

# LMI Tests for Positive Definite Polynomials: Slack Variable Approach

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## Abstract

The considered problem is assessing non-negativity of a function's values when indeterminates are in domains constrained by scalar polynomial inequalities. The tested functions are multi-indeterminates polynomial matrices which are required to be positive semi-definite. For such problems new tests based on linear matrix inequalities are provided in a Slack Variables type approach. The results are compared to those obtained via the Sum-Of-Squares approach, are proved to be equivalent in case of unbounded domains and less conservative if polytopic-type bounds are known.

## Index Terms

LMI, Matrix-valued polynomials, Robustness, Positivity, Sum-of-Squares

## I. INTRODUCTION

About ten years after Shor has proposed the Sums-Of-Squares (SOS) approach for obtaining the global minimum of polynomial functions [25], Chesi et al. [2], Lasserre [7] and Parrilo [13] have independently shown that SOS decomposition can be cast to semi-definite programming. After these first results, many publications taking advantage of SOS decomposition technique have been reported, e.g. nonlinear system analysis, hybrid system analysis, etc., some of which are surveyed in [12], [6]. But all positive polynomials are not SOS and Motzkin polynomials

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are examples of this fact. More precisely, as enunciated in Hilbert problems, the set of SOS polynomials coincides with the set of non-negative polynomials only in three very simple cases: single indeterminate (whatever degree), quadratic form (whatever number of indeterminates), and quartic form in two indeterminates [25], [17]. However, under some realistic restrictions, the gap between SOS polynomials and non-negative polynomials disappears. Lasserre has proved that all non-SOS polynomials, which are non-negative, can be approximated by other polynomials of higher degree which can be verified to be non-negative with theory of moments [7], and they can also be approximated by some SOS polynomials with small SOS polynomial perturbations [8]. Further, polynomial functions, which are non-negative on unit ball, can be approximated by some SOS polynomials with small SOS polynomial perturbations [9]. Most results are derived assuming scalar-valued polynomials but Scherer and Hol [24] have provided as well a matrix-valued version of SOS approximation with use of Putinar's SOS representation.

On the other hand, in the last decade, much research for robust stability/performance analysis using Slack Variable (SV) approach via parametrically affine Linear Matrix Inequalities (LMIs) has been conducted, e.g. [3], [15], [5]. Based on Finsler lemma, the approach creates new variables that allow without much mathematical complications to derive LMI results with reduced conservatism. In addition to simplicity in matrix manipulations, the approach benefits of extensions for control design, [1], [14], [4]. Yet, the question whether conservatism of the SV approach can be further reduced remains an open problem. Partial answers have been provided. One [10], [11], using the fact that SV approach is in general applied to polynomials with indeterminates constrained in the unit simplex, is based on a matrix-valued version of Pólya's theorem [23]. Another [16], [19], [20] uses redundant system modeling which may be interpreted as taking advantage of polynomial representations in basis of higher degree. All these publications illustrate the influence of representations of the robustness analysis problems as polynomial matrix inequalities, however, to the authors' knowledge, no result has been reported for the relationship between SOS approach and SV approach. Based on preliminary results [22], [21] this relationship is investigated in the present paper and SV approach is shown to encompass SOS approach.

The two upper cited techniques to be compared may be mathematically summarized as follows. For a given choice of a basis of monomials gathered in a vector  $\check{\theta}$  (in the present paper the

elements of  $\check{\theta}$  are of the form  $\prod_k \theta_k^{m_k}$  where the  $\theta_k$  are the scalar indeterminates), a matrix valued polynomial has at least one representation such that  $F(\theta) = (\check{\theta} \otimes \mathbf{1})^T \hat{F} (\check{\theta} \otimes \mathbf{1})$ , where  $\mathbf{1}$  stands for the identity matrix of the dimension of  $F(\theta)$ . The elements of the  $\hat{F}$  matrix are the coefficient matrices of the polynomial. SOS result states that  $F(\theta)$  is positive semi-definite for all indeterminates  $\theta$  if there exists a matrix  $\hat{L}$  such that  $(\check{\theta} \otimes \mathbf{1})^T \hat{L} (\check{\theta} \otimes \mathbf{1}) = \mathbf{0}$  and  $\hat{F} + \hat{L}$  is positive semi-definite.  $F(\theta)$  has in such case a factorization as sum of squares of polynomials. SV technique is based on Finsler lemma and on the fact that there exists an affine with respect to  $\theta$  matrix  $\Psi(\theta)$  such that  $\Psi^T(\theta)(\check{\theta} \otimes \mathbf{1}) = \mathbf{0}$ . SV result states that  $F(\theta)$  is positive semi-definite for  $\theta$  in the convex set of a finite number of vertices  $\theta^{[p]}$  if there exists a matrix  $N$  such that  $\hat{F} + \Psi(\theta^{[p]})N + N^T\Psi^T(\theta^{[p]})$  is positive semi-definite for all vertices. In the paper we extend this last SV result for the case where the indeterminates are unbounded (not in a polytope) and prove the direct relation between the  $N$  and  $\hat{L}$  matrices.

The outline is as follows. The next section is devoted to (inevitably intricate) notations and to the problem definition: proving a polynomial matrix values are positive semi-definite for indeterminates constrained in the intersection of polytopes and scalar polynomial inequalities. Section III states the main results, namely the LMI formulas obtained via the SV approach, and shows equivalence of SV and SOS methods in case of unbounded polytopic constraints. The by-products of the SV approach is a less conservative result in case of bounded indeterminate domains and an easy to code reformulation of SOS results. Section IV considers the mixed bounded/unbounded case and formulates results for solving robust polynomial semi-definite programs. A conclusion ends the paper indicating prospective research directions.

## II. PRELIMINARIES

### A. General notations

$\mathbb{R}^{n \times m}$ ,  $S^n$  and  $\mathbb{Q}^n$ , denote respectively the sets of  $n \times m$  real, symmetric  $n \times n$  real and skew-symmetric  $n \times n$  real matrices. For two symmetric matrices,  $A$  and  $B$ ,  $A > (\geq) B$  means that  $A - B$  is positive (semi-) definite.  $A^T$  denotes the transpose of  $A$ .  $A^\dagger$  is the Moore-Penrose inverse of  $A$ .  $\mathbf{1}_n$ ,  $\mathbf{0}_{m,n}$  and  $\mathbf{0}_n$  denote respectively the identity matrix of size  $n$ , null matrix of size  $m \times n$  and the square null matrix of size  $n \times n$ . If the context allows it, the dimensions of these matrices are sometimes omitted. For a given matrix  $B \in \mathbb{R}^{m \times n}$  such that  $\text{rank}(B) = r$ ,

define  $B^\perp \in \mathbb{R}^{n \times (n-r)}$  the right orthogonal complement of  $B$  by  $BB^\perp = \mathbf{0}$  and  $B^\perp B^{\perp T} > \mathbf{0}$ . The notation  $\langle A \rangle$  stands for the symmetric matrix  $A + A^T$ . Trivially  $\langle A \rangle = \mathbf{0}$  if and only if  $A$  is skew-symmetric and  $\langle A \rangle = 2A$  if and only if  $A$  is symmetric.  $\otimes$  stands for the Kronecker product (matrix direct product). For  $A = [a_{ij}] \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{p \times q}$  it is such that  $C = A \otimes B$  is a  $\mathbb{R}^{np \times mq}$  matrix with block entries  $C = [a_{ij}B]$ . For matrices of appropriate dimensions, the Kronecker product is such that  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ .

### B. Problem definition

Let  $\bar{k}$  real scalar independent indeterminates  $\theta_1, \dots, \theta_{\bar{k}}$  and let  $\theta = \left( \theta_1 \ \dots \ \theta_{\bar{k}} \right)^T$ . Consider  $F$  a symmetric-matrix valued polynomial function of the indeterminates,  $F(\theta) \in \mathbb{S}^n$ . The goal of the paper is to give LMI tests to prove  $F(\theta)$  is positive semi-definite for  $\theta$  constrained in the intersection of two type of sets

$$F(\theta) \geq \mathbf{0}_n \quad , \quad \forall \theta \in \mathcal{G} \cap \mathcal{P} \quad (1)$$

defined as follows.

- Let  $\bar{q}$  scalar polynomials of the indeterminates  $\theta_1, \dots, \theta_{\bar{k}}$ ,  $g_q(\theta) \in \mathbb{R}$ ,  $q = 1 \dots \bar{q}$ . The first type of set is such that these polynomials are non-negative:

$$\mathcal{G} = \left\{ \theta \in \mathbb{R}^{\bar{k}} \quad : \quad g_q(\theta) \geq 0 \quad , \quad q = 1 \dots \bar{q} \right\} . \quad (2)$$

Such set is possibly unbounded and may be non-connected.

- Let  $\mathcal{K}_b \subset \{1 \dots \bar{k}\}$  be a set of indeterminates' indexes and let  $\bar{k}_b$  be the number of elements in the set. Define  $\vartheta \in \mathbb{R}^{\bar{k}_b}$  the vector of indeterminates  $\theta_{k \in \mathcal{K}_b}$  ordered with growing index. The second type of set is a semi-polytope defined as the intersection of  $\mathbb{R}^{\bar{k}}$  with a polytope of the indeterminates  $\vartheta$  having  $\bar{p}$  vertices  $\vartheta^{[p]}$  :

$$\mathcal{P} = \left\{ \theta \in \mathbb{R}^{\bar{k}} \quad : \quad \vartheta = \sum_{p=1}^{\bar{p}} \zeta_p \vartheta^{[p]} \quad , \quad \zeta_p \geq 0 \quad , \quad \sum_{p=1}^{\bar{p}} \zeta_p = 1 \right\} . \quad (3)$$

In such set, the indeterminates  $\theta_{k \in \mathcal{K}_b}$  are bounded.

Two special and important cases are studied in the next section: first, the (fully) bounded case where  $\bar{k}_b = \bar{k}$ ; second, the unbounded case where  $\bar{k}_b = 0$ . Based on these results, section IV gives extensions to the mixed bounded/unbounded case ( $0 < \bar{k}_b < \bar{k}$ ).

### C. Notations for polynomials manipulations

Define for each indeterminate  $\check{\theta}_k^{m_k} = \left( 1 \ \theta_k \ \dots \ \theta_k^{m_k} \right)^T \in \mathbb{R}^{m_k+1}$  the vector of powers up to power  $m_k$ . The notation  $\check{\theta} = \check{\theta}_1^{m_1} \otimes \check{\theta}_2^{m_2} \dots \otimes \check{\theta}_{\bar{k}}^{m_{\bar{k}}}$  produces a vector of all monomial obtained as products of elements  $\theta_k^j$  with indeterminates indices  $k$  ranging from 1 to  $\bar{k}$  and powers going up to  $m_k$ . In the following, the polynomials are expressed in this basis of monomials. Note that this basis is very general and maybe over-dimensioned for given polynomials and lead to increased computational effort. To handle this issue one may consider Groebner bases methods. But for the purpose of comparison between SV and SOS the general basis is needed and the use of Groebner bases is left for future work.

For the sake of readability of formulas define the notation  $\sigma_{k_1, k_2} = \prod_{k=k_1}^{k_2} (m_k + 1)$  for all  $1 \leq k_1 \leq k_2 \leq \bar{k}$  and let  $\sigma_{1,0} = 1$  and  $\sigma_{\bar{k}+1, \bar{k}} = 1$ . Moreover, let  $\sigma = \sigma_{1, \bar{k}}$  and define  $\pi_k = \sigma m_k / (m_k + 1)$  for all  $k \in \{1 \dots \bar{k}\}$ .

For each  $k \in \{1 \dots \bar{k}\}$  define  $\bar{m}_k$  the maximal degree of the entries of the matrix polynomial  $F$  and of the  $g_q$  polynomials with respect to the indeterminate  $\theta_k$ . Then, for any sequence  $\{m_k\}_{k=1 \dots \bar{k}}$  such that  $2m_k \geq \bar{m}_k$ , the  $F$  and  $g_q$  polynomials can be expressed as

$$F(\theta) = (\check{\theta}^T \otimes \mathbf{1}_n) \hat{F}(\check{\theta} \otimes \mathbf{1}_n) \quad , \quad g_q(\theta) = \check{\theta}^T \hat{G}_q \check{\theta} \quad (4)$$

where  $\hat{F} \in \mathbb{S}^{n\sigma}$  and  $\hat{G}_q \in \mathbb{S}^\sigma$ . Note that the coefficients matrices  $\hat{F}$  and  $\hat{G}_q$  are non unique, especially if the  $m_k$  are chosen large. Given the sequence  $\{m_k\}_{k=1 \dots \bar{k}}$ , construction of an admissible choice of  $\hat{F}$  and  $\hat{G}_q$  matrices may be tedious but does not make any problem.  $\hat{F}$  and  $\hat{G}_q$  are further assumed to be given.

Slack Variable approach is based on the use of elimination lemma [26] (a variation on Finsler lemma) that states inequality  $B^{\perp*}QB^{\perp} > \mathbf{0}$  is equivalent to the existence of a matrix  $N$  such that  $Q + \langle NB \rangle > \mathbf{0}$ . The key idea for producing LMI results is that "creating" the additional variable  $N$  may transform a polynomial inequality ( $B(\theta)^{\perp}$  is a polynomial with respect to indeterminates  $\theta$ ) into a linear inequality ( $B(\theta)$  is affine with respect to  $\theta$ ). This linearization is usually achieved at the expense of some conservatism because  $N$  is chosen independent of  $\theta$ . Yet, in the present paper the inherent conservatism of the method is reduced thanks to the use of multiple slack variables, one per indeterminate. Notations for this multiple slack variables creation are as follows.

For each  $k \in \{1 \dots \bar{k}\}$ , define the following two constant matrices

$$\eta_k^{[0]} = \begin{bmatrix} \mathbf{0}_{1,m_k} \\ -\mathbf{1}_{m_k} \end{bmatrix}, \quad \eta_k^{[\infty]} = \begin{bmatrix} \mathbf{1}_{m_k} \\ \mathbf{0}_{1,m_k} \end{bmatrix}.$$

The vector  $\check{\theta}_k^{m_k}$  of power series of  $\theta_k$  happens to be a right orthogonal complement of an affine expression of these two matrices:  $\check{\theta}_k^{m_k} = ((\eta_k^{[0]} + \theta_k \eta_k^{[\infty]})^T)^\perp$ . Based on equation (12), one therefore gets that the following equations on the vector of all monomials  $\check{\theta}$  hold

$$(\check{\theta}^T \otimes \mathbf{1}_n) \left( \mathbf{1}_{\sigma_{1,k-1}} \otimes (\eta_k^{[0]} + \theta_k \eta_k^{[\infty]}) \otimes \mathbf{1}_{n\sigma_{k+1,\bar{k}}} \right) = \mathbf{0} \quad (5)$$

whatever  $k = 1 \dots \bar{k}$  and whatever  $\theta \in \mathbb{R}^{\bar{k}}$ . Rather than considering  $\check{\theta}$  as a right orthogonal complement for a matrix dependent of all indeterminates  $\theta_1 \dots \theta_{\bar{k}}$ , the upper formula is used in the paper to create separate slack variables with respect to each indeterminate.

For compactness of formulas define the following matrices

$$\Psi_k^{[0]} = \left( \mathbf{1}_{\sigma_{1,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{n\sigma_{k+1,\bar{k}}} \right), \quad \Psi_k^{[\infty]} = \left( \mathbf{1}_{\sigma_{1,k-1}} \otimes \eta_k^{[\infty]} \otimes \mathbf{1}_{n\sigma_{k+1,\bar{k}}} \right)$$

and  $\Psi_k(\theta_k) = \Psi_k^{[0]} + \theta_k \Psi_k^{[\infty]}$ . With this notations, formula (5) read as  $(\check{\theta}^T \otimes \mathbf{1}_n) \Psi_k(\theta_k) = \mathbf{0}$ . Moreover, in case  $k \in \mathcal{K}_b$ , define  $\Psi_k^{[p]} = \Psi_k(\theta_k^{[p]})$  which is such that  $\Psi_k(\theta_k) = \sum_{p=1}^{\bar{p}} \zeta_p \Psi_k^{[p]}$ .

### III. SLACK VARIABLES RESULTS

#### A. Bounded case ( $\bar{k}_b = \bar{k}$ )

The first result is for the case of bounded indeterminates in a polytope. This case corresponds to the usual framework in which SV approach has been developed [3], [15], [5]. Yet, at the difference of these references, the Theorem generates several slack variables ( $N_k$ ), rather than just one. This modification, that increases the numerical complexity, provides new degrees of freedom useful in the following to prove new properties.

*Theorem 1:* Assume  $\bar{k}_b = \bar{k}$ , all indeterminates are bounded in a given polytope. Property (1) holds if there exist  $\bar{q}\bar{p}$  symmetric matrices  $H_q^{[p]} \in \mathcal{S}^n$  and  $\bar{k}$  matrices  $N_k \in \mathbb{R}^{n\pi_k \times n\sigma}$  such that for all  $(p, q) \in \{1 \dots \bar{p}\} \times \{1 \dots \bar{q}\}$  the following LMI conditions are feasible

$$\hat{F} + \left\langle \sum_{k=1}^{\bar{k}} \Psi_k^{[p]} N_k \right\rangle \geq \sum_{q=1}^{\bar{q}} \hat{G}_q \otimes H_q^{[p]}, \quad H_q^{[p]} \geq \mathbf{0}_n \quad (6)$$

*Proof:* Define  $\theta \in \mathcal{P}$  any element of the polytope parameterized by the  $\bar{p}$  scalars  $\zeta_p$  as in (3). Let  $H_q(\theta) = \sum_{p=1}^{\bar{p}} \zeta_p H_q^{[p]}$ . The inequalities (6) are linear with respect to the vertices  $\theta^{[p]}$  and  $H_q^{[p]}$  therefore by convexity they hold as well for all values  $\theta$  in the polytope  $\mathcal{P}$ :

$$\hat{F} + \left\langle \sum_{k=1}^{\bar{k}} \Psi_k(\theta_k) N_k \right\rangle \geq \sum_{q=1}^{\bar{q}} \hat{G}_q \otimes H_q(\theta) \quad , \quad H_q(\theta) \geq \mathbf{0}_n \quad . \quad (7)$$

Pre and post-multiplying inequalities (7) by  $\check{\theta}^T \otimes \mathbf{1}_n$  and its transpose respectively one gets due to formulas (5):

$$\forall \theta \in \mathcal{P} \quad : \quad F(\theta) \geq \sum_{q=1}^{\bar{q}} g_q(\theta) H_q(\theta) \quad , \quad H_q(\theta) \geq \mathbf{0}_n \quad .$$

Hence  $F(\theta) \geq \mathbf{0}_n$  for all  $\theta \in \mathcal{G} \cap \mathcal{P}$  (constraints  $g_q(\theta) \geq 0$  and  $\theta \in \mathcal{P}$ ). ■

### B. Unbounded case ( $\bar{k}_b = 0$ )

The unbounded case is now considered. It is to our knowledge the first time such SV result is produced without constraining indeterminates inside polytopes. As suggested by the notations it can be interpreted as taking vertices at infinity.

*Theorem 2:* Assume  $\bar{k}_b = 0$ , no bound is a priori known on the indeterminates. Property (1) holds if there exist  $\bar{q}$  symmetric matrices  $H_q \in \mathbb{S}^n$  and  $\bar{k}$  skew-symmetric matrices  $\hat{N}_k \in \mathbb{Q}^{n \times n}$  such that for all  $q \in \{1 \dots \bar{q}\}$  the following LMI conditions are feasible

$$\hat{F} + \left\langle \sum_{k=1}^{\bar{k}} \Psi_k^{[0]} \hat{N}_k \Psi_k^{[\infty]T} \right\rangle \geq \sum_{q=1}^{\bar{q}} \hat{G}_q \otimes H_q \quad , \quad H_q \geq \mathbf{0}_n \quad (8)$$

*Proof:* Consider the following quantity

$$\left\langle (\Psi_k(\theta_k) \hat{N}_k \Psi_k^{[\infty]T}) \right\rangle = \left\langle \Psi_k^{[0]} \hat{N}_k \Psi_k^{[\infty]T} \right\rangle + \theta_k \Psi_k^{[\infty]} \left\langle \hat{N}_k \right\rangle \Psi_k^{[\infty]T} \quad . \quad (9)$$

The  $\hat{N}_k$  matrices being skew-symmetric the last term is zero. The expression is therefore independent of  $\theta$  and hence for all  $\theta \in \mathbb{R}^{\bar{k}}$  and one has  $\left\langle \Psi_k^{[0]} \hat{N}_k \Psi_k^{[\infty]T} \right\rangle = \left\langle (\Psi_k(\theta_k) \hat{N}_k \Psi_k^{[\infty]T}) \right\rangle$ . The matrices  $N_k = \hat{N}_k \Psi_k^{[\infty]T}$  and  $H_q(\theta) = H_q$  therefore satisfy conditions (7) for all  $\theta \in \mathbb{R}^{\bar{k}}$ . The remaining follows the lines of Theorem's 1 proof. ■

One major issue of the paper is to give a comparison between SV and SOS approaches. For the unbounded case the two happen to be exactly the same as stated by the next Theorem. Therefore

it implies that all results known in the SOS context may be generalized to the SV approach. In particular, when  $\mathcal{G} \cap \mathcal{P} = \mathbb{R}^{\bar{k}}$  is unconstrained, exactness of the LMI tests is guaranteed for the following cases:  $\bar{k} = 1$  (single indeterminate polynomials);  $m_k \leq 1$  (quadratic forms);  $\bar{k} = 2$  and  $m_k \leq 2$  (quartic forms of two indeterminates). Moreover asymptotic convergence of LMI conditions to exact tests for problem (1) as the degree of the basis of monomials grows (under particular assumptions on the constraints, see [24]) keeps true for the SV approach. To be more exact, to obtain the exact equivalence of the SV approach with results for the SOS approach in [24], it is needed to extend the upper results to  $H_q$  matrix SOS polynomials. This may be done without any theoretical obstacle but needs involved mathematical manipulations (see for example papers [21], [18]). We have limited our study to the case of zero degree  $H_q$  to limit this mathematical manipulation complexity.

*Theorem 3:* The LMI conditions (8) hold if and only if the matrix-valued polynomial  $F(\theta) - \sum_{q=1}^{\bar{q}} g_q(\theta)H_q$  is sum-of-squares.

To prove the Theorem, recall (see [24]) that a polynomial  $S(\theta) = (\check{\theta}^T \otimes \mathbf{1}_n)\hat{S}(\check{\theta} \otimes \mathbf{1}_n)$  is sum-of-squares if and only if  $\hat{S} + \hat{L}$  is positive semi-definite for some matrix  $\hat{L}$  such that  $(\check{\theta}^T \otimes \mathbf{1}_n)\hat{L}(\check{\theta} \otimes \mathbf{1}_n)$  is identically zero for all  $\theta \in \mathbb{R}^{\bar{k}}$ . Theorem 3 needs therefore to prove that all such matrices  $\hat{L}$  have a multiple Slack Variables formulation ( $\hat{L} = \left\langle \sum_{k=1}^{\bar{k}} \Psi_k^{[0]} \hat{N}_k \Psi_k^{[\infty]T} \right\rangle$  for some skew-symmetric matrices  $\hat{N}_k$ ). This corresponds to the following lemma (with  $\alpha = 1$  and  $\beta = n$ ).

*Lemma 1:* Let  $\alpha$  and  $\beta$  be two integers. The multi-indeterminate matrix polynomial  $L(\theta) = (\mathbf{1}_\alpha \otimes \check{\theta}^T \otimes \mathbf{1}_\beta)\hat{L}(\mathbf{1}_\alpha \otimes \check{\theta} \otimes \mathbf{1}_\beta)$  is identically zero for all  $\theta \in \mathbb{R}^{\bar{k}}$  if and only if there exists  $\bar{k}$  skew-symmetric matrices  $\hat{N}_k$  such that

$$\hat{L} = \left\langle \sum_{k=1}^{\bar{k}} (\mathbf{1}_{\alpha\sigma_{1,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{\beta\sigma_{k+1,\bar{k}}}) \hat{N}_k (\mathbf{1}_{\alpha\sigma_{1,k-1}} \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_{\beta\sigma_{k+1,\bar{k}}}) \right\rangle. \quad (10)$$

*Proof:* The 'if' part is similar to the proof of Theorem 2 and is therefore omitted. The 'only if' part of the proof needs many more derivations and is therefore given in appendix at the end of the paper. ■

Lemma 1 can also be seen as a simple to code formula for generating the matrix  $\hat{L}$  of the SOS approach. Unfortunately the drawback is that the number of decision variables contained

in the  $\hat{N}_k$  matrices is larger than those needed in SOS. It is thus a simple to code but over-parameterized formula. To reduce the numerical complexity it is also possible to structure some more the  $\hat{N}_k$  matrices following the ideas in [22], [21]. The reason for these structures can be found in the details of the proof given in appendix.

### C. Mixed bounded/unbounded case

*Theorem 4:* Property (1) holds if there exist  $\bar{q}\bar{p}$  symmetric matrices  $H_q^{[p]} \in \mathbb{S}^n$ ,  $\bar{k}_b$  matrices  $N_k \in \mathbb{R}^{n\pi_k \times n\sigma}$  without any structure and  $(\bar{k} - \bar{k}_b)$  skew-symmetric matrices  $\hat{N}_k \in \mathbb{Q}^{n\pi_k}$ , such that for all  $(p, q) \in \{1 \dots \bar{p}\} \times \{1 \dots \bar{q}\}$  the following LMI conditions are feasible

$$\hat{F} + \left\langle \sum_{k \notin \mathcal{K}_b} \Psi_k^{[0]} \hat{N}_k \Psi_k^{[\infty]T} + \sum_{k \in \mathcal{K}_b} \Psi_k^{[p]} N_k \right\rangle \geq \sum_{q=1}^{\bar{q}} \hat{G}_q \otimes H_q^{[p]} \quad , \quad H_q^{[p]} \geq \mathbf{0}_n \quad (11)$$

Theorem 4 is a trivial combination of both previous result. It allows to reduce conservatism of the unbounded case if some of the indeterminates are known to be constrained in bounded sets as stated in the following corollary.

*Corollary 1:* If the set  $\mathcal{G}$  is known to be bounded or partially bounded with respect to some indeterminates  $\theta_{k \in \mathcal{K}_b}$ , then, let any over-bounding semi-polytopic set,  $\mathcal{P} \supset \mathcal{G}$ , the LMIs (11) are feasible if (8) has a solution.

This corollary indicates that if the constraints  $\mathcal{G}$  define a bounded (or partially bounded) region, then the SV result is less conservative than the SOS result (assuming identical representations  $\hat{F}$ ,  $\hat{G}_q$  of the polynomials). As said previously (just before Theorem 3) this comparison is here done for the case when one restricts the variables  $H_q$  to be indeterminate independent. These can also be chosen as polynomials (of degree such that  $g_q(\theta)H_q(\theta)$  are of degrees  $2m_k$  with respect to each indeterminate  $\theta_k$ ) in which case the comparison between SV and SOS holds as well for the same reasons. Still, conservatism reduction provided by the slack variables for bounded indeterminates is at the expense of increasing drastically the number of variables and the size of the LMIs (by a factor  $\bar{p}$ ). Not to make this factor too large, one may consider cross-polytopes with  $\bar{p} = 2\bar{k}$  vertices defined as  $\theta^{[p]} = \left( 0 \quad \dots \quad 0 \quad \pm\rho \quad 0 \quad \dots \quad 0 \right)^T$  with  $\rho$  sufficiently large.

Numerical examples with comparisons of the SV and SOS methods can be found in [22], [21]. They illustrate both the efficiency of the contribution and its drawbacks in terms of numerical

complexity. For space reasons these numerical examples are not reproduced in the present paper.

#### IV. CONCLUSIONS

Aiming at the origin to compare Slack Variables (SV) and Sum-Of-Squares (SOS) approaches, we finally obtain in this paper both, an easy to code formulation of SOS results, and a generalization of the SV approach for non-bounded indeterminates. SV results moreover prove to be less conservative than SOS in case of bounded indeterminates, yet more demanding in terms of numerical burden. As attested by the numerous papers adopting SV approach (sometimes implicitly without direct references to the method), this technique is quite popular for robustness analysis in case of polytopic uncertainties but up to now seemed not to have extensions for other types of uncertainties. The exposed results allow these extensions where, as suggested by the notations, vertices of the polytope are replaced by matrices describing the parameters at the origin and at infinity.

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## APPENDIX

This appendix is dedicated to the proof of Lemma 1. For that, consider the following notations and preliminary technical results.

Define  $\theta_{[k_1, k_2]} = \left( \theta_{k_1} \ \dots \ \theta_{k_2} \right)^T$  the vector of indeterminates with indices  $k$  ranging from  $k_1$  to  $k_2$ . As previously  $\check{\theta}_k^{m_k} = \left( 1 \ \theta_k \ \dots \ \theta_k^{m_k} \right)^T \in \mathbb{R}^{m_k+1}$  is the vector of powers of each indeterminate  $\theta_k$  up to power  $m_k$ . Let  $\check{\theta}_{[k_1, k_2]} = \check{\theta}_{k_1}^{m_{k_1}} \otimes \check{\theta}_{k_1+1}^{m_{k_1+1}} \dots \otimes \check{\theta}_{k_2}^{m_{k_2}}$  be the vectors of all monomials obtained as products of elements  $\theta_k^j$  with indeterminates indices  $k$  ranging from  $k_1$  to  $k_2$  and powers going up to  $m_k$ . The vectors  $\check{\theta}_{[k_1, k_2]}$  have exactly  $\sigma_{k_1, k_2} = \prod_{k=k_1}^{k_2} (m_k + 1)$  elements. Recall that by definition  $\sigma_{1,0} = 1$  and  $\sigma_{\bar{k}+1, \bar{k}} = 1$ . Recall as well the notation  $\pi_k = \sigma m_k / (m_k + 1)$ . For compactness of formulas, we have adopted the following notation in the main corpus of the paper  $\theta = \theta_{[1, \bar{k}]}$ ,  $\check{\theta} = \check{\theta}_{[1, \bar{k}]}$  and  $\sigma = \sigma_{1, \bar{k}}$ . The notations are such that whatever  $k \in \{1 \dots \bar{k}\}$  one gets

$$\begin{aligned} \check{\theta} &= \check{\theta}_{[1, k-1]} \otimes \check{\theta}_k^{m_k} \otimes \check{\theta}_{[k+1, \bar{k}]} = (\check{\theta}_{[1, k-1]} \otimes \mathbf{1}_{\sigma_{k,k}} \otimes \check{\theta}_{[k+1, \bar{k}]}) (1 \otimes \check{\theta}_k^{m_k} \otimes 1) \\ &= (\mathbf{1}_{\sigma_{1, k-1}} \otimes \check{\theta}_k^{m_k} \otimes \mathbf{1}_{\sigma_{k+1, \bar{k}}}) (\check{\theta}_{[1, k-1]} \otimes 1 \otimes \check{\theta}_{[k+1, \bar{k}]}). \end{aligned} \quad (12)$$

This type of factorization of one among all indeterminates is of major importance for deriving the results.

*Lemma 2:* Let  $\alpha$  and  $\beta$  two integers and assume  $\hat{L}(\theta)$  is a matrix polynomial of several scalar indeterminates. If the following two conditions hold

$$(\mathbf{1}_\alpha \otimes e_1^T \otimes \mathbf{1}_\beta) \hat{L}(\theta) (\mathbf{1}_\alpha \otimes e_1 \otimes \mathbf{1}_\beta) = \mathbf{0} \quad , \quad (\mathbf{1}_\alpha \otimes e_{l+1}^T \otimes \mathbf{1}_\beta) \hat{L}(\theta) (\mathbf{1}_\alpha \otimes e_{l+1} \otimes \mathbf{1}_\beta) = \mathbf{0} \quad (13)$$

where  $e_1 = \left( 1 \ 0 \ \dots \ 0 \right)^T \in \mathbb{R}^{l+1}$  and  $e_{l+1} = \left( 0 \ \dots \ 0 \ 1 \right)^T \in \mathbb{R}^{l+1}$ , then there exists a polynomial matrix  $M(\theta)$  of same degree as  $\hat{L}(\theta)$  such that

$$\hat{L}(\theta) = \left\langle (\mathbf{1}_\alpha \otimes \begin{bmatrix} \mathbf{0}_{1,l} \\ -\mathbf{1}_l \end{bmatrix} \otimes \mathbf{1}_\beta) M(\theta) (\mathbf{1}_\alpha \otimes \begin{bmatrix} \mathbf{1}_l & \mathbf{0}_{l,1} \end{bmatrix} \otimes \mathbf{1}_\beta) \right\rangle$$

*Proof:* The proof is direct application of Theorem 2.3.7 in [26]. Details are omitted for space limitation reasons. ■

*Lemma 3:* Let  $\alpha$  and  $\beta$  two integers and the mono-indeterminate matrix polynomial of order  $2m_k$ :  $L(\theta_k) = (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k T} \otimes \mathbf{1}_\beta) \hat{L}(\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k} \otimes \mathbf{1}_\beta)$ . The polynomial  $L(\theta_k)$  is identically zero for

all  $\theta_k \in \mathbb{R}$ , if and only if there exists a skew-symmetric matrix  $M_k$  such that

$$\hat{L} = \left\langle (\mathbf{1}_\alpha \otimes \eta_k^{[0]} \otimes \mathbf{1}_\beta) M_k (\mathbf{1}_\alpha \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_\beta) \right\rangle .$$

*Proof:* The 'if' part of the proof is omitted because similar to that of Theorem 2. The 'only if' part is now produced. For the time of the proof define the following matrices

$$\Upsilon_l^{[0]} = \mathbf{1}_\alpha \otimes \begin{bmatrix} \mathbf{0}_{1,l} \\ -\mathbf{1}_l \end{bmatrix} \otimes \mathbf{1}_\beta \quad , \quad \Upsilon_l^{[\infty]} = \mathbf{1}_\alpha \otimes \begin{bmatrix} \mathbf{1}_l \\ \mathbf{0}_{l,1} \end{bmatrix} \otimes \mathbf{1}_\beta$$

which coincide with  $\mathbf{1}_\alpha \otimes \eta_k^{[0]} \otimes \mathbf{1}_\beta$  and  $\mathbf{1}_\alpha \otimes \eta_k^{[\infty]} \otimes \mathbf{1}_\beta$  respectively when  $l = m_k$ .

The lowest degree and the highest degree terms of  $L$  should be zeros which writes

$$(\mathbf{1}_\alpha \otimes e_1^T \otimes \mathbf{1}_\beta) \hat{L} (\mathbf{1}_\alpha \otimes e_1 \otimes \mathbf{1}_\beta) = \mathbf{0} \quad , \quad (\mathbf{1}_\alpha \otimes e_{m_k+1}^T \otimes \mathbf{1}_\beta) \hat{L} (\mathbf{1}_\alpha \otimes e_{m_k+1} \otimes \mathbf{1}_\beta) = \mathbf{0}$$

which due to Lemma 2 implies the existence of a matrix  $M$  such that  $\hat{L} = \left\langle \Upsilon_{m_k}^{[0]} M \Upsilon_{m_k}^{[\infty]T} \right\rangle$ . Decompose  $M = M^{skew} + M^{sym}$  in the skew-symmetric and symmetric parts. The structures of  $\Upsilon_{m_k}^{[0]}$  and  $\Upsilon_{m_k}^{[\infty]}$  are such that

$$(\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k T} \otimes \mathbf{1}_\beta) \Upsilon_{m_k}^{[0]} = -\theta_k (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k-1 T} \otimes \mathbf{1}_\beta) \quad , \quad \Upsilon_{m_k}^{[\infty]T} (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k} \otimes \mathbf{1}_\beta) = \mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k-1} \otimes \mathbf{1}_\beta$$

therefore, the  $2m_k$  order polynomial  $L$  being identically zero implies

$$\begin{aligned} L(\theta_k) &= (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k T} \otimes \mathbf{1}_\beta) \hat{L} (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k} \otimes \mathbf{1}_\beta) \\ &= -2\theta_k \underbrace{(\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k-1 T} \otimes \mathbf{1}_\beta) M^{sym} (\mathbf{1}_\alpha \otimes \check{\theta}_k^{m_k-1} \otimes \mathbf{1}_\beta)}_{N(\theta_k)} = \mathbf{0} \end{aligned}$$

for all  $\theta_k \in \mathbb{R}$ , *i.e.* the polynomial  $N(\theta_k)$  is identically zero. The remaining of the proof is done by induction on  $m_k$  the maximal degree in the vector  $\check{\theta}_k^{m_k} = \left( 1 \quad \theta_k \quad \dots \quad \theta_k^{m_k} \right)^T$ .

- Start with  $m_k = 1$ . Then  $\check{\theta}_k^{m_k-1} = 1$  and  $N(\theta_k) = M^{sym} = \mathbf{0}$ . Hence it is proved that  $\hat{L} = \left\langle \Upsilon_1^{[0]} M_k \Upsilon_1^{[\infty]T} \right\rangle$  with  $M_k = M^{skew}$  skew-symmetric.
- Assume Lemma 3 holds for  $m_k - 1$  and apply it to the polynomial  $N$  (which is of degree  $2(m_k - 1)$ ). It implies there exists a skew-symmetric matrix  $P$  such that  $M^{sym} = \left\langle \Upsilon_{m_k-1}^{[0]} P \Upsilon_{m_k-1}^{[\infty]T} \right\rangle$ . At this stage of the proof one gets that

$$\hat{L} = \left\langle \Upsilon_{m_k}^{[0]} M^{skew} \Upsilon_{m_k}^{[\infty]T} \right\rangle + \left\langle \Upsilon_{m_k}^{[0]} \left\langle \Upsilon_{m_k-1}^{[0]} P \Upsilon_{m_k-1}^{[\infty]T} \right\rangle \Upsilon_{m_k}^{[\infty]T} \right\rangle$$

with  $M^{skew}$  and  $P$  skew-symmetric. Simple manipulations show that  $\Upsilon_{m_k}^{[0]} \Upsilon_{m_k-1}^{[\infty]} = \Upsilon_{m_k}^{[\infty]} \Upsilon_{m_k-1}^{[0]}$ ,

therefore one gets after developing the expression and re-factorizing it properly that  $\hat{L} =$

$\langle \Upsilon_{m_k}^{[0]} M_k \Upsilon_{m_k}^{[\infty]T} \rangle$  where  $M_k = M^{skew} + \Upsilon_{m_k-1}^{[0]} P \Upsilon_{m_k-1}^{[\infty]T} + \Upsilon_{m_k-1}^{[\infty]} P \Upsilon_{m_k-1}^{[0]T}$ . This matrix happens to be skew-symmetric:

$$\langle M_k \rangle = \langle M^{skew} \rangle + \Upsilon_{m_k-1}^{[0]} \langle P \rangle \Upsilon_{m_k-1}^{[\infty]T} + \Upsilon_{m_k-1}^{[\infty]} \langle P \rangle \Upsilon_{m_k-1}^{[0]T} = \mathbf{0}$$

since both matrices  $M^{skew}$  and  $P$  are skew-symmetric ( $\langle M^{skew} \rangle = \mathbf{0}$  and  $\langle P \rangle = \mathbf{0}$ ). ■

Due to Lemma 2, if  $\hat{L}$  is a polynomial matrix of some other indeterminates,  $M$  may be chosen to be a polynomial with same degree with respect to these indeterminates. This fact is used in the following.

*Proof of Lemma 1:* Demonstration is done by induction on the number of indeterminates.

- For  $\bar{k} = 1$ , Lemma 1 holds because is the same as Lemma 3.
- Assume Lemma 1 holds for polynomials with  $\bar{k}-1$  indeterminates and consider a polynomial  $L$  of  $\bar{k}$  indeterminates assumed to be identically zero. Perform the following factorization

$$\begin{aligned} L(\theta) &= (\mathbf{1}_\alpha \otimes \check{\theta}_{[1,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) \hat{L}(\mathbf{1}_\alpha \otimes \check{\theta}_{[1,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) \\ &= (\mathbf{1}_\alpha \otimes \check{\theta}_1^{m_k T} \otimes \mathbf{1}_\beta) L_1(\theta_{[2,\bar{k}]}) (\mathbf{1}_\alpha \otimes \check{\theta}_1^{m_k} \otimes \mathbf{1}_\beta) \end{aligned}$$

where  $L_1$  is the following polynomial of  $\bar{k} - 1$  indeterminates

$$L_1(\theta_{[2,\bar{k}]}) = (\mathbf{1}_{\alpha\sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) \hat{L}(\mathbf{1}_{\alpha\sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta). \quad (14)$$

Apply Lemma 3 to the polynomial  $L$  viewed as a polynomial of the first indeterminate, it implies existence of a skew-symmetric polynomial matrix  $M_1$  of order  $2m_k$  with respect to each other indeterminates  $\theta_k$  ( $k = 2 \dots \bar{k}$ ) such that:

$$L_1(\theta_{[2,\bar{k}]}) = \left\langle (\mathbf{1}_\alpha \otimes \eta_1^{[0]} \otimes \mathbf{1}_\beta) M_1(\theta_{[2,\bar{k}]}) (\mathbf{1}_\alpha \otimes \eta_1^{[\infty]T} \otimes \mathbf{1}_\beta) \right\rangle. \quad (15)$$

The polynomial  $M_1$  admits a representation in the basis of monomials as

$$M_1(\theta_{[2,\bar{k}]}) = (\mathbf{1}_{\alpha m_1} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) \hat{M}_1(\mathbf{1}_{\alpha m_1} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta). \quad (16)$$

Decompose  $\hat{M}_1 = \hat{M}_1^{skew} + \hat{M}_1^{sym}$  in its skew-symmetric and symmetric parts. The constraint that  $M_1(\theta_{[2,\bar{k}]})$  is skew-symmetric implies

$$\langle M_1(\theta_{[2,\bar{k}]}) \rangle = \mathbf{0} + 2(\mathbf{1}_{\alpha m_1} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) \hat{M}_1^{sym}(\mathbf{1}_{\alpha m_1} \otimes \check{\theta}_{[2,\bar{k}]^T}^T \otimes \mathbf{1}_\beta) = \mathbf{0}.$$

$\hat{M}_1^{sym}$  therefore defines a polynomial of  $\bar{k} - 1$  indeterminates which is identically zero. Lemma 1 is assumed to hold for polynomials of  $\bar{k} - 1$  indeterminates, thus

$$\hat{M}_1^{sym} = \left\langle \sum_{k=2}^{\bar{k}} (\mathbf{1}_{\alpha m_1 \sigma_{2,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \hat{N}_{1k} (\mathbf{1}_{\alpha m_1 \sigma_{2,k-1}} \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \right\rangle.$$

with skew-symmetric matrices  $\hat{N}_{1k}$ . Simple manipulations show that whatever  $k = 2 \dots \bar{k}$  and whatever  $(i, j) \in \{0, \infty\}^2$  the following formulas hold

$$(\mathbf{1}_\alpha \otimes \eta_1^{[i]} \otimes \mathbf{1}_\beta) (\mathbf{1}_{\alpha m_1 \sigma_{2,k-1}} \otimes \eta_k^{[j]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) = (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[j]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) (\mathbf{1}_\alpha \otimes \eta_1^{[i]} \otimes \mathbf{1}_{\tilde{\beta}_k})$$

where  $\tilde{\beta}_k = \beta \sigma_{2,\bar{k}} m_k / (m_k + 1)$ . Therefore with a quite similar operations as in the proof of Lemma 3, one gets that

$$\begin{aligned} & \left\langle (\mathbf{1}_\alpha \otimes \eta_1^{[0]} \otimes \mathbf{1}_\beta) \hat{M}_1^{sym} (\mathbf{1}_\alpha \otimes \eta_1^{[\infty]T} \otimes \mathbf{1}_\beta) \right\rangle \\ &= \left\langle \sum_{k=2}^{\bar{k}} (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \tilde{N}_k (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \right\rangle \end{aligned} \quad (17)$$

where  $\tilde{N}_k =$

$$(\mathbf{1}_\alpha \otimes \eta_1^{[0]} \otimes \mathbf{1}_{\tilde{\beta}_k}) \hat{N}_{1,k} (\mathbf{1}_\alpha \otimes \eta_1^{[\infty]T} \otimes \mathbf{1}_{\tilde{\beta}_k}) + (\mathbf{1}_\alpha \otimes \eta_1^{[\infty]T} \otimes \mathbf{1}_{\tilde{\beta}_k}) \hat{N}_{1,k} (\mathbf{1}_\alpha \otimes \eta_1^{[0]} \otimes \mathbf{1}_{\tilde{\beta}_k}).$$

The matrices  $\tilde{N}_k$  happen to be skew-symmetric because  $\hat{N}_{1,k}$  are skew-symmetric. Denote  $\tilde{N}_1 = \hat{M}_1^{skew}$ . At this stage, combining equations (15), (16) and (17) gives

$$L_1(\theta_{[2,\bar{k}]}) = (\mathbf{1}_{\alpha \sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]^T} \otimes \mathbf{1}_\beta) \hat{L}_1 (\mathbf{1}_{\alpha \sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]} \otimes \mathbf{1}_\beta) \quad (18)$$

where  $\hat{L}_1 = \left\langle \sum_{k=1}^{\bar{k}} (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \tilde{N}_k (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \right\rangle$  with all matrices  $\tilde{N}_k$  being skew-symmetric. Subtracting expression (18) to (14) gives

$$(\mathbf{1}_{\alpha \sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]^T} \otimes \mathbf{1}_\beta) (\hat{L} - \hat{L}_1) (\mathbf{1}_{\alpha \sigma_{1,1}} \otimes \check{\theta}_{[2,\bar{k}]} \otimes \mathbf{1}_\beta) = \mathbf{0}$$

for all  $\theta_{[2,\bar{k}]} \in \mathbb{R}^{\bar{k}-1}$ . This new polynomial matrix of  $\bar{k} - 1$  indeterminates is identically zero.

Lemma 3 is assumed to hold for such polynomials of  $\bar{k} - 1$  indeterminates, therefore there exist  $\bar{k} - 1$  skew-symmetric matrices  $\hat{N}_{2,k}$  ( $k = 2 \dots \bar{k}$ ) such that

$$\hat{L} - \hat{L}_1 = \left\langle \sum_{k=2}^{\bar{k}} (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[0]} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \hat{N}_{2,k} (\mathbf{1}_{\alpha \sigma_{1,k-1}} \otimes \eta_k^{[\infty]T} \otimes \mathbf{1}_{\beta \sigma_{k+1,\bar{k}}}) \right\rangle. \quad (19)$$

Combining (19) and the formula for  $\hat{L}_1$  proves the result where  $\hat{N}_1 = \tilde{N}_1 = \hat{M}_1^{skew}$  and  $\hat{N}_k = \hat{N}_{2,k} + \tilde{N}_k$  for  $k = 2 \dots \bar{k}$ . ■