LMI Results for Resilient State-Feedback with $H_\infty$ Performance

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Considered problem: Resilient state-feedback design with $H_\infty$ performance.

Fragile: Uncertainties on the control destroy expected closed-loop properties.

Uncertainties on the control:

- Tuning margins for practical implementation
- Implementation errors
- Finite-precision elements and computation.

Resilient: Not fragile
Notations

State-space LTI systems in continuous-time with performance output/input signals:

\[
\Sigma \sim \begin{pmatrix}
\dot{x} \\
z \\
y
\end{pmatrix} =
\begin{bmatrix}
A & B_w & B \\
C_z & D_{zw} & D_{zu} \\
C & D_{yw} & D
\end{bmatrix}
\begin{pmatrix}
x \\
w \\
u
\end{pmatrix}
\]

Alternative representation:

\[
Q_x \begin{pmatrix}
x \\
-\dot{x}
\end{pmatrix} + Q_w \begin{pmatrix}
w \\
-z
\end{pmatrix} + Q_u \begin{pmatrix}
u \\
-y
\end{pmatrix} = 0, \quad Q_x =
\begin{bmatrix}
A & 1 \\
C_z & 0 \\
C_y & 0
\end{bmatrix}
\]

Define \( \Sigma \star K \) the closed-loop system with \( u = Ky \).
Preliminary result: Ellipsoidal sets of controllers

Derived from results published at IFAC’02 Barcelona, ECC’03 Cambridge:

**Theorem 1** There exists a solution to the following matrix inequality problem

\[
Q_x \begin{bmatrix} O & P \\ P & O \end{bmatrix} Q_x^T < Q_w \begin{bmatrix} -I & O \\ O & \gamma^2 I \end{bmatrix} Q_w^T + Q_u \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} Q_u^T
\]

\[
P > O, \quad Z > O, \quad X \leq YZ^{-1}Y^T
\]

if and only if all controls \( u = Ky \) that satisfy the quadratic constraint

\[
X + KY^T + YK^T + KZK^T \leq 0
\]

perform the closed-loop \( H_\infty \) performance \( \| \Sigma * K \|_\infty \leq \gamma \)

Remarks:

→ Non-linear matrix inequalities, NP-hard in general
→ Design of sets of controllers (matrix \( \{X, Y, Z\} \) — ellipsoids)
**Theorem 2** Let the following matrices

\[
\hat{Q}_x = \begin{bmatrix}
A & I \\
C_z & 0
\end{bmatrix}, \quad \hat{Q}_w = \begin{bmatrix}
B_w & 0 \\
D_{zw} & I
\end{bmatrix}, \quad \hat{Q}_u = \begin{bmatrix}
B & I \\
D_{zu} & 0
\end{bmatrix}
\]

There exists a solution to the following LMI problem

\[
\begin{bmatrix}
\hat{Q}_x & P \\
P & 0
\end{bmatrix} \hat{Q}_x^T < \begin{bmatrix}
-1 & 0 \\
0 & \gamma^2 I
\end{bmatrix} \begin{bmatrix}
\hat{Q}_w^T + \hat{Q}_u & \hat{X} & \hat{Y} \\
\hat{Y}^T & -\hat{Z}
\end{bmatrix} \hat{Q}_u^T
\]

\[P > 0, \quad \hat{Z} > 0, \quad \hat{X} \leq 0\]

if and only if all state-feedback gains \( u = Kx \) of the \( \{X, Y, Z\} \)-ellipsoid

where

\[
X = \hat{X} + \hat{Y} \hat{Z}^{-1} \hat{Y}^T, \quad Y = \hat{Y} \hat{Z}^{-1} P, \quad Z = P \hat{Z} P
\]

perform the closed-loop \( H_\infty \) performance \( \|\Sigma \ast K\|_\infty \leq \gamma \)
Main result: The state-feedback case

Remarks:

→ Finding ellipsoidal set of state-feedback controllers is purely LMI.

→ $K_0 = -YZ^{-1} = -\hat{Y}P^{-1}$ is the center of the ellipsoidal set.

→ The classical LMI for the design of a unique control gain is a sub-case where

$$\hat{X} = 0, \quad K = K_0$$

and $\hat{Z}$ is removed because redundant in the constraints.

Can the LMI design of ellipsoidal sets solve the resilience problem?

→ Avoid sets close to singletons.

→ Specify the set geometry for additive or multiplicative uncertainty on $K$.

→ Find a set that includes a known feedback gain $K_n$. 
Corollary 1

If the solution to theorem 2 satisfies simultaneously the LMI constraints

\[
\begin{bmatrix}
\tau_a & P \\
P & \hat{Z}
\end{bmatrix} \geq 0 , \quad \hat{X} \leq -\tau_a \delta_a^2 I
\]

then \(\|\Sigma \ast K(\Delta)\|_\infty \leq \gamma\) for any additive norm-bounded uncertainty on \(K_o\) such that:

\[K(\Delta) = K_o + \Delta : \quad \Delta \Delta^T \leq \delta_a^2 I\]

Remarks:

\(\rightarrow\) Generate ellipsoidal sets that contain a "sphere" of radius \(\delta_a\).

\(\rightarrow\) If \(\delta_a = 1/2\) it implies that the gain build out of the rounded coefficients of \(K_o\) still guarantees the closed-loop stability and \(H_\infty\) performance.
Resilience w.r.t. multiplicative uncertainty

Corollary 2

*If the solution to theorem 2 satisfies simultaneously the LMI constraints*

$$
\begin{bmatrix}
\tau_m I & S_2 \hat{Y} \\
\hat{Y}^T S_2^T & \hat{Z}
\end{bmatrix} \geq O , \quad \hat{X} \leq -\tau_m \delta_m^2 S_1 S_1^T
$$

*then* $$\| \Sigma * K(\Delta) \|_\infty \leq \gamma$$ for any multiplicative norm-bounded uncertainty on $$K_o$$ *such that:*

$$K(\Delta) = (I + S_1 \Delta S_2) K_o : \Delta \Delta^T \leq \delta_m^2 I$$

Remarks:

→ In this case the uncertainty is not totally known *a priori* but related to the designed value $$K_o$$.

→ Uncertainty is known to have influence only on some elements of $$K_o$$.

→ Modeling found in [Yee et al., IJSS 2001]
Corollary 3

If the solution to theorem 2 satisfies simultaneously the LMI constraints

\[
\begin{bmatrix}
\hat{X} & \delta \hat{Y} \\
\delta \hat{Y}^T & -\hat{Z}
\end{bmatrix} \leq 0
\]

then \( \| \Sigma \ast K(\Delta) \|_\infty \leq \gamma \) for any scalar multiplicative uncertainty on \( K_o \) such that:

\[ K(\Delta) = (1 + \delta)K_o : |\delta| \leq \bar{\delta} \]

Remarks:

→ Allows a set of proportional state-feedback gains.
→ \( \delta \) is a design parameter for fine tuning.
→ Modeling found in [Corrado and Haddad, ACC 1999]
Example 1: purely academic to illustrate ellipsoids

\[ A = \begin{bmatrix} 1 & 1 \\ -10 & 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \gamma = 1 \]

\[ C_z = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_{zw} = 0.5, \quad D_{zu} = 0.1 \]

→ Classical design (single state-feedback):

\[ K_1 = \begin{bmatrix} -10.2357 & 12.2276 \end{bmatrix} : \quad \| \Sigma \ast K_1 \|_\infty = 0.8307 \]

Fragile to \( \bar{\delta} = 0.5 \) and \( \delta_a = 10 \):

\[ \| \Sigma \ast (1 - 0.5)K_1 \|_\infty = 1.1160, \quad \| \Sigma \ast \begin{bmatrix} -2 & 12 \end{bmatrix} \|_\infty = \infty \]

→ New ellipsoidal set design without constraints on the set. Center:

\[ K_2 = \begin{bmatrix} -14.5908 & 15.6533 \end{bmatrix} : \quad \| \Sigma \ast K_2 \|_\infty = 0.7555 \]

the obtained set has a volume of 5.62 times the unit circle.
Example 1: purely academic to illustrate ellipsoids

→ Ellipsoidal set constrained by $\bar{\delta} = 0.5$ and $\delta_\alpha = 10$ gives a center:

$$K_3 = \begin{bmatrix} -68.8418 & 68.1654 \end{bmatrix}$$

$$\|\Sigma \ast K_3\|_\infty = 0.6977$$
LMI results for $H_{\infty}$ state-feedback design can be extended to resilient design.

Remains an LMI problem, few additional variables.

Same procedure can be applied for other state-feedback problems (pole location, $H_2$, ...)

Prospective: extend to full-order dynamic output-feedback.