Slack variable approach for robust stability analysis of switching discrete-time systems

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Abstract
Robust stability analysis is investigated for discrete-time linear systems with rational dependency with respect to polytopic type uncertainties. Two type of uncertainties are considered: constant parametric uncertainties and time-varying switching uncertainties. Results are in LMI formalism and proofs involve parameter-dependent, quadratic in the state, Lyapunov functions. The new proposed conditions are shown to extend and merge two important existing results. Conservatism reduction is tackled via a model augmentation technique. Numerical complexity is contained by exploiting the structure of the models with respect to the uncertainties.

1 Introduction

In the early 2000, two results have been produced that we aim at studying and merging in the present paper. Both results consider linear systems with uncertainties of affine polytopic type. The first one, [14], that extends results of [10, 12], is dedicated to robust analysis assuming uncertainties are constant over time. The second, [5], assumes time-varying uncertainties with possibly unbounded time-variations. To distinguish both cases, the former is designated at the the parametric case, while the latter is the switching uncertainty case. Results of [5] have indeed been mainly used for the study of switching systems, see for example [6]. An intermediate case between the two is when uncertainties are time-varying with bounded rates. This intermediate case is not considered in the present paper.

In both parametric and switching cases, the upper cited papers prove stability using Lyapunov certificates of the same polytopic type. Results are formulated in terms of a finite number of LMI constraints, yet these are quite different. The contribution of the present paper is to explore the links between the two results. A by product is the illustration that all these results can be easily extended to systems rationally-dependent on the polytopic uncertainties. To do so, the contribution is to consider descriptor type models. The models are such that the E matrix is parameter-dependent and left-hand side invertible. These two features for E differ from assumptions in [1, 2].

The other difference with the last cited papers is that we provide conditions that relax the assumption of parameter-indepedent slack variables. Such assumption happens to be needed only for parametric uncertainties and can be relaxed when considering switching uncertainties. The resulting conservatism reduction is at the expense of increased numerical burden. To limit the effect of this increased burden, we provide methods that exploit the structure of the uncertain models. The methods limit the size of the slack variables. These results are improved versions of that in [13].

The outline of the paper is as follows. Preliminaries are devoted to exposure of the two central results from [14] and [5]. The following section is devoted to the main results for rationally uncertainty dependent switching systems. The fourth section gives the techniques for reducing the size of LMI problems both in terms of number of variables and of size of the constraints. The fifth section treats the mixed switching and parametric uncertainties case and recalls a simple technique for conservatism reduction. It is followed by an illustrative numerical example. Some conclusions are given in the closing section.

2 Preliminaries

Notation: I stands for the identity matrix. A^T is the transpose of the matrix A. $\{A\}^{\mathcal{S}}$ stands for the symmetric matrix $\{A\}^{\mathcal{S}} = A + A^T$. For a matrix $A \in \mathbb{R}^{n\times m}$ or rank r, $A^\perp \in \mathbb{R}^{(n-r)\times n}$ stands for the matrix of maximal rank such that $A^\perp A = 0$. A^+ stands for the Moore-Penrose of A. $A \prec B$ is the matrix inequality stating that $A - B$ is negative definite. $\Xi_{\bar{v}} = \{\theta_{v=1\dots\bar{v}} \geq 0,\ \sum_{v=1}^{\bar{v}} \theta_v = 1\}$ is the unit simplex in $\mathbb{R}^{\bar{v}}$. Its vertices are the $\bar{v}$ vectors $\theta^{[v]}$ with all zeros coefficients except one equal to 1.

The considered systems are linear discrete-time:

$$x_{k+1} = A(\theta_k)x_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the vector state at time $k \in \mathbb{N}$. The matrix $A(\theta_k)$ is assumed to depend of a vector of uncertainties $\theta_k \in \Xi_{\bar{v}}$ and is for a start considered to be affine in the uncertainties:

$$A(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{k,v} A^{[v]}. \tag{2}$$

$A^{[v=1\dots\bar{v}]}$ are given vertex matrices. The system is said to be the affine polytopic. The important key feature of this uncertain model is that $A(\theta)$ lies for all $\theta \in \Xi_{\bar{v}}$ in the convex hull of the finite number of vertices. The is the key feature that allows to prove robust stability, that is stability for all the infinitely many possibles realizations of the uncertainties, by LMI feasibility tests that involve only the vertices.

The state of the art at the end of the twentieth century for addressing this problem, known as the "quadratic stability" result of [3], was to search for a parameter independent quadratic Lyapunov function $V_k = x_k^T P x_k$ and states as follows:

Theorem 1 *If there exists a matrix $P = P^T \succ 0 \in \mathbb{R}^{n\times n}$ such that the following LMI conditions hold for all vertices $v = 1\dots\bar{v}$*

$$A^{[v]T} P A^{[v]} - P \prec 0 \tag{3}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any time varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof The proof is well known. We reproduce it here only for pedagogical purpose to illustrate that the next to come proofs follow similar lines. First note that the constraints $A^T PA - P \prec 0$ are convex with respect to the matrices A. To check this fact apply a Schur complement argument to get the equivalent constraint $\begin{bmatrix} -P & PA \\ A^T P & -P \end{bmatrix} \prec 0$. It is a linear matrix inequality with respect to A, hence, convex in A. Assuming (3) holds, since, as stated upper, it is convex in the $A^{[v]}$s, the inequality also holds for any convex combination of the vertices. That is, for all $\theta_k \in \Xi_{\bar{v}}$ one has $A(\theta_k)^T PA(\theta_k) - P \prec 0$. Pre and post multiply this inequality by x_k^T and its transpose to get along the trajectories of (1) $x_{k+1}^T P x_{k+1} - x_k^T P x_k < 0$ for all non-zero x_k. Stability is hence proved whatever sequence $\{\theta_k\}_{k\geq 0}$ with the positive definite Lyapunov function $V_k = x_k^T P x_k$. ■

Theorem 1 is known to be conservative. In the case of parametric ($\theta_k = \theta$ constant) conservatism comes from the choice of a parameter-independent Lyapunov matrix P proving stability for all values inside the polytopic convex set. A result that allows to reduce the conservatism is as follows ([14])

Theorem 2 *If there exist $\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$ and a matrix $G \in \mathbb{R}^{2n\times n}$ such that the following LMI conditions hold for all vertices $v = 1\dots\bar{v}$*

$$\begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^{\mathcal{S}} \tag{4}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any parametric uncertainty $\theta_k = \theta \in \Xi_{\bar{v}}$. Moreover, if conditions (3) hold, then conditions (4) hold as well.

Proof The proof of robust stability starts as upper by noticing that the LMI constraints are affine, and hence convex, in both the matrices $P^{[v]}$ and $A^{[v]}$. Ddefining the affine polytopic matrix $P(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P^{[v]}$ one therefore gets for all $\theta \in \Theta$:

$$\begin{bmatrix} P(\theta) & 0 \\ 0 & -P(\theta) \end{bmatrix} \prec \left\{ G \begin{bmatrix} I & -A(\theta) \end{bmatrix} \right\}^{\mathcal{S}}.$$

Pre and post multiply this matrix inequality by $\left(\begin{array}{cc} x_{k+1}^T & x_k^T \end{array} \right)$ and its transpose respectively to get exactly $x_{k+1}^T P(\theta) x_{k+1} - x_k^T P(\theta) x_k < 0$ along non zero trajectories of (1). Proof of stability is as upper but with the parameter-dependent Lyapunov function $V_k(\theta) = x_k^T P(\theta) x_k$.

Now we prove that conditions (4) are no more conservative than that of (3). Assume the latter hold and apply the Schur complement argument to get

$$\left[\begin{array}{cc} -P & PA^{[v]} \\ A^{[v]T}P & -P \end{array} \right] \prec 0.$$

This inequality happens to be exactly that of (4) when choosing $P^{[v]} = P$ and $G^T = \left[\begin{array}{cc} P & 0 \end{array} \right]$. ■

As seen from the proof, the conservatism reduction of Theorem 2 is thanks to the decoupling of P and A matrices that allows the introduction of the slack-variables G. Thus obtaining convexity. Yet, if looking at (3) the LMIs are already convex in both A and P as soon as one or the other is fixed. This fact is at the core of the following result:

Theorem 3 *If there exist $\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n \times n}$ such that the following LMI conditions hold for all pairs of vertices $v = 1 \ldots \bar{v}$, $w = 1 \ldots \bar{v}$*

$$A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0 \tag{5}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$. Moreover, if conditions (3) hold, then conditions (5) hold as well.

This result is originated from [5]. As a matter of fact, in that paper the result is not formulated that simply. It involves some unnecessary additional variables. As already noticed in [6], and as shown in the following section these slack variables are useless in this case. It should be noted as well that [5] proves that the condition is not only sufficient but also necessary, as long as one restricts the Lyapunov function to the following polytopic in the uncertainties, quadratic in the state form:

$$V_k(\theta) = x_k^T P(\theta_k) x_k \quad : \quad P(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{v,k} P^{[v]}. \tag{6}$$

Proof The proof of robust stability starts as previously in terms of convexity. First, notice the LMIs are convex in both the vertex matrices $P^{[v]}$ and $A^{[v]}$ with indexes v. Hence, defining the affine polytopic matrix $P(\theta_k)$ as in (6), one gets for all $w = 1 \ldots \bar{v}$ and all $\theta_k \in \Xi_{\bar{v}}$:

$$A(\theta_k) P^{[w]} A(\theta_k) - P(\theta_k) \prec 0$$

These inequalities being convex in the matrices $P^{[w]}$, one gets for all $\theta_k \in \Xi_{\bar{v}}$ and all $\theta_{k+1} \in \Xi_{\bar{v}}$

$$A(\theta_k) P(\theta_{k+1}) A(\theta_k) - P(\theta_k) \prec 0.$$

Pre and post multiply this matrix inequality by x_k^T and its transpose respectively to get exactly $x_{k+1}^T P(\theta_{k+1}) x_{k+1} - x_k^T P(\theta_k) x_k < 0$ along trajectories of (1). Proof of stability is as previously but with the parameter-dependent time-varying Lyapunov function given in (6).

The last part of the theorem that states that it is no more conservative than Theorem 1 is trivial taking $P^{[v]} = P$ for all vertices. ■

The goal of this paper is to analyze the links and differences of the results of Theorems 2 and 3 which both improve results of Theorem 1 using similar polytopic Lyapunov functions but for different assumptions on time evolutions of the uncertainty θ.

3 Main results

3.1 Descriptor models of systems with rationally dependent switching parameters

Let us consider now that the system (1) depends rationally of the uncertainties θ. In such case, using classical linear-fractional transformation (LFT), it can be equivalently rewritten in the following feedback-loop configuration (see [7])

$$\left\{ \begin{array}{l} x_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + D_k \end{array} \right. \quad w_k = \Delta(\theta_k) z_k \tag{7}$$

where $\Delta(\theta)$ is affine in the uncertainties and can be written as $\Delta(\theta) = \sum_{v=1}^{\bar{v}} \theta_v \Delta^{[v]}$. By construction, the LFT is said well posed if the matrix $I - D\Delta(\theta)$ is invertible for all $\theta \in \Xi_{\bar{v}}$ and the model (1) is recovered by the following formula:

$$A(\theta) = A + B\Delta(\theta)(I - D\Delta(\theta))^{-1}C$$

An alternative to that modeling, is the following descriptor representation in which uncertainties enter in an affine fashion:

$$\begin{bmatrix} I \\ 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} -B\Delta(\theta_k) \\ I - D\Delta(\theta_k) \end{bmatrix} z_k = \begin{bmatrix} A \\ C \end{bmatrix} x_k.$$

This model happens to be a sub case of more general descriptor models as proposed by [4, 9] :

$$E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k \tag{8}$$

where $\pi \in \mathbb{R}^q$ are fictive signals used for rendering the model affine. All matrices may be considered as affine polytopic

$$\begin{bmatrix} E_x(\theta) & E_\pi(\theta) & -F(\theta) \end{bmatrix} = \sum_{v=1}^{\bar{v}} \theta_v \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix}. \tag{9}$$

In the present paper, we consider only systems that are originally in non-descriptor form. In such case, as seen upper when considering models issued from well-posed LFT representations, the matrix $E(\theta) = \begin{bmatrix} E_x(\theta) & E_\pi(\theta) \end{bmatrix}$ is square and invertible for all $\theta \in \Xi_{\bar{v}}$. This assumption guarantees that x_{k+1} and π_k are well defined for all k, and that the system is causal without impulsive modes. Extensions to more general descriptor models are possible following the lines of [1, 2].

3.2 Slack variables result

Theorem 4 *If there exist $2\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $G^{[v]} \in \mathbb{R}^{(2n+q)\times(n+q)}$ such that the following LMI conditions hold for all pairs of vertices $v = 1\dots\bar{v}$, $w = 1\dots\bar{v}$*

$$\begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G^{[w]} \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix} \right\}^{\mathcal{S}} \tag{10}$$

then the uncertain system defined by (8-9) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof The LMIs are convex in both the vertex matrices $P^{[v]}$, $E_x^{[v]}$, $E_\pi^{[v]}$ and $F^{[v]}$ with indexes v. Hence, defining the affine polytopic matrix $P(\theta_k)$ as in (6) one gets for all $w = 1\dots\bar{v}$ and all $\theta_k \in \Xi_{\bar{v}}$:

$$\begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \prec \left\{ G^{[w]} \begin{bmatrix} E_x(\theta_k) & E_\pi(\theta_k) & -F(\theta_k) \end{bmatrix} \right\}^{\mathcal{S}}.$$

These inequalities being convex in the matrices $P^{[w]}$ and $G^{[w]}$, defining $G(\theta_{k+1}) = \sum_{w=1}^{\bar{w}} \theta_{w,k+1} G^{[w]}$ one gets for all $\theta_k \in \Xi_{\bar{v}}$ and all $\theta_{k+1} \in \Xi_{\bar{v}}$

$$\begin{array}{c} \begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \\ \prec \left\{ G(\theta_{k+1}) \begin{bmatrix} E_x(\theta_k) & E_\pi(\theta_k) & -F(\theta_k) \end{bmatrix} \right\}^{\mathcal{S}}. \end{array}$$

Pre and post multiply this matrix inequality by $\begin{pmatrix} x_{k+1}^T & \pi_k^T & x_k^T \end{pmatrix}$ and its transpose respectively to get exactly $x_{k+1}^T P(\theta_{k+1}) x_{k+1} - x_k^T P(\theta_k) x_k < 0$ along trajectories of (8). ∎

As illustrated by the proof, Theorem 4 is a direct extension of Theorem 3 for rationally-dependent uncertain systems. The extension is made possible thanks to the affine descriptor modeling of the systems and by introducing as in Theorem 2 some slack variables. A question that arises naturally is whether the additional variables are necessary or do they artificially complexify the numerical problem to solve. The following section aims at giving some answers to this question.

Before that, let us study the conservatism of Theorem 4. One source of conservatism is the choice of a quadratic in the state Lyapunov function $V_k(\theta) = x_k^T P(\theta_k) x_k$. Assuming this choice is done, let us look at the other possible sources of conservatism. Lyapunov stability implies that for all $\theta_k \in \Xi_{\bar{v}}$ and $\theta_{k+1} \in \Xi_{\bar{v}}$ the following quadratic form is negative for all vectors satisfying the linear constraint:

$$\eta_k^T \begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \eta_k < 0$$
$$\forall \eta_k = \begin{pmatrix} x_{k+1} \\ \pi_k \\ x_k \end{pmatrix} \neq 0 \ : \ \begin{bmatrix} E_x(\theta_k) & E_\pi(\theta_k) & -F(\theta_k) \end{bmatrix} \eta_k = 0.$$

Equivalently, by Finsler lemma (see [16]), it writes as the existence of a parameter-dependent $G(\theta_k, \theta_{k+1})$ matrix such that

$$\begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix}$$
$$\prec \left\{ G(\theta_k, \theta_{k+1}) \begin{bmatrix} E_x(\theta_k) & E_\pi(\theta_k) & -F(\theta_k) \end{bmatrix} \right\}^{\mathcal{S}}.$$

The second source of conservatism appears at this point. It amounts to looking for θ_k independent slack matrices $G(\theta_k, \theta_{k+1}) = G(\theta_{k+1})$. Assume this conservative choice is made. Let $P^{[v]} = P(\theta^{[v]})$ and $G^{[w]} = G(\theta^{[w]})$ be the parameter dependent matrices evaluated at the vertices of the simplex. Since the upper defined conditions holds for all $\theta_k \in \Xi_{\bar{v}}$ and $\theta_{k+1} \in \Xi_{\bar{v}}$ it also holds for all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$ which is exactly the condition of Theorem 4.

As for the classical slack variable results of Theorem 2 the upper discussion indicates that the only source of conservatism comes from imposing the parameter-dependent slack variable G not to depend of the uncertainties. Here the restriction is only on part of this dependency: dependency to the current uncertainty value θ_k. The fact that the Lyapunov matrix is of affine polytopic-type is a consequence of this choice. That constatation is classical to slack variable results (see for example [11]) and is extended here to the switching uncertainty case.

4 Reducing the number of decision variables

First, let us consider the formula (10) applied to the case of affine polytopic uncertain systems (1-2) it reads as follows:

$$\begin{bmatrix} P^{[w]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G^{[w]} \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^{\mathcal{S}}. \tag{11}$$

When compared to (5) the LMIs contain many more decision variables and are of doubled size. The upper formulated question is whether this increased numerical complexity is useful in terms of conservatism reduction or not. The answer is clearly no. Indeed, pre and post multiply (11) by $\begin{bmatrix} A^{[v]T} & I \end{bmatrix}$ and its transpose respectively. The result is exactly (5). Conversely, assume (5) holds, then, by a Schur complement argument it reads equivalently as

$$\begin{bmatrix} -P^{[w]} & P^{[w]}A^{[v]} \\ A^{[v]T}P^{[w]} & -P^{[v]} \end{bmatrix} \prec 0$$

which is exactly (11) for the choice $G^{[w]T} = \begin{bmatrix} P^{[w]} & 0 \end{bmatrix}$. Conditions (5) and (11) are equivalent and the former is preferable numerically since it is of reduced dimensions and contains much less decision variables. This same discussion also applies to the results given in [5]. The additional variables it contains are useless, at least for the stability analysis issue.

4.1 Parameter-independent columns

The upper discussion is now generalized allowing to reduce systematically the size of the LMIs (10). Unfortunately, except for the upper case of affine systems (1-2), we were not able to prove that the procedure is fully lossless. The overall procedure contains two steps. The first one concerns parameter independent columns of the $E(\theta)$ matrix. It may be conservative in some cases. The second one, that is exposed later on, is lossless and concerns parameter-independent rows.

Lemma 1 *Assume there exists an invertible matrix T such that for all $v = 1 \dots \bar{v}$*

$$E^{[v]}T = \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} \end{bmatrix} T = \begin{bmatrix} E_1 & E_2^{[v]} \end{bmatrix} \ , \ E_1 \in \mathbb{R}^{(n+q)\times p}.$$

Based on this factorization define

$$N_1^{[v]} = E_1^{+} \begin{bmatrix} E_2^{[v]} & -F^{[v]} \end{bmatrix}, \ \ N_2^{[v]} = E_1^{\perp} \begin{bmatrix} E_2^{[v]} & -F^{[v]} \end{bmatrix}.$$

Moreover, let the following notations

$$\begin{bmatrix} M_{11}(P^{[w]}) & M_{12}(P^{[w]}) \\ M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) \end{bmatrix} = T^T \begin{bmatrix} P^{[w]} & 0 \\ 0 & 0 \end{bmatrix} T \ \ , \ \ M_{11} \in \mathbb{R}^{p\times p}$$

$$\hat{M}(P^{[w]}, P^{[v]}) = \left[\begin{array}{c|cc} M_{11}(P^{[w]}) & M_{12}(P^{[w]}) & 0 \\ \hline M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) & 0 \\ 0 & 0 & -P^{[v]} \end{array}\right]$$

If the following LMIs in the decisions variables $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $\hat{G}^{[v]} \in \mathbb{R}^{(2n+q-p)\times(n+q-p)}$ hold for all all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$\begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \hat{M}(P^{[w]}, P^{[v]}) \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} \prec \left\{ \hat{G}^{[w]} N_2^{[v]} \right\}^{\mathcal{S}} \tag{12}$$

then conditions of Theorem 4 are feasible, and hence the uncertain system defined by (8-9) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof Starting from conditions (12) by a small perturbation argument, one gets that there exists $\epsilon > 0$ such that:

$$\begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \left[\begin{array}{c|cc} M_{11}(P^{[w]}) + \epsilon I & M_{12}(P^{[w]}) & 0 \\ \hline M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) & 0 \\ 0 & 0 & -P^{[v]} \end{array}\right] \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} \prec \left\{ \hat{G}^{[w]} N_2^{[v]} \right\}^{\mathcal{S}}.$$

Applying a Schur complement argument, it reads also as

$$\begin{bmatrix} -(M_{11}(P^{[w]}) + \epsilon I) & -(M_{11}(P^{[w]}) + \epsilon I) N_1^{[v]} \\ -N_1^{[v]T}(M_{11}(P^{[w]}) + \epsilon I) & (*) \end{bmatrix} \prec 0$$

where the bottom-right block is $(*) =$

$$\begin{bmatrix} M_{22}(P^{[w]}) & 0 \\ 0 & -P^{[v]} \end{bmatrix} - \left\{ \hat{G}^{[w]} N_2^{[v]} + \begin{bmatrix} M_{12}^T(P^{[w]}) \\ 0 \end{bmatrix} N_1^{[v]} \right\}^{\mathcal{S}}.$$

After some manipulations the inequalities also write as

$$\hat{M}(P^{[w]}, P^{[v]}) \prec \left\{ \breve{G}^{[w]} \begin{bmatrix} E_1 & E_2^{[v]} & -F^{[v]} \end{bmatrix} \right\} - \begin{bmatrix} \epsilon I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\breve{G}^{[w]} = \begin{bmatrix} (M_{11}(P^{[w]}) + \epsilon I) E_1^{+} \\ \hat{G} E_1^{\perp} + \begin{bmatrix} M_{12}^T(P^{[w]}) \\ 0 \end{bmatrix} E_1^{+} \end{bmatrix}$. The final step of the proof needs to pre and post multiply the inequality by $\breve{T}^T = \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix}$ and $\breve{T}$ respectively. It implies exactly (10) with $G^{[w]} = \breve{T}^T \breve{G}^{[w]}$. ■

The reduction of numerical complexity from Theorem 4 to Lemma 1 can be measured in terms of number of decisions variables and number of rows of the LMIs. The difference of the number of variables is

$$\bar{v}((3n + 2q)p - p^2)$$

which is positive since $p \le n + q$. The reduction of the number of rows in the LMIs is $\bar{v}^2 p$. These values are non negligible, especially when the number of vertices $\bar{v}$ is large. It is often the case, for example when the model is composed of N independent parameters in intervals. In that case $\bar{v} = 2^N$.

4.2 Parameter-independent rows

The upper defined method for reducing the dimensions of the LMI problem is based on a factorization of the $E(\theta)$ matrix taking advantage of parameter independent columns. That procedure can be combined to the next lemma that takes advantage of possible knowledge about parameter independent rows.

Lemma 2 *Assume there exists an invertible matrix S such that for all $v = 1 \dots \bar{v}$*

$$S \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \quad , \quad \begin{array}{l} rank(F_1) = r \\ F_1 \in \mathbb{R}^{s \times (2n+q)} \end{array}$$

then the conditions of Theorem 4 are feasible, if and only if, there exists $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n \times n}$, $\tilde{G}^{[v]} \in \mathbb{R}^{(2n+q-r)\times(n+q-s)}$ such that the following conditions hold for all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$X^T \begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v]} \end{bmatrix} X \prec \left\{ \tilde{G}^{[w]} F_2^{[v]} X \right\}^{\mathcal{S}} \tag{13}$$

where $X = F_1^{T\perp T} \in \mathbb{R}^{(2n+q)\times(2n+q-r)}$ is such that $F_1 X = 0$.

Lemma 3 *Assume there exists an invertible matrix $\tilde{S}$ such that for all $v = 1 \dots \bar{v}$*

$$S N_2^{[v]} = \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \quad , \quad \begin{array}{l} rank(F_1) = r \\ F_1 \in \mathbb{R}^{s \times (2n+q-p)} \end{array}$$

then the conditions of Lemma 1 are feasible, if and only if, there exists $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n \times n}$, $\tilde{G}^{[v]} \in \mathbb{R}^{(2n+q-p-r)\times(n+q-p-s)}$ such that the following conditions hold for all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$X^T \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \hat{M}(P^{[w]}, P^{[v]}) \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} X \prec \left\{ \tilde{G}^{[w]} F_2^{[v]} X \right\}^{\mathcal{S}} \tag{14}$$

where $X = F_1^{T\perp T} \in \mathbb{R}^{(2n+q-p)\times(2n+q-p-r)}$ is such that $F_1 X = 0$.

Proof Only the proof of Lemma 3 is detailed. The proof of Lemma 2 follows exactly the same lines.

Fist we prove that (12) implies (14). By definition of S the right hand side of (12) is $\left\{ \hat{G}^{[w]} S^{-1} \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \right\}^{\mathcal{S}}$. Therefore, pre and post multiplying (12) by X^T and X respectively, one gets exactly (14) with $\tilde{G}^{[w]} = X^T \hat{G}^{[w]} S^{-1} \begin{bmatrix} 0 \\ I_{n+q-p-s} \end{bmatrix}$.

Conversely, since X is full column rank one has $X^T X^{+T} = I$, and (14) can rewritten as:

$$X^T \left(\Psi^{[v,w]} - \left\{ X^{+T} \tilde{G}^{[w]} F_2^{[v]} \right\}^{\mathcal{S}} \right) X \prec 0$$

where $\Psi^{[v,w]}$ is the left-hand side term of (12). Finsler lemma [16] implies the existence of positive scalars $\epsilon^{[v,w]}$ such that

$$\Psi^{[v,w]} - \left\{ X^{+T} \tilde{G}^{[w]} F_2^{[v]} \right\}^{\mathcal{S}} \prec \epsilon^{[v,w]} F_1^T F_1 \prec \epsilon^{[w]} F_1^T F_1.$$

The right-hand side inequality is obtained taking $\epsilon^{[w]} > \epsilon^{[v,w]}$ for all $v = 1 \dots \bar{v}$. This last inequality is exactly (12) for the choice $\hat{G}^{[w]} = \begin{bmatrix} \frac{\epsilon^{[w]}}{2} F_1^T & X^{+T} \tilde{G}^{[w]} \end{bmatrix} S$. ■

Only the reduction of numerical complexity from Theorem 4 to Lemma 2 is stated since that to Lemma 3 is the combination of two lemmas but is not the sum of the two. The difference of the number of variables is

$$\bar{v}((2n+q)s + r(n+q) - rs)$$

and the reduction of the number of rows in the LMIs is $\bar{v}^2 r$. Again, these values are non negligible.

5 Robustness and conservatism reduction

5.1 Extension to robust analysis of uncertain switching systems

In the previous section it is assumed that all uncertainties are time varying. This is a very general case that includes the case of constant, parametric, uncertainties. But, as seen in the preliminaries, there exist as well some slack variable results specific for the situation where all uncertain parameters are constant. The goal of this subsection is to consider the combined case when some uncertainties $\theta_k \in \Xi_{\bar{v}}$ are time varying, while other, $\phi \in \Xi_{\bar{\mu}}$ are constant.

The uncertain models are again assumed in descriptor form

$$E_x(\theta_k,\phi)x_{k+1} + E_\pi(\theta_k,\phi)\pi_k = F(\theta_k,\phi)x_k \tag{15}$$

with left invertible $E(\theta_k,\phi) = \left[\begin{array}{cc} E_x(\theta_k,\phi) & E_x(\theta_k,\phi) \end{array}\right]$ for all uncertainties. The model is considered as affine polytopic in both the uncertainties

$$\begin{array}{l} \left[\begin{array}{ccc} E_x(\theta,\phi) & E_\pi(\theta,\phi) & -F(\theta,\phi) \end{array}\right] \\ = \sum_{\mu=1}^{\bar{\mu}} \sum_{v=1}^{\bar{v}} \phi_\mu \theta_v \left[\begin{array}{ccc} E_x^{[\mu,v]} & E_\pi^{[\mu,v]} & -F^{[\mu,v]} \end{array}\right]. \end{array} \tag{16}$$

Without any difficulty, for these models one gets the following general slack variables result:

Theorem 5 *If there exist $\bar{v}\bar{\mu}$ matrices $P^{[\mu,v]} = P^{[\mu,v]T} \succ 0 \in \mathbb{R}^{n\times n}$ and $\bar{v}$ matrices $G^{[v]} \in \mathbb{R}^{(2n+q)\times(n+q)}$ such that the following LMI conditions hold for all triples of vertices $\mu = 1\dots\bar{\mu}$, $v = 1\dots\bar{v}$, $w = 1\dots\bar{v}$*

$$\left[\begin{array}{ccc} P^{[\mu,w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[\mu,v]} \end{array}\right] \prec \left\{ G^{[w]} \left[\begin{array}{ccc} E_x^{[\mu,v]} & E_\pi^{[\mu,v]} & -F^{[\mu,v]} \end{array}\right] \right\}^{\mathcal{S}} \tag{17}$$

then the uncertain system defined by (15-16) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$ and any parametric uncertainty $\phi \in \Xi_{\bar{\mu}}$.

The proof follows exactly the lines of the previous ones and is therefore not reproduced here. Moreover, similar results as Lemmas 1, 2, 3 are applicable. They are not included in the manuscript for evident reasons of lack of space.

Theorem 5 is the result that merges the two type of results defined in the preliminaries. They happen to be complementary and thanks to the slack variables approach to have a simple mathematical formulation (at least before applying the lemmas for numerical complexity reduction).

5.2 Conservatism reduction

All slack-variable results exposed up to this point are potentially conservative. Following the methodology of [8, 15] it is yet possible to derive very simply some less conservative results. These papers considered continuous-time systems and derived less conservative results by adding artificially derivatives of the state of higher degree in the model. For the considered discrete-time systems, the same strategy amounts to increasing the model with the equations describing x_{k+2}, x_{k+3} and further time instants. For the clarity of the exposure, we limit the description of the methodology to the first level of conservatism reduction that involves only x_{k+2}.

By simply repeating the equation (15), the uncertain model still reads as a descriptor model affine in the uncertainties

$$\hat{E}_x(\theta_{k+1},\theta_k,\phi)\hat{x}_{k+1} + \hat{E}_\pi(\theta_{k+1},\theta_k,\phi)\hat{\pi}_k = \hat{F}(\theta_{k+1},\theta_k,\phi)\hat{x}_k \tag{18}$$

where $\hat{x}_k^T = \left(\begin{array}{cc} x_{k+1}^T & x_k^T \end{array}\right)$, $\hat{\pi}_k^T = \left(\begin{array}{cc} \pi_{k+1}^T & \pi_k^T \end{array}\right)$,

$$\hat{E}_x(\theta_{k+1},\theta_k,\phi) = \left[\begin{array}{cc} E_x(\theta_{k+1},\phi) & 0 \\ 0 & E_x(\theta_k,\phi) \\ 0 & I \end{array}\right],$$

$$\hat{E}_\pi(\theta_{k+1},\theta_k,\phi) = \left[\begin{array}{cc} E_\pi(\theta_{k+1},\phi) & 0 \\ 0 & E_\pi(\theta_k,\phi) \\ 0 & 0 \end{array}\right],$$

$$\hat{F}(\theta_{k+1},\theta_k,\phi) = \left[\begin{array}{cc} F(\theta_{k+1},\phi) & 0 \\ 0 & F(\theta_k,\phi) \\ I & 0 \end{array}\right].$$

All slack variable results exposed to this point apply trivially to that augmented model (18). This is of course at the expense of the increase of the number of variables (number of rows and columns is roughly doubled both for the P and the G matrices), of the size of LMIs (doubled) and of the number of vertices involved (goes from $\bar{v}^2$ to $\bar{v}^3$ since conditions concern θ_k, θ_{k+1} and θ_{k+2}).

6 Numerical example

Let the system described by

$$a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0. \tag{19}$$

It admits the following usual state-space representation, rational in the uncertainties $a_k \neq 0$ and b_k

$$x_{k+1} = \begin{pmatrix} y_{k+2} \\ y_{k+1} \end{pmatrix} = \begin{bmatrix} -b_k^2/a_k & -b_k \\ 1 & 0 \end{bmatrix} x_k$$

and the following affine descriptor representation

$$\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k.$$

The uncertainties are assumed to belong to intervals

$$a_k \in [\,1\,,\,2\,]\;,\; b_k \in [\,-0.5\,,\,\beta].$$

The LMI conditions are tested for different values of β in order to measure their conservatism.

Conditions of Theorem 4 are feasible up to $\beta_1 = 0.81090$, the number of decision variables is 72 and the number of lines of the LMIs (10) is 80. For values of β larger than 0.81096 the LMIs are found unfeasible. In between the two given values, the SDPT3 solver of [17] does not conclude due to numerical problems (we used default settings).

Conditions of Lemma 2 are then tested. From a theoretical point of view these are equivalent to that of Theorem 4. Yet, they are of smaller dimensions (44 variables, 64 rows). This has the effect of reducing the possible numerical problems. Indeed, the LMIs are feasible up to $\beta_2 = 0.81094$ and unfeasible for $\beta = 0.81095$. For results further on, no numerical problems were found and we therefore give only the upper bounds for with the LMIs are feasible. They are unfeasible as soon as the last digit is increased.

For the considered example conditions of Lemma 1 are exactly the same as those of 2 (at the expense of a permutation of some rows and columns). Results are hence trivially identical.

Now we consider results of Theorem 5 (applying of course as before the Lemma 2 technique). As a start we assume $a_k = a$ is constant and keep b_k as a time varying uncertainty. In that case the LMIs are feasible up to $\beta_3 = 0.89027$. The fact that $\beta_3 > \beta_2$ is not surprising since the set of uncertainties is somehow reduced: constant a parameters is a sub case of time-varying uncertainties. Notice that although $\beta_3 > \beta_2$ is what is expected, there is yet no guarantee to have this relation based on the theoretical results. Both values are lower bounds on the actual critical value due to conservatism of results and there is no known relation between the two. The LMIs solved in this case have 28 variables and 32 rows.

Considering the reverse case of constant $b_k = b$ and time varying a_k, the LMIs are feasible up to $\beta_4 = 0.82658$. Again one gets $\beta_4 > \beta_2$. The size of the LMI problem is identical to the previous one.

Finally we consider the fully parametric case when both uncertainties are constant: $a_k = a$, $b_k = b$. The LMI are then feasible up to $\beta_5 = 0.98059$. One gets here as well $\beta_5 > \beta_3$ and $\beta_5 > \beta_4$ which is coherent when reducing the sets of uncertainties from time-varying to constant. The LMI problem has 20 decisions variables and 16 rows. Simple analysis of the system indicates that the actual robust bound for constant parameters is $\beta_5^\star = 1$.

To reduce conservatism we now follow the strategy exposed in subsection 5.2. Let the model increased by one sample of time:

$$\begin{array}{c} a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0 \\ a_{k+1} y_{k+3} + b_{k+1}^2 y_{k+2} + a_{k+1} b_{k+1} y_{k+1} = 0. \end{array} \tag{20}$$

Conditions of Lemma 2 are tested. They are feasible up to $\beta_2^{aug} = 0.84677$. As expected $\beta_2^{aug} \geq \beta_2$. Applying the results to the augmented model allows to reduce conservatism. It is at the expense of increased numerical burden. The number of variables is 480 and the LMIs have 1536 rows. This LMI problem is the largest of all tested ones and it takes less that 5 seconds to be solved with the SDPT3 solver of [17] (on a personal laptop computer).

β (nb vars/nb rows)	syst. (19)	syst. (20)
a_k, b_k	0.81094 (44/64)	0.84677 (480/1536)
a, b_k	0.89027 (28/32)	0.90293 (144/192)
a_k, b	0.82658 (28/32)	0.85375 (144/192)
a, b	0.98059 (20/16)	0.99519 (48/24)

Table 1: Summary of the numerical results

Testing conditions of Theorem 5 for the augmented plant (20) provides as well some results with reduced conservatism. When $a_k = a$ is assumed constant and b_k time-varying, the LMIs are feasible up to $\beta_3^{aug} = 0.90293 > \beta_3$. When $b_k = b$ is assumed constant and a_k time-varying, the LMIs are feasible up to $\beta_4^{aug} = 0.85375 > \beta_4$. The number of decision variables is 144 and the number of rows is 192. When both $a_k = a$ and $b_k = b$ are constant, the LMI tests are feasible up to $\beta_5^{aug} = 0.99519 > \beta_5$. The LMI problem has 48 decisions variables and 24 rows.

All results are summarized in Table 1. They illustrate the high flexibility of the slack-variable approach to analyze systems rational in the uncertainties that could be switching (no bounds on their variations), parametric (constant) or a combination of the two. Results are shown to be conservative in general and a technique is shown to provide less conservative results by simply applying the formulas to models augmented with repeated dynamics further ahead in time.

The drawback of the slack variable approach is to introduce many decision variables. As seen on the examples this increase of the numerical burden comes mainly from the switching uncertainties. Some lemmas are provided to contain this numerical complexity augmentation. These prove to be efficient both in reducing the computation time and the robustness to numerical errors.

7 Conclusions

The contribution of the paper is to provide a flexible general methodology for the analysis of discrete-time systems with uncertainties. Uncertainties can be either of switching type or parametric. Results are LMI based and both numerical complexity and conservatism issues inherent to LMI results are addressed. Further research will be devoted to extending these analysis results to design problems.

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