Structured adaptive control for solving LMIs

Alexandru-Razvan Luzi,
Alexander L. Fradkov, Jean-Marc Biannic,
Dimitri Peaucelle

IFAC-ALCOSP Caen, july 2013

By-product of research work on adaptive satellite attitude control:
”Structured adaptive attitude control of a satellite”,
A.R. Luzi, D. Peaucelle, J.-M. Biannic, Ch. Pittet, J. Mignot,
International Journal of Adaptive Control and Signal Processing 2013
What are LMIs?

**LMIs: Linear Matrix Inequalities**

\[
\max \sum b_i y_i : \quad F_0 + \sum y_i F_i \prec 0
\]

**LMIs are SDP: Semi-Definite Programming**

\[
\min c^T x : \quad Ax = b, \quad \text{mat}(x) \succ 0
\]

- Primal-dual, convex, solvers in polynomial-time [Nesterov, ...]
- Nice parser: YALMIP
- Many control problems have LMI formulations, mainly in robust control

\[
P \succ 0, \quad A^T P + PA \prec 0
\]

- New results for: combinatorial optimization, robust optimization, algebraic geometry, cryptography, optimal control...
Introduction

- Direct adaptive control:
  Adaptation of control gains done directly based on measurements.

- Indirect adaptive control:
  Estimator of model parameters + scheduled control gain

- Feedback-loop stabilizing gains, MRAC not considered

- Lyapunov based stability proofs, not gradient approximation ‘MIT rule’

- Framework initiated by V.A. Yakubovich in the late 1960’s

- Contributions: new adaptive control law with asymptotic structure
  + may solve LMIs
1 Passivity-based adaptive control
2 LMIs are strict-passifiable systems
3 Structured adaptive control
4 Numerical Example
Passivity-based adaptive control of LTI systems

Theorem

The following two conditions are equivalent:

1. There exists a static control $u(t) = Fy(t) + w(t)$ for the system

   $$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad y(t) = Cx(t) \quad , \quad z(t) = y(t)$$

   that makes the closed-loop strictly passive (with respect to $w/z$).

2. For all $\Gamma \succ 0$ the following adaptive control

   $$u(t) = K(t)y(t) + w(t) \quad , \quad \dot{K}(t) = -y(t)y^T(t)\Gamma$$

   makes the closed-loop globally strictly-passive.
• Strict-passivity includes asymptotic stability of $x = 0$

• Adaptive control converges to $K(\infty)$: strictly-passifying static gain

△ Theorem for square systems - extensions exist for non-square systems

△ Not all stabilizable systems are strictly-passifiable

- modified adaptive laws exist for stabilizable systems

• Condition 1 also reads in terms of matrix inequalities as

$$\exists Q \succ 0 \quad : \quad (A + BF C)^T Q + Q (A + BF C) \prec 0 \quad , \quad QB = C^T$$

It happens to be an LMI constraint!

$$\exists Q \succ 0 \quad : \quad A^T Q + QA + C^T (F^T + F)C \prec 0 \quad , \quad QB = C^T$$

■ Finding $F$ solution to the LMI is equivalent to simulating the system with the adaptive control law and taking $F = K(\infty)$. 
All LMIs define strict-passifiable systems

Let us consider an example:

LMIs for an upper bound on the $H_{\infty}$ norm of $G(s) \sim (A, B, C, D)$

$$
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D \\
B^T P + D^T C & -\gamma^2 I + D^T D
\end{bmatrix} \prec 0 , \quad P = P^T \succ 0.
$$
LMIs are strict-passifiable systems

All LMIs define strict-passifiable systems

Let us consider an example:
LMIs for an upper bound on the $H_\infty$ norm of $G(s) \sim (A, B, C, D)$

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D \\
B^T P + D^T C & -\gamma^2 1 + D^T D
\end{bmatrix} \prec 0, \quad P = P^T \succ 0.
\]

Converted with simple manipulations into one simple LMI

\[A + B^T F B \prec 0\]

with structural equality constraints on $F$

\[
F = \begin{bmatrix}
F_P & 0 \\
0 & F_\gamma
\end{bmatrix}, \quad F_P = \begin{bmatrix}
0 & P & 0 \\
P & 0 & 0 \\
0 & 0 & -P
\end{bmatrix}, \quad P = P^T, \quad F_\gamma = -\gamma^2 1
\]

\[
A = \begin{bmatrix}
C^T C & C^T D & 0 \\
D^T C & D^T D & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
A & B & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]
Let us consider an example:

LMIs for an upper bound on the $H_\infty$ norm of $G(s) \sim (A, B, C, D)$

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D \\
B^T P + D^T C & -\gamma^2 I + D^T D
\end{bmatrix} \prec 0, \quad P = P^T > 0.
\]

Converted with simple manipulations into one simple LMI

\[A + B^T F B \prec 0\]

with structural equality constraints on $F$

\[
F = \begin{bmatrix}
F_P & 0 \\
0 & F_\gamma
\end{bmatrix}, \quad F_P = \begin{bmatrix}
0 & P & 0 \\
P & 0 & 0 \\
0 & 0 & -P
\end{bmatrix}, \quad P = P^T, \quad F_\gamma = -\gamma^2 I
\]

The constraint $A + B^T F B \prec 0$ holds iff

$(A, B, C = B^T)$ is strictly-passifiable by $F$ (condition 1).

LMI converted to strict-passification problem, with equality constraints.
LMIs are strict-passifiable systems

- Procedure applies to any LMI:
  - Concludes with search of passifying gain \( F = \begin{bmatrix} F_1 & 0 \\ \vdots & \ddots \\ 0 & F_N \end{bmatrix} \)
  - for a (symmetric) system \((A, B, C = B^T)\)
  - with additional structural equality constraints that can be compacted in
    \[ U_i \text{vec}(F_i) = 0 \]
    (Where \( \text{vec}(F_i) \) is the vector composed of stacked columns of \( F_i \).)

- All constraints \( U_i \text{vec}(F_i) = 0 \) include the constraint \( F_i = F_i^T \).
Block-diagonal adaptive control with asymptotic structure

Theorem

Assume \( A = A^T \) and \( C = B^T \), then the following two are equivalent:

1. There exists a symmetric decentralized static control \( u_i(t) = F_i y_i(t) \) satisfying structural constraints \( U_i \text{vec}(F_i) = 0 \) that stabilizes asymptotically

\[
\dot{x}(t) = Ax(t) + \sum B_i u_i(t) \quad , \quad y_i(t) = C_i x(t).
\]

2. For all \( \Gamma_i \succ 0 \), \( \alpha_i > 0 \) the following adaptive control

\[
\begin{align*}
    u_i(t) &= K_i(t) y_i(t) + w_i(t) \\
    \dot{K}_i(t) &= -y_i(t) y_i^T(t) \Gamma_i - \alpha_i \cdot \text{mat} \left( U_i^T U_i \cdot \text{vec}(K_i(t)) \right) \Gamma_i
\end{align*}
\]

makes the closed-loop globally asymptotically stable and the adaptive gains converge to constant values \( F_i = K_i(\infty) \) solution to condition 1.

(‘mat’ is the function such that \( \text{mat}(\text{vec}(F)) = F \)
Proof of \( \textbullet \Rightarrow \textbullet \)

- Stability of a symmetric matrix \( A + BFC \) proved by \( V(x) = \frac{1}{2}x^Tx \), i.e. \( \textbullet \) implies
  \[
  \exists F : (A + BFC)^T + (A + BFC) < 0, \quad \text{such that} \quad F = \text{diag}[\cdots F_i \cdots], \quad U_i \cdot \text{vec}(F_i) = 0 \tag{1}
  \]

- Let the Lyapunov function for the non-linear system (with adaptive law)
  \[
  V(x, K) = \frac{1}{2} \left( x^Tx + \sum_i \text{Tr} \left( (K_i - F_i)\Gamma^{-1}(K_i - F_i)^T \right) \right)
  \]

- After manipulations, using \( B = C^T \), \( U_i \cdot \text{vec}(F_i) = 0 \), we get:
  \[
  \dot{V}(x, K) = x^T (A + BFC)^T x - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)).
  \]
Proof of $1 \Rightarrow 2$ (continued)

\[ \dot{V}(x, K) = x^T (A + BF_C)^T x - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)). \]

- First term is strictly negative due to (1), until $x = 0$,
- Last term is strictly negative, until $U_i \cdot \text{vec}(K_i) = 0$.

- The system converges to the attractor

\[ A = \{(x, K) : x = 0, \ U_i \cdot \text{vec}(K_i) = 0\} \]

- Reasoning in [Ioannou&Sun 96] allows to conclude that $K_i(t)$ converges to a constant gain $K_i(\infty)$. 

The system with adaptive control is globally asymptotically stable, it converges to an asymptotically stable equilibrium:

\[ F_i = K_i(\infty) \] are stabilizing gains
Summary

- All LMI problems are equivalent to static output-feedback strict-passification problems with structure constraints:
  - $\mathbb{A} = \mathbb{A}^T$
  - gain $F$ is block-diagonal
  - sub-blocks should satisfy $U_i \text{vec}(F_i) = 0$.

- If a structured strict-passification problem admits solutions, the block-diagonal adaptive law with asymptotic structure will converge to one of these.

- The LMIs can be solved by simulating the adaptive controlled systems.

- If the system converges $K_i(\infty) = F_i$ are solutions of the LMIs.

- If does not converges the LMIs are infeasible.
Numerical example

Consider the transfer function:

\[ G(s) = \frac{s^2 + s + 1}{s^2 + s + 2} \]

Problem: compute the \( H_{\infty} \) norm (or at least an upper bound).

In Matlab: \( \text{norm}(G, \text{Inf}, 1e^{-4}) = 1.3251 \)
Numerical example

Consider the transfer function:

$$G(s) = \frac{s^2 + s + 1}{s^2 + s + 2}$$

Problem: compute the $H_\infty$ norm (or at least an upper bound).

In Matlab: \[ \text{norm}(G, \text{Inf}, 1e-4) = 1.3251 \]

LMI problem converted to adaptive passification

$$\dot{K}_i = -y_i y_i^T \Gamma_i - \alpha_i \cdot \text{mat} \left( U_i^T U_i \cdot \text{vec}(K_i) \right) \Gamma_i, \quad y_1 \in \mathbb{R}^6, \quad y_2 \in \mathbb{R}$$

with structural asymptotic constraints:

$$F_1 = \begin{bmatrix} 0 & P & 0 \\ P^T & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} \quad P = P^T \in \mathbb{R}^{2 \times 2}, \quad F_2 = -\gamma^2 1 = -\gamma^2.$$
Numerical Example

- Parameters for simulating the adaptive law (simulation in Simulink)
  - Initial conditions $x = (1 \ldots 1)^T$ and $K_i = 0$
  - $\Gamma_1 = 1000 \cdot 1$, $\Gamma_2 = 10$, $\alpha_1 = \alpha_2 = 1$
- Convergence to zero of the ‘outputs’ $y_i$
Convergence to structured values of the adapted gains $K_i$.

$$\|U_1^T \text{vec}(K_i)\|_2$$

$$K_1(\infty) = \begin{bmatrix} 0 & 0 & 4.6330 & 1.0671 & 0 & 0 & 0 \\ 0 & 0 & 1.0671 & 10.7960 & 0 & 0 & 0 \\ 4.6330 & 1.0671 & 0 & 0 & 0 & 0 & 0 \\ 1.0671 & 10.7960 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{-}4.6330 & \text{-}1.0671 \\ 0 & 0 & 0 & 0 & \text{-}1.0671 & \text{-}10.7960 \end{bmatrix}$$

$$K_2(\infty) = \text{-}7.1307$$
**Evolution of the (1 : 2, 3 : 4) elements of $K_1$ that converge to $P$**

**Solution of the LMIs**

\[
P = \begin{bmatrix} 4.6330 & 1.0671 \\ 1.0671 & 10.7960 \end{bmatrix}, \quad \gamma = 2.6703 \geq 1.3251 = \gamma_{opt}
\]
Test for feasible / unfeasible cases

Only $K_1$ is adapted, $\gamma$ is slowly linearly modified

Unstable behavior when $\gamma < 1.3251 = \gamma_{opt}$. 
Conclusions

Conclusions et perspectives

- LMI feasibility problems can be solved by simulating systems
  - Need for a parser to convert LMIs to adaptive control problem
  - Simulation time is large - what is the best implementation?
  - Is simulation time polynomial w.r.t. size of problem?

- What about LMI optimization problems?
  - Decreasing parameters until system becomes unstable?
  - Minimizing gap with dual LMI problem (it works).
  - Other?

- Solving time-varying LMI problems?