Robust $\mathcal{H}_2$ performance analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs

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Abstract

A particular class of uncertain linear discrete-time periodic systems is considered. The problem of robust stabilization of real polytopic linear discrete-time periodic systems via a periodic state-feedback control law is tackled here, along with $\mathcal{H}_2$ performance optimization. Using additional slack variables and the periodic Lyapunov lemma, an extended sufficient condition of robust $\mathcal{H}_2$ stabilization is proposed. Based on periodic parameter-dependent Lyapunov functions, this last condition is shown to be always less conservative than the more classic one based on the quadratic stability framework. This is illustrated on numerical examples from the literature.

Key words: Linear periodic discrete-time systems, Polytopic uncertainty, Robust stability, Quadratic stability, Parameter-dependent Lyapunov functions, $\mathcal{H}_2$ performance, LMI

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Preprint submitted to Elsevier Science *14 September 2005*

1 Introduction

One century after the pioneering works on periodic differential equations with periodic coefficients [23, 16, 19], the eighties have witnessed a renewed interest in the modelling, analysis and control of periodic systems (see [10] and the references therein). This interest is mainly due to the variety and the originality of the possible applications of a control theory dedicated to this class of systems. One can first recall the classic examples of control of vibrations in helicopters [7] as well as the attitude control of satellites equipped with magnetorquers [22, 31].

Another interesting and original application of linear periodic systems control theory concerns autonomous orbit control. During the last years, the problem of autonomous orbit control for spacecraft has been largely addressed for various applications ranging from geostationary station keeping to formation flying earth orbiters. For circular orbits, the synthesis problems (compute an adequate control law) are generally tackled via the use of Hill's equations leading to a complete linear time-invariant formulation. On the contrary, for elliptical orbits, the discrete-time approximation of the linearized equations of relative motion of a spacecraft in the orbit plane yields discrete-time linear periodic model. In [26], a time-invariant reformulation [5] is utilized to design a discrete-time optimal periodic controller. Those results are completed in [27] by considering pertubations resulting from atmospheric drag.

When dealing with such intrinsically periodic systems, it is natural to wonder if the well-established analysis and synthesis framework for linear-time invariant systems may be extended to this peculiar class of time-varying models. Indeed, time-invariant reformulations for discrete-time systems [5] paved the way for the development of a broad variety of tools (periodic Lyapunov and Riccati equations) directly extrapolated from the LTI set-up. Structural properties, stability analysis as well as the most popular synthesis techniques (pole placement, Linear Quadratic optimal control) were therefore extended to discrete-time and continuous-time periodic systems [8, 6, 9, 31]. Surprisingly, few results exist that extend the robust control framework to periodic systems [18], [13], [20].

In this paper, a particular class of uncertain linear discrete-time periodic systems, similar to the one presented in [13], is considered. The contribution of this study is the design of robustly stablizing state-feedback controllers for this class of systems and focuses on $\mathcal{H}_2$ performance optimization. As presented in [2], the $\mathcal{H}_2$ performance of a periodic system is measured through the generalized $\mathcal{H}_2$ norm of the transfer between exogeneous inputs and outputs. This norm characterizes the sensitivity of the system with respect to perturbations (gaussian noise, impulses) on the input signals. An $\mathcal{LMI}$ formulation for the computation of this norm has been proposed in [30] for the case of certain systems. Following a classical methodology in the robust LTI context, a quadratic approach is first proposed to extend results of [30] to the case of uncertain polytopic systems. Then, strongly related to results presented in [11], a new extended framework based on periodic parameter-dependent Lyapunov functions, is presented. Using additional variables as proposed by [17, 12, 24], extended sufficient conditions for robust stabilization with $\mathcal{H}_2$ performance optimization are given. Those are always less conservative than the ones developed in the quadratic context and improve the performance of the controller. All stabilization methods presented in this paper are $\mathcal{LMI}$ formulated and can therefore be solved in polynomial time with Semi-Definite Programming (*SDP*) solvers. A numerical example illustrates the relevance of these new conditions and discusses their efficiency.

Notations: The transpose of a matrix A is denoted A'. For symmetric matrices, $>$ $(\geq)$ denotes the Löwner partial order, i.e. $A > (\geq)\ B$ iff $A - B$ is positive (semi) definite. $\mathbf{1}$ stands for the identity matrix and $\mathbf{0}$ for the zero matrix with the appropriate dimensions. $\mathbb{S}^n$ denotes the set of symmetric matrices of $\mathbb{R}^{n\times n}$. $\mathbb{N}$ is the set of natural integers. The symmetric part of a square matrix A is denoted $\langle A\rangle$, i.e. $\langle A\rangle = A + A'$. $\mathrm{co}\left\{A^{[1]}, \cdots, A^{[N]}\right\}$ is the convex hull of the collection of N vertices $A^{[1]}, \cdots, A^{[N]}$ such that its elements are parameterized by λ such that

$$A(\lambda) = \sum_{i=1}^{N} \lambda_i A^{[i]}\ ,\ \lambda \in \Lambda = \left\{\lambda \in \mathbb{R}^N :\ \lambda_i \geq 0,\ \sum_{i=1}^{N} \lambda_i = 1\right\}$$

$\{A_k\}_{k\in\mathbb{N}}$ is a sequence whose elements at time k are A_k. Such sequence is said T-periodic if for all $k \geq 0$ one has $A_{k+T} = A_k$. $\Delta_k = \Delta(t-k)$ denotes the shifted unit pulse applied at some time k. Δ_k^i is the vector signal Δ_k which has a shifted unit pulse in the i-th position and zeros in the other positions. $\Sigma[u]$ denotes the output of system Σ for a given input u.

2 Problem Statement

Let the linear uncertain discrete-time time-varying system $\Sigma(\lambda)$ defined by the following state-space realization:

$$\begin{pmatrix} x_{k+1} \\ z_k \end{pmatrix} = \begin{bmatrix} A_k(\lambda) & B_{wk}(\lambda) & B_{uk}(\lambda) \\ C_{zk}(\lambda) & D_{zwk}(\lambda) & D_{zuk}(\lambda) \end{bmatrix} \begin{pmatrix} x_k \\ w_k \\ u_k \end{pmatrix} = M_k(\lambda) \begin{pmatrix} x_k \\ w_k \\ u_k \end{pmatrix} \qquad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control vector, $w_k \in \mathbb{R}^{m_w}$ is the disturbance vector, $z_k \in \mathbb{R}^{p_z}$ is the controlled output vector and $\lambda \in \Lambda$

is a vector of parametric uncertainties. For each k, the parameter-dependent system matrix $M_k(\lambda)$ belongs to the convex polytope $\mathcal{M}_k$ defined by:

$$\mathcal{M}_k = \mathrm{co}\left\{M_k^{[1]}, \cdots, M_k^{[N]}\right\} \quad , \quad M_k^{[i]} = \begin{bmatrix} A_k^{[i]} & B_{wk}^{[i]} & B_{uk}^{[i]} \\ C_{zk}^{[i]} & D_{zwk}^{[i]} & D_{zuk}^{[i]} \end{bmatrix}$$

The sequence of polytopes $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ is assumed to be T-periodic. The resulting uncertain system is said to be polytopic T-periodic.

The paper is devoted to giving constructive conditions for robust stabilizing memoryless T-periodic state-feedback of the form $u_k = K_k x_k$ that minimizes the worst-case $\mathcal{H}_2$ norm of the uncertain closed-loop system $\Sigma_{cl}(\lambda)$:

$$\begin{aligned} \begin{pmatrix} x_{k+1} \\ z_k \end{pmatrix} &= \begin{bmatrix} A_k(\lambda) + B_{uk}(\lambda)K_k & B_{wk}(\lambda) \\ C_{zk}(\lambda) + D_{zuk}(\lambda)K_k & D_{zwk}(\lambda) \end{bmatrix} \begin{pmatrix} x_k \\ w_k \end{pmatrix} \\ &= \begin{bmatrix} A_k^{cl}(K_k,\lambda) & B_{wk}(\lambda) \\ C_{zk}^{cl}(K_k,\lambda) & D_{zwk}(\lambda) \end{bmatrix} \begin{pmatrix} x_k \\ w_k \end{pmatrix} = M_k^{cl}(K_k,\lambda) \begin{pmatrix} x_k \\ w_k \end{pmatrix} \end{aligned} \tag{2}$$

The closed-loop matrices $M_k^{cl}(\lambda)$ belong to the T-periodic sequences of polytopes $\{\mathcal{M}_k^{cl}\}_{k\in\mathbb{N}}$ defined by $\mathcal{M}_k^{cl} = \mathrm{co}\left\{M_k^{cl[1]}, \cdots, M_k^{cl[N]}\right\}$ and

$$M_k^{cl[i]} = \begin{bmatrix} A_k^{[i]} + B_{uk}^{[i]}K_k & B_{wk}^{[i]} \\ C_{zk}^{[i]} + D_{zuk}^{[i]}K_k & D_{zwk}^{[i]} \end{bmatrix} = \begin{bmatrix} A_k^{cl[i]} & B_{wk}^{[i]} \\ C_{zk}^{cl[i]} & D_{zwk}^{[i]} \end{bmatrix}$$

Robust stability and $\mathcal{H}_2$ performance are tackled using the extended Lyapunov framework proposed in [4]. Robust stability is proved equivalent to the existence of a positive quadratic parameter-dependent periodic Lyapunov function:

$$V_k(x_k,\lambda) = x_k' P_k(\lambda) x_k \ , \quad P_{k+T}(\lambda) = P_k(\lambda) > \mathbf{0} \tag{3}$$

decreasing along the trajectories of the system for all admissible uncertainties. If a T-periodic state-feedback sequence $\{K_k\}_{k\in\mathbb{N}}$ exists such that Σ_{cl} is robustly stable then Σ is said to be robustly stabilizable via a T-periodic state-feedback. Let $\mathcal{K}$ be the set of all robustly stabilizing state-feedback T-periodic sequences. For a particular sequence $\{K_k\}_{k\in\mathbb{N}} \in \mathcal{K}$, and a particular realization of the uncertain vector $\lambda \in \Lambda$, the generalized $\mathcal{H}_2$ norm of the periodic system $\Sigma_{cl}(\lambda)$ as defined in [2, 30]) is:

$$\|\Sigma_{cl}(\lambda)\|_2^2 = \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{m_w} \|\Sigma_{cl}(\lambda)[\Delta_k^i]\|_2^2 \tag{4}$$

i.e. the mean of all the responses corresponding to impulsive inputs applied at each time k of the period on each of the m_w channels. The $\mathcal{H}_2$ norm obviously depends on the value of the vector of uncertain parameters λ. Therefore, the squared worst-case $\mathcal{H}_2$ norm of Σ_{cl} may be defined as:

$$\gamma_{wc}(\{K_k\}_{k\in\mathbb{N}}) = \max_{\lambda\in\Lambda} \|\Sigma_{cl}(\lambda)\|_2^2 \tag{5}$$

Which mean that for all uncertainties: $\|\Sigma_{cl}(\lambda)\|_2^2 \leq \gamma_{wc}(\{K_k\}_{k\in\mathbb{N}})$. The aim of the paper is to compute a state-feedback controller that minimizes $\gamma_{wc}(\{K_k\}_{k\in\mathbb{N}})$.

Problem 1 *Worst-case $\mathcal{H}_2$ state-feedback stabilization:*

$$\{K_k\}_{k\in\mathbb{N}}^{opt} = \arg\left\{\min_{\{\mathbf{K_k}\}_{k\in\mathbb{N}}\in\mathcal{K}} \max_{\boldsymbol{\lambda}\in\Lambda} \|\Sigma_{cl}(\boldsymbol{\lambda})\|_2^2\right\} \tag{6}$$

that is find $\{K_k\}_{k\in\mathbb{N}}^{opt} \in \mathcal{K}$ such that $\gamma_{wc}(\{K_k\}_{k\in\mathbb{N}}^{opt})$ is minimum.

For a given $\{K_k\}_{k\in\mathbb{N}}$, the computation of the generalized $\mathcal{H}_2$ norm of periodic systems has been formulated in terms of linear matrix inequalities in [30]. This formulation is extended here to handle polytopic uncertain systems. In order to lighten the notation, the dependency of variables γ upon the sequence $\{K_k\}_{k\in\mathbb{N}}$ will be subsequently considered to be implicit except when explicitly needed. For a given $\lambda \in \Lambda$, the computation of $\|\Sigma_{cl}(\lambda)\|_2$ may be recast as:

Lemma 1 *The $\mathcal{H}_2$ cost $\|\Sigma_{cl}(\lambda)\|_2$ of system (2) for a given sequence $\{K_k\}_{k\in\mathbb{N}}$ and a given realization of the uncertain vector $\lambda \in \Lambda$ is solution of the following optimization problem:*

$$\|\Sigma_{cl}(\lambda)\|_2^2 = \min_{\mathbf{X_k}(\lambda)\in\mathbb{S}^n} \boldsymbol{\gamma} \tag{7}$$

constrained by the $\mathcal{LMI}$s (with $k \in \{1\cdots T\}$)

$$A_k^{cl}(\lambda)\mathbf{X_k}(\lambda)A_k^{cl'}(\lambda) - \mathbf{X_{k+1}}(\lambda) + B_{wk}(\lambda)B'_{wk}(\lambda) < \mathbf{0} \tag{8}$$

$$\text{Trace}\sum_{k=1}^{T}\left\{C_{zk}^{cl}(\lambda)\mathbf{X_k}(\lambda)C_{zk}^{cl'}(\lambda) + D_{zwk}(\lambda)D'_{zwk}(\lambda)\right\} < T\boldsymbol{\gamma} \tag{9}$$

$$\mathbf{X_k}(\lambda) > \mathbf{0} \quad , \quad \mathbf{X_{T+1}}(\lambda) = \mathbf{X_1}(\lambda) \tag{10}$$

The existence of a solution to the matrix inequalities (8-10) simultaneously proves the stability of the closed-loop system (attested by the parameter-dependent Lyapunov function $V_k(x_k, \lambda) = x'_k X_k^{-1}(\lambda)x_k$) and guarantees that the square of $\mathcal{H}_2$ norm is less than a given γ. For a given $\lambda \in \Lambda$ and a given sequence $\{K_k\}_{k\in\mathbb{N}}$, the previous problem may be easily solved by usual SDP Tools [21]. Applying a classical linearizing change of variables [3], a sequence $\{K_k\}_{k\in\mathbb{N}}$ minimizing the $\mathcal{H}_2$ norm for a given parametric realization of the closed-loop system Σ_{cl} can be computed by solving again an $\mathcal{LMI}$ optimization problem. When λ is considered to be unknown, even if the sequence $\{K_k\}_{k\in\mathbb{N}}$ is known (worst-case analysis), the problem is much harder since it amounts to solve the following max-min problem. Moreover, the considered Problem 1 amounts to finding the controller that achieves χ^*_{wc}:

$$\gamma_{wc} = \max_{\boldsymbol{\lambda}\in\Lambda} \min_{\mathbf{X_k}(\boldsymbol{\lambda})\text{s.t. } (8-10)} \boldsymbol{\gamma} \qquad , \qquad \chi^*_{wc} = \min_{\{\mathbf{K_k}\}_{k\in\mathbb{N}}\in\mathcal{K}} \gamma_{wc} \tag{11}$$

Throughout the paper, γ denotes a squared $\mathcal{H}_2$ cost obtained in analysis (solution or relaxation of a max-min optimization problem) and χ denotes a squared $\mathcal{H}_2$ cost obtained in synthesis (solution or relaxation of a min-max-min optimization problem). Even in the simpler case of LTI systems, this problem is known to be hard to solve exactly. Except for special simple cases, one has to resort to using some relaxations (corresponding to subsets of controllers $\mathcal{K}_{sub} \in \mathcal{K}$) to get conditions with associated SDP numerical procedures giving a suboptimal solution to problem 1.

Problem 2 *Worst-case guaranteed $\mathcal{H}_2$ state-feedback stabilization:*

$$\{K_k\}^{sub}_{k\in\mathbb{N}} = \arg\left\{ \min_{\{\mathbf{K_k}\}_{k\in\mathbb{N}}\in\mathcal{K}_{sub}} \max_{\boldsymbol{\lambda}\in\Lambda} \|\Sigma_{cl}(\boldsymbol{\lambda})\|_2^2 \right\} \tag{12}$$

where $\mathcal{K}_{sub}$ is a subset of robustly stabilizing controllers such that (12) is possibly described by $\mathcal{LMI}$ constraints.

Denote $\chi_{sub} = \gamma(\{K_k\}^{sub}_{k\in\mathbb{N}})$, the goal is to get the tightest possible gap between χ_{sub} and χ^*_{wc}. Next section is devoted to $\mathcal{LMI}$ analysis methods for computing upper bounds on γ_{wc} for a given $\{K_k\}_{k\in\mathbb{N}}$. Then, in section 4, those analysis tools are used to derive solutions to problem 2.

3 Robust $\mathcal{H}_2$ analysis

3.1 Quadratic $\mathcal{H}_2$ cost

Mimicking a well-known relaxation for robust stabilization of LTI uncertain systems, the authors of [13] define the quadratic stability concept for uncertain periodic linear discrete-time systems. In this section, we extend this result to compute a bound for the $\mathcal{H}_2$ norm. Quadratic stability comes from the particular choice of periodic quadratic Lyapunov functions (3) that are independent of the uncertain parameters: $P_k(\lambda) = P_k$. Using this restricted parameterization of the Lyapunov function, and denoting $X_k = P_k^{-1}$, we get the following upper-bound for γ_{wc} when the sequence $\{K_k\}_{k\in\mathbb{N}}$ is given.

Theorem 1 *Let the following optimization problem:*

$$\gamma^q_{sub} = \min_{\mathbf{X_k}\in\mathbb{S}^n,\ \mathbf{Z_k}\in\mathbb{S}^{p_z}} \left(\frac{1}{T} \text{Trace} \sum_{k=1}^{T} \mathbf{Z_k} \right) \tag{13}$$

constrained by the $\mathcal{LMI}s$ ($k \in \{1 \cdots T\}$, $i \in \{1 \cdots N\}$)

$$A_k^{cl[i]} \mathbf{X_k} A_k^{cl[i]\prime} - \mathbf{X_{k+1}} + B_{wk}^{[i]} B_{wk}^{[i]\prime} < \mathbf{0} \tag{14}$$

$$C_{zk}^{cl[i]} \mathbf{X_k} C_{zk}^{cl[i]\prime} + D_{zwk}^{[i]} D_{zwk}^{[i]\prime} < \mathbf{Z_k} \tag{15}$$

$$\mathbf{X_k} > \mathbf{0} \quad , \quad \mathbf{X_{T+1}} = \mathbf{X_1} \tag{16}$$

γ_{sub}^q is a the squared quadratic guaranteed $\mathcal{H}_2$ cost for (2) and $\gamma_{wc} \leq \gamma_{sub}^q$.

Proof Suppose $(X_k,\, Z_k)$ solution of (15-16). Applying a Schur complement argument [28], (15) and (14) may be respectively written as:

$$\begin{bmatrix} -X_{k+1} & A_k^{cl[i]} X_k & B_{wk}^{[i]} \\ X_k A_k^{cl[i]\prime} & -X_k & \mathbf{0} \\ B_{wk}^{[i]\prime} & \mathbf{0} & -\mathbf{1} \end{bmatrix} < \mathbf{0} \quad , \quad \begin{bmatrix} -Z_k & C_{zk}^{cl[i]} X_k & D_{zwk}^{[i]} \\ X_k C_{zk}^{cl[i]\prime} & -X_k & \mathbf{0} \\ D_{zwk}^{[i]\prime} & \mathbf{0} & -\mathbf{1} \end{bmatrix} < \mathbf{0} \tag{17}$$

Computing k convex combinations of (17) and applying a Schur complement argument leads for all $k \in \{1 \cdots T\}$ and all $\lambda \in \Lambda$:

$$A_k^{cl}(\lambda) X_k A_k^{cl\prime}(\lambda) + B_{wk}(\lambda) B_{wk}^{\prime}(\lambda) < \mathbf{0} \tag{18}$$

$$C_{zk}^{cl}(\lambda) X_k(\lambda) C_{zk}^{cl\prime}(\lambda) + D_{zwk}(\lambda) D_{zwk}^{\prime}(\lambda) < Z_k \tag{19}$$

where $X_k > \mathbf{0}$ and $X_{T+1} = X_1$. Noting $\gamma = \dfrac{1}{T} \mathrm{Trace} \sum_{k=1}^{T} Z_k$ and summing up inequalities (19) over a period, inequalities (8-10) are retrieved with $X_k(\lambda) = X_k$. Finally, inequality $\gamma_{wc} \leq \gamma_{sub}^q$ comes from the relaxation of max-min by min-max. $\square$

3.2 Polytopic $\mathcal{H}_2$ cost

Based on some recent results [17, 12, 24], a new optimization problem involving additional slack variables is introduced and is shown to be always less conservative than the quadratic one. The conservatism of the quadratic stability relaxation mainly comes from the use of a single periodic Lyapunov function for the whole set of uncertainty. Overcoming this drawback requires to optimize over parameter-dependent periodic Lyapunov functions. Even when the uncertain model is known to have a polytopic structure, it is hardly possible to *a priori* know the dependency of the Lyapunov function upon the unknown parameters. Nevertheless, a natural sub-optimal problem is to seek for polytopic periodic Lyapunov functions of the form (3) where:

$$P_k^{-1}(\lambda) = X_k(\lambda) = \sum_{i=1}^{N} \lambda_i X_k^{[i]} \tag{20}$$

Trivially, such a choice of Lyapunov function will lead to a tighter bound on γ_{wc} than γ^q_{sub}. This new suboptimal bound can be called a *polytopic $\mathcal{H}_2$ cost.* Next theorem gives an $\mathcal{LMI}$ method for computing an upper bound of the polytopic $\mathcal{H}_2$ cost.

Theorem 2 *Let the following optimization problem:*

$$\gamma^e_{sub} = \min_{\mathbf{X}^{[\mathbf{i}]}_{\mathbf{k}} \in \mathbb{S}^n,\ \mathbf{Z}_{\mathbf{k}} \in \mathbb{S}^{p_z},\ \mathbf{H}_{\mathbf{k}} \in \mathbb{R}^{n\times 2n},\ \mathbf{S}_{\mathbf{k}} \in \mathbb{R}^{n\times(p_z+n)}} \left(\frac{1}{T} \mathrm{Trace} \sum_{k=1}^{T} \mathbf{Z}_{\mathbf{k}} \right) \tag{21}$$

constrained by the $\mathcal{LMI}$s ($k \in \{1 \cdots T\}$, $i \in \{1 \cdots N\}$)

$$\begin{bmatrix} -\mathbf{X}^{[\mathbf{i}]}_{\mathbf{k+1}} + B^{[i]}_{wk} B^{[i]\prime}_{wk} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{[\mathbf{i}]}_{\mathbf{k}} \end{bmatrix} + \left\langle \begin{bmatrix} A^{cl[i]}_{k} \\ -\mathbf{1} \end{bmatrix} \mathbf{H}_{\mathbf{k}} \right\rangle < \mathbf{0} \tag{22}$$

$$\begin{bmatrix} -\mathbf{Z}_{\mathbf{k}} + D^{[i]}_{zwk} D^{[i]\prime}_{zwk} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{[\mathbf{i}]}_{\mathbf{k}} \end{bmatrix} + \left\langle \begin{bmatrix} C^{cl[i]}_{zk} \\ -\mathbf{1} \end{bmatrix} \mathbf{S}_{\mathbf{k}} \right\rangle < \mathbf{0} \tag{23}$$

$$\mathbf{X}^{[\mathbf{i}]}_{\mathbf{k}} > \mathbf{0} \quad , \quad \mathbf{X}^{[\mathbf{i}]}_{\mathbf{T+1}} = \mathbf{X}^{[\mathbf{i}]}_{\mathbf{1}} \tag{24}$$

then γ^e_{sub} is the squared extended guaranteed $\mathcal{H}_2$ cost for (2) and $\gamma_{wc} \le \gamma^e_{sub}$.

Proof Suppose a solution ($X^{[i]}_k$, Z_k, H_k, S_k) to (23-24). Then, the computation of the convex combinations over the N vertices allows to write for all uncertainties (the quadratic terms $B_{wk}(\lambda)B'_{wk}(\lambda)$ in (8) are treated using Schur transformation as in the proof of Theorem 1):

$$\begin{bmatrix} -X_{k+1}(\lambda) + B_{wk}(\lambda)B'_{wk}(\lambda) & \mathbf{0} \\ \mathbf{0} & X_k(\lambda) \end{bmatrix} + \left\langle \begin{bmatrix} A^{cl}_k(\lambda) \\ -\mathbf{1} \end{bmatrix} H_k \right\rangle < \mathbf{0}$$

Applying elimination lemma [28] to the last inequality leads to:

$$\begin{bmatrix} \mathbf{1} & A^{cl}_k(\lambda) \end{bmatrix} \begin{bmatrix} -X_{k+1}(\lambda) + B_{wk}(\lambda)B'_{wk}(\lambda) & \mathbf{0} \\ \mathbf{0} & X_k(\lambda) \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ A^{cl\prime}_k(\lambda) \end{bmatrix} < \mathbf{0}$$

which are exactly (8). The same procedure applies to (23) to get (9). □

The main reason to introduce slack variables consists in the decoupling between the periodic Lyapunov matrices $X^{[i]}_k$ and the system matrices thus allowing the use of parameter-dependent periodic Lyapunov functions of the form (20) where $X^{[i]}_k$ is defined at each vertex of the polytope $\mathcal{M}^{cl}_k$. It is then possible to show that:

Lemma 2 *The new bound for $\mathcal{H}_2$ norm computed with theorem 2 is always better than the one elaborated in the quadratic context: $\gamma_{wc} \le \gamma^e_{sub} \le \gamma^q_{sub}$.*

Proof The proof is straightforward. If there exist T matrices X_k and T matrices Z_k such that (14-16), then the set of matrices such that:

$$X_k^{[i]} = X_k \ , \quad H_k = \begin{bmatrix} \mathbf{0} \ X_k \end{bmatrix} \ , \quad S_k = \begin{bmatrix} \mathbf{0} \ X_k \end{bmatrix} \ , \quad Z_k$$

are solution of the $\mathcal{LMI}s$ (22-24). □

4 Robust $\mathcal{H}_2$ synthesis

For the same reasons as developed in the analysis section, the sequence of controllers $\{K_k^{opt}\}_{k\in\mathbb{N}}$ ensuring that the closed-loop system has a minimal $\mathcal{H}_2$ norm γ_{wc}^* cannot be computed in the general case. The objective of this section is to extend analysis results to find a suboptimal sequence of controllers such that the $\mathcal{H}_2$ norm of the closed-loop system is less than a particular bound, as close as possible to γ_{wc}^*. To do so, results based on theorem 2 are expected to be better than design methods based on quadratic stability. The adopted methodology for deriving $\mathcal{LMI}$ conditions is in accordance with the linearizing change of variables of [3]. Moreover, as in [1] it needs some arbitrary factorization of the H_k and S_k matrices which is in the present case parameterized by a couple of sequences $\Upsilon = (\{A_k^0\}_{k\in\mathbb{N}}, \{C_{zk}^0\}_{k\in\mathbb{N}})$ whose elements belong to $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{p_z\times n}$ respectively.

Theorem 3 *Let Υ be a given couple of sequences as defined above and define the following optimization problem:*

$$\chi_{sub}^e(\Upsilon) = \min_{\mathbf{X_k^{[i]}}\in\mathbb{S}^\mathbf{n},\ \mathbf{Z_k}\in\mathbb{S}^{\mathbf{p_z}},\ \mathbf{G_k}\in\mathbb{R}^{\mathbf{n}\times\mathbf{n}},\ \mathbf{Y_k}\in\mathbb{R}^{\mathbf{m}\times\mathbf{n}}} \left(\frac{1}{T} \mathrm{Trace} \sum_{k=1}^{T} \mathbf{Z_k} \right) \tag{25}$$

constrained by the Υ dependent $\mathcal{LMI}s$ ($k \in \{1\cdots T\}$, $i \in \{1\cdots N\}$)

$$\begin{bmatrix} -\mathbf{X_{k+1}^{[i]}} + B_{wk}^{cl[i]} B_{wk}^{cl[i]\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{X_k^{[i]}} \end{bmatrix} + \left\langle \begin{bmatrix} -A_k^{[i]}\mathbf{G_k} - B_{uk}^{[i]}\mathbf{Y_k} \\ \mathbf{G_k} \end{bmatrix} \begin{bmatrix} A_k^{0\prime} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0} \tag{26}$$

$$\begin{bmatrix} -\mathbf{Z_k} + D_{zwk}^{[i]} D_{zwk}^{[i]\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{X_k^{[i]}} \end{bmatrix} + \left\langle \begin{bmatrix} -C_{zk}^{[i]}\mathbf{G_k} - D_{zuk}^{[i]}\mathbf{Y_k} \\ \mathbf{G_k} \end{bmatrix} \begin{bmatrix} C_{zk}^{0\prime} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0} \tag{27}$$

$$\mathbf{X_k^{[i]}} > \mathbf{0} \quad , \quad \mathbf{X_{T+1}^{[i]}} = \mathbf{X_1^{[i]}} \tag{28}$$

then the T-periodic controller $u_k = K_k^e(\Upsilon)x_k$ defined by $K_k^e(\Upsilon) = Y_k G_k^{-1}$ for $k \in \{1\cdots T\}$ is such that $\chi_{sub}^e(\Upsilon)$ is a squared extended guaranteed $\mathcal{H}_2$ cost for (2) and

$$\chi_{wc}^* \leq \gamma_{sub}^e(\{K_k^e(\Upsilon)\}_{k\in\mathbb{N}}) \leq \chi_{sub}^e(\Upsilon) \ .$$

Proof Suppose a solution ($X_k^{[i]}$, Z_k, G_k, Y_k) to (27-28). Then the controller definition implies that $Y_k = K_k^e(\Upsilon)G_k$, which allows to write:

$$\begin{bmatrix} -X_{k+1}^{[i]} + B_{wk}^{[i]}B_{wk}^{[i]\prime} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \left\langle \begin{bmatrix} A_k^{[i]} + B_{uk}^{[i]}K_k^e(\Upsilon) \\ -\mathbf{1} \end{bmatrix} (-G_k) \begin{bmatrix} A_k^{0\prime} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0}$$
$$\begin{bmatrix} -Z_k + D_{zwk}^{[i]}D_{zwk}^{[i]\prime} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \left\langle \begin{bmatrix} C_{zk}^{[i]} + D_{zuk}^{[i]}K_k^e(\Upsilon) \\ -\mathbf{1} \end{bmatrix} (-G_k) \begin{bmatrix} C_{zk}^{0\prime} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0}$$

Defining $S_k = -G_k \begin{bmatrix} C_{zk}^{0\prime} & -\mathbf{1} \end{bmatrix}$ and $H_k = -G_k \begin{bmatrix} A_k^{0\prime} & -\mathbf{1} \end{bmatrix}$ it gives

$$\begin{bmatrix} -X_{k+1}^{[i]} + B_{wk}^{[i]}B_{wk}^{[i]\prime} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \left\langle \begin{bmatrix} A_k^{[i]} + B_{uk}^{[i]}K_k^e(\Upsilon) \\ -\mathbf{1} \end{bmatrix} H_k \right\rangle < \mathbf{0} \tag{29}$$
$$\begin{bmatrix} -Z_k + D_{zwk}^{[i]}D_{zwk}^{[i]\prime} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \left\langle \begin{bmatrix} C_{zk}^{[i]} + D_{zuk}^{[i]}K_k^e(\Upsilon) \\ -\mathbf{1} \end{bmatrix} S_k \right\rangle < \mathbf{0} \tag{30}$$

Theorem 2 proves that the optimization problem defined by equations (25,28) and (29,30) implies that $\gamma_{sub}^e(\{K_k^e(\Upsilon)\}_{k\in\mathbb{N}})$ is a squared extended $\mathcal{H}_2$ cost of the closed-loop system. Moreover, $\chi_{sub}^e(\Upsilon)$ is always greater or equal to $\gamma_{sub}^e(\{K_k^e(\Upsilon)\}_{k\in\mathbb{N}})$ because matrices S_k and H_k are not free but constrained by Υ. □

The main characteristic of Theorem 3 is the degrees of freedom offered by the choice of matrices A_k^0 and C_{zk}^0. Considering these matrices as decisions variables renders the problem non-LMI. But appropriate choices of these allow to efficiently take advantage of the LMI conditions. To manage this appropriate choice the following result is obtained.

Lemma 3 $\chi_{sub}^e(\Upsilon) \geq \gamma_0$ *where* γ_0 *is the squared polytopic* $\mathcal{H}_2$ *cost of the following stable* T*-periodic system:*

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A_k^0 & B_{wk}(\lambda) \\ C_{zk}^0 & D_{zwk}(\lambda) \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{31}$$

Proof Elimination lemma applied to $\mathcal{LMI}s$ (4) implies that matrices A_k^0 and C_{zk}^0 must verify:

$$\begin{aligned} A_k^0 X_k^{[i]} A_k^{0\prime} - X_{k+1}^{[i]} + B_{wk}^{[i]}B_{wk}^{[i]\prime} &< \mathbf{0} \\ C_{zk}^0 X_k^{[i]} C_{zk}^{0\prime} + D_{zwk}^{[i]}D_{zwk}^{[i]\prime} &< Z_k \end{aligned}$$

for all $k \in \{1 \cdots T\}$ and for all $i \in \{1 \cdots N\}$. Convex combinations over the N vertices show that these $\mathcal{LMI}s$ coupled with (25-28) imply that $\frac{1}{T}\text{Trace}\sum_{k=1}^{T} Z_k$ is an over-bound on the $\mathcal{H}_2$ cost of (31) for a choice of polytopic $X_k(\lambda)$ Lyapunov matrices. $\square$

This last result shows that $\{A_k^0\}_{k\in\mathbb{N}}$ must be chosen as a stable periodic sequence. Moreover, the pair of sequences $\Upsilon = (\{A_k^0\}_{k\in\mathbb{N}}, \{C_{zk}^0\}_{k\in\mathbb{N}})$ generates a lower bound to $\chi_{sub}^e(\Upsilon)$ and should not be chosen too large. Consequently, a natural choice of Υ consists in retaining a pair providing the lowest bound $\gamma_0 = 0$ and this can be obtained by the simple choice of a hyper-stable system $\{A_k^0 = \mathbf{0}\}_{k\in\mathbb{N}}$ with $\{C_{zk}^0 = \mathbf{0}\}_{k\in\mathbb{N}}$. Define Υ_ρ as the particular choice of sequences:

$$\Upsilon_\rho = (\{A_k^0 = \rho\mathbf{1}\}_{k\in\mathbb{N}}, \{C_{zk}^0 = \mathbf{0}\}_{k\in\mathbb{N}})$$

The intuitively interesting condition based on the use of Υ_0 can be compared to the quadratic framework approach as shown in the following Theorem.

Theorem 4 *Let the following optimization problem:*

$$\chi_{sub}^q = \min_{\mathbf{X_k}\in\mathbb{S}^n,\ \mathbf{Z_k}\in\mathbb{S}^{p_z},\ \mathbf{Y_k}\in\mathbb{R}^{m\times n}} \left(\frac{1}{T}\text{Trace}\sum_{k=1}^{T}\mathbf{Z_k}\right) \tag{32}$$

constrained by the $\mathcal{LMI}s$ ($k \in \{1\cdots T\}$, $i \in \{1\cdots N\}$)

$$\begin{bmatrix} -\mathbf{X_{k+1}} + B_{wk}^{[i]}B_{wk}^{[i]\prime} & A_k^{[i]}\mathbf{X_k} + B_{uk}^{[i]}\mathbf{Y_k} \\ (A_k^{[i]}\mathbf{X_k} + B_{uk}^{[i]}\mathbf{Y_k})' & -\mathbf{X_k} \end{bmatrix} < \mathbf{0} \tag{33}$$

$$\begin{bmatrix} -\mathbf{Z_k} + D_{zwk}^{[i]}D_{zwk}^{[i]\prime} & C_{zk}^{[i]}\mathbf{X_k} + D_{zuk}^{[i]}\mathbf{Y_k} \\ (C_{zk}^{[i]}\mathbf{X_k} + D_{zuk}^{[i]}\mathbf{Y_k})' & -\mathbf{X_k} \end{bmatrix} < \mathbf{0} \tag{34}$$

$$\mathbf{X_k} > \mathbf{0} \quad , \quad \mathbf{X_{T+1}} = \mathbf{X_1} \tag{35}$$

then the T-periodic controller $u_k = K_k^q x_k$ defined by $K_k^q = Y_k G_k^{-1}$ for $k \in \{1\cdots T\}$ is such that χ_{sub}^q is the squared quadratic guaranteed $\mathcal{H}_2$ cost for (2) and

$$\chi_{wc}^* \leq \chi_{sub}^e(\Upsilon_0) \leq \chi_{sub}^q = \gamma_{sub}^q(\{K_k^q\}_{k\in\mathbb{N}}) \tag{36}$$

Proof The proof is straightforward: First notice that $\mathcal{LMI}s$ (32-35) are a sub case of (25-28) when taking $A_k^0 = \mathbf{0}$, $C_{zk}^0 = \mathbf{0}$, unique Lyapunov matrices $\mathbf{X_k^{[i]}} = \mathbf{X_k}$, $\forall\, i \in \{1\cdots N\}$ and choosing $\mathbf{G_k} = \mathbf{X_k}$. This proves that $\chi_{sub}^e(\Upsilon_0) \leq \chi_{sub}^q$. Second, notice that the change of variables $K_k^q G_k = Y_k$ makes (32-35) exactly equal to (13-16), which proves $\chi_{sub}^q = \gamma_{sub}^q(\{K_k^q\}_{k\in\mathbb{N}})$. $\square$

At this time, there is no other theoretical methodology to *a priori* make a good choice in terms of conservatism of the obtained bound for the sequences in Υ

but it is possible to show on examples that the proposed choice may not be the best (see numerical experiments of section 5). One way to look for "good" values would be to take Υ_ρ with $|\rho| < 1$ to ensure the stability of (31). Then the design amounts to perform multiple $\mathcal{LMI}$ optimizations along with a line search over ρ. Unfortunately, such a design procedure is numerically heavy and quite time-consuming. Such choice may be found in [14, 15] and examples of a line search over ρ can be found in papers by E. Fridman and U. Shaked, for example [29]. A discussion on this question for the case of uncertain time-delay systems was published in [25].

Rather than restricting the search of $\{A_k^0\}_{k\in\mathbb{N}}$ matrices to the form of $\{\rho\mathbf{1}\}$, a simple algorithm based on iterative analysis and design steps is proposed:

Algorithm 1

- *Step 0 - Compute a stabilizing controller (either $\{K_k^q\}_{k\in\mathbb{N}}$ using Theorem 4 or $\{K_k^e(\Upsilon_0)\}_{k\in\mathbb{N}}$ using Theorem 3).*
- *Step 1 - Perform the robust analysis of the closed-loop system using theorem 2 and get sequences $\{H_k\}_{k\in\mathbb{N}}$ and $\{S_k\}_{k\in\mathbb{N}}$.*
- *Step 2 - Partition H_k and S_k such as $H_k = \begin{bmatrix} F_k & G_k \end{bmatrix}$, $S_k = \begin{bmatrix} Q_k & R_k \end{bmatrix}$. Let Υ such that $A_k^0 = -F_k' G_k^{-T}$ and $C_{zk}^0 = -Q_k' G_k^{-T}$.*
- *Step 3 - Compute a new controller $\{K_k^e(\Upsilon)\}_{k\in\mathbb{N}}$ from Theorem 3.*
- *Step 4 - Go back to step 1 to improve on the $\mathcal{H}_2$ cost or stop if the closed-loop robust $\mathcal{H}_2$ cost is not significantly improved.*

Under mild conditions on step 1, it is possible to prove that the sequence of $\mathcal{H}_2$ costs generated by the algorithm is always decreasing.

5 Illustrative example

Consider the 3-periodic system presented in [13] defined by its state-space matrices:

$$A_1 = \begin{bmatrix} -3-\alpha & 2 \\ -3 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1-\alpha & 2 \\ 0.5 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1-\alpha & 2 \\ 2.5 & 3 \end{bmatrix}$$

$$B_{u1} = B_{w1} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \quad B_{u2} = B_{w2} = \begin{bmatrix} 1 \\ -\frac{3\beta+2}{10} \end{bmatrix} \quad B_{u3} = B_{w3} = \begin{bmatrix} 0.5(\beta+1) \\ 1 \end{bmatrix}$$

$C_{zk} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D_{zuk} = D_{zwk} = 0$ for all $k = 1, 2, 3$. The two uncertain parameters $|\alpha| \leq \bar{\alpha}$ and $0 \leq \beta \leq 1$ define a polytope of system matrices with 4 vertices. Note that in [13] the example is treated without uncertainties on the control input matrix B_u.

Results from theorem 4 are applied to find $\mathcal{H}_2$ suboptimal robustly stabilizing controllers for different upper-bounds $\bar{\alpha}$ on the uncertain parameter α. For each synthesis method, a robustly stabilizing controller ($\{K_k\}^q_{k\in\mathbb{N}}$ or $\{K^e_k(\Upsilon_0)\}_{k\in\mathbb{N}}$) is computed and the associated guaranteed $\mathcal{H}_2$ cost at the optimum of the optimization problem (χ^q_{sub} or $\chi^e_{sub}(\Upsilon_0)$) is presented in the following table. In addition, analysis of the closed-loop system is performed to compute the closed-loop extended $\mathcal{H}_2$ cost γ^e_{sub} by applying the results of theorem 2. The table clearly confirms the advantages of the extended framework proposed in this paper, both for robust synthesis and robust analysis.

Method \ $\bar{\alpha}$	0	0.01	0.015775	0.1	0.3	0.497629
Theorem 4	$\chi^q_{sub} = 17.57$	25.60	$2.5 \cdot 10^3$	fail	fail	fail
Theorem 2	$\gamma^e_{sub} = 7.43$	9.14	11.64	×	×	×
Theorem 3	$\chi^e_{sub}(\Upsilon_0) = 1.56$	1.57	1.58	1.71	2.49	$1.3 \cdot 10^3$
Theorem 2	$\gamma^e_{sub} = 1.28$	1.29	1.29	1.38	1.61	2.04

Consider now a particular value of $\bar{\alpha} = 0.015775$. The following controllers are computed with the quadratic and extended approaches respectively:

$$\begin{aligned} K^q_1 &= 10^{-5} \cdot \begin{bmatrix} 0.1648 & -0.1722 \end{bmatrix} & K^e_1(\Upsilon_0) &= \begin{bmatrix} 3.0286 & -2.3929 \end{bmatrix} \\ K^q_2 &= \begin{bmatrix} 0.8529 & -2.6864 \end{bmatrix} & K^e_2(\Upsilon_0) &= \begin{bmatrix} 0.9825 & -2.2140 \end{bmatrix} \\ K^q_3 &= \begin{bmatrix} -5.0225 & -3.7207 \end{bmatrix} & K^e_3(\Upsilon_0) &= \begin{bmatrix} -2.3053 & -2.3518 \end{bmatrix} \end{aligned}$$

These controllers are then applied to the system and the $\mathcal{H}_2$ norm of the closed-loop systems are computed for a grid on α and β. The result is plotted on figure 1 and confirms the γ^e_{sub} values given in the table.

For the same value of $\bar{\alpha}$, previous results can be further improved using Theorem 3. A grid of values from -0.9 to 0.9 every 0.1 is performed for ρ. Only the values from -0.4 to 0.5 give feasible LMI conditions and the best $\chi^e_{sub}(\Upsilon_\rho)$ is 1.5308, obtained for $\rho = 0.1$ with the controller:

$$K^e_1(\Upsilon_{0.1}) = \begin{bmatrix} 3.0120 & -2.3841 \end{bmatrix} \quad , \quad K^e_2(\Upsilon_{0.1}) = \begin{bmatrix} 0.9747 & -2.1660 \end{bmatrix}$$
$$K^e_3(\Upsilon_{0.1}) = \begin{bmatrix} -2.1790 & -2.3687 \end{bmatrix} \quad .$$

Robust analysis performed on the closed-loop system shows that this controller improves the extended $\mathcal{H}_2$ bound up to $\gamma^e_{sub} = 1.2489$. This proves that the choice Υ_0 is not optimal. However finding $\rho = 0.1$ is far from trivial, it needed to solve 19 LMIs in the present case.

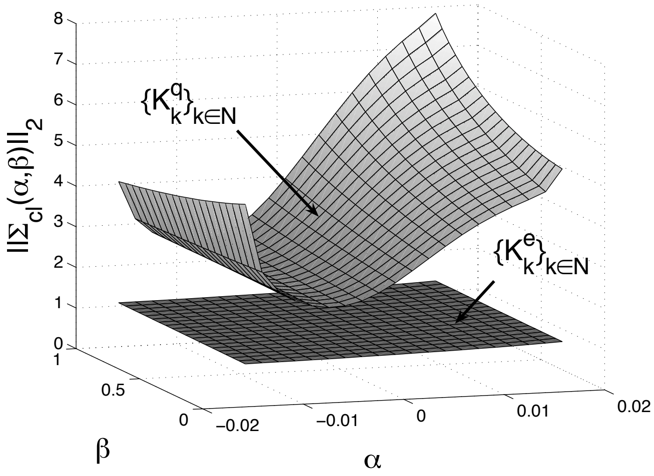

Fig. 1. $\mathcal{H}_2$ *norm of the closed-loop system with* $\{K_k^q\}_{k\in\mathbb{N}}$ *and* $\{K_k^e(\Upsilon_0)\}_{k\in\mathbb{N}}$

Now apply the iterative algorithm 1 initialized with controller $\{K_k^e(\Upsilon_0)\}_{k\in\mathbb{N}}$. At the first iteration it gives on step 2 another choice for Υ such that

$$A_1^0 = \begin{bmatrix} -0.1934 & -0.1665 \\ 0.2320 & 0.3652 \end{bmatrix} \; A_2^0 = \begin{bmatrix} -0.1414 & 0.0276 \\ 0.3307 & 0.3407 \end{bmatrix} \; A_3^0 = \begin{bmatrix} -0.8606 & 0.0668 \\ -0.4653 & -0.0453 \end{bmatrix}$$
$$C_1^0 = \begin{bmatrix} -0.3883 & 0.9898 \end{bmatrix} \; C_2^0 = \begin{bmatrix} 0.6741 & 0.0635 \end{bmatrix} \; C_3^0 \begin{bmatrix} 0.7784 & -0.1610 \end{bmatrix}$$

and on step 3 a new controller:

$$K_1^e(\Upsilon) = \begin{bmatrix} 3.0294 & -2.3603 \end{bmatrix} \quad , \quad K_2^e(\Upsilon) = \begin{bmatrix} 0.9379 & -2.1529 \end{bmatrix}$$
$$K_3^e(\Upsilon) = \begin{bmatrix} -2.3123 & -2.4278 \end{bmatrix}$$

with an associated bound $\chi_{sub}^e(\Upsilon) = 1.2914 < \chi_{sub}^e(\Upsilon_{0.1}) = 1.5308$. Starting a second iteration one gets at the analysis step 1 $\gamma_{sub}^e(\{K_k^e(\Upsilon)\}_{k\in\mathbb{N}}) = 1.2649$. Repeating this procedure one gets after 9 additional iterations (*i.e.* after 19 LMI optimization problems) a new controller such that $\gamma_{sub}^e(\{K_k^e\}_{k\in\mathbb{N}}) = 1.2137$ illustrating that the algorithm does improve the $\mathcal{H}_2$ guaranteed cost.

6 Conclusion

The problem of state-feedback stabilization of linear discrete-time polytopic systems along with $\mathcal{H}_2$ performance optimization is solved. First results are obtained in the classical quadratic context, then a new extended framework based on parameter-dependent Lyapunov functions is proposed. The use of additional variables allows to decouple the computation of the state-feedback from the computation of the Lyapunov matrices. It therefore leads to less conservative conditions. A numerical example has illustrated the relevancy of the results and discussed the efficiency of the different approaches. The advantages of the methods developed in the extended framework have been pointed out. First, they allow to stabilize the system with respect to larger set of uncertainties. Then, for a given set of uncertainties, they drastically improve the performance of the controlled system. Extension of these results to the case of $\mathcal{H}_\infty$ performance is under investigation.

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