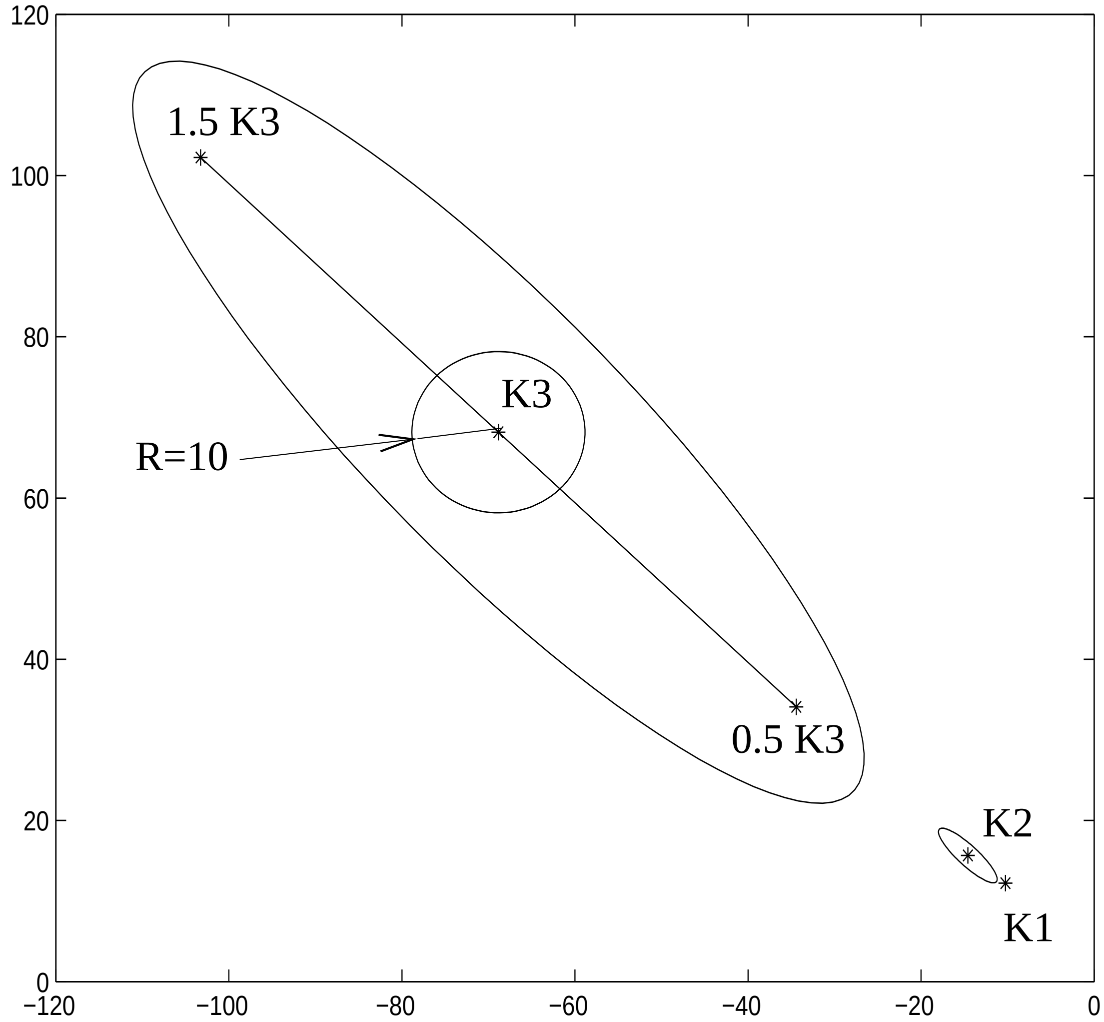
$$K_2 = \begin{bmatrix} -14.5908 & 15.6533 \end{bmatrix} \quad : \quad \|\Sigma \star K_2\|_\infty = 0.7555$$

the obtained set has a volume of 5.62 times the unit circle.

→ Ellipsoidal set constrained by $\bar{\delta} = 0.5$ and $\delta_a = 10$ gives a center:

$$K_3 = \left[\begin{array}{cc} -68.8418 & 68.1654 \end{array} \right]$$

$$\|\Sigma \star K_3\|_\infty = 0.6977$$



Conclusion

→ LMI results for H_∞ state-feedback design can be extended to resilient design.

→ Remains an LMI problem, few additional variables.

→ Same procedure can be applied for other state-feedback problems (pole location, H_2, ...)

→ Prospective: extend to full-order dynamic output-feedback.