Cours EDSYS - Commande Adaptative

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INDIRECT ADAPTIVE CONTROL

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OUTLINE

1. INTRODUCTION

Dynamic systems are characterized by their **structures** and **parameters**:

Linear:

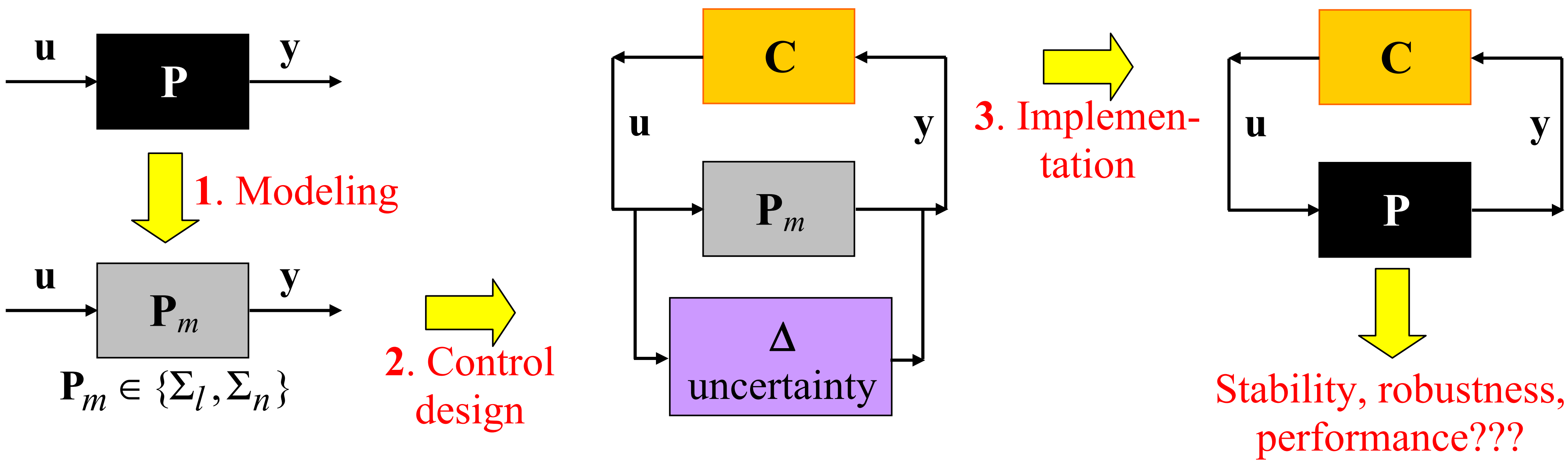
$$\Sigma_l : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}(\boldsymbol{\theta})\mathbf{u} + \mathbf{d}; \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v}, \end{cases}$$

Nonlinear:

$$\Sigma_n : \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{d}, \boldsymbol{\theta}); \\ \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}) + \mathbf{v}, \end{cases}$$

$\mathbf{x}$ is state vector, $\mathbf{u}$ is control input, $\mathbf{d}$ is disturbance, $\mathbf{y}$ is output, $\mathbf{v}$ is noise, $\boldsymbol{\theta}$ is parameters.

Control system design steps:

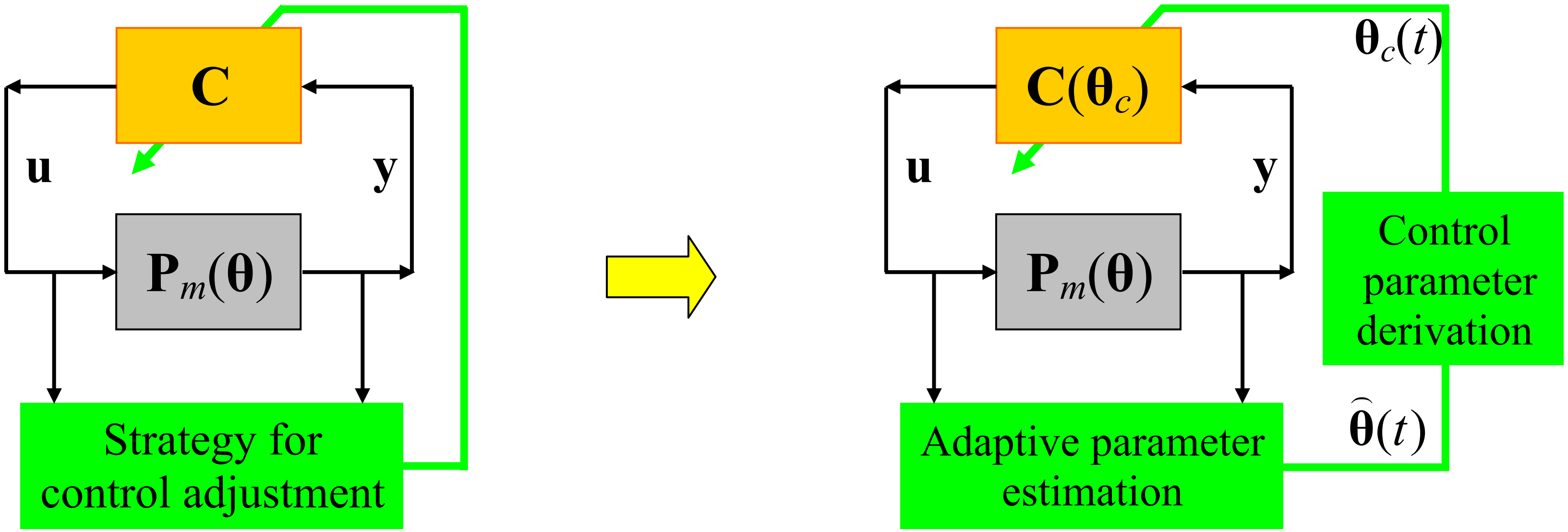


a. Main properties

Parameter estimation is to use a collection of available system signals $\mathbf{y}$ and $\mathbf{u}$, based on certain system structure information Σ_l or Σ_n, to produce estimates $\hat{\boldsymbol{\theta}}(t)$ of the system parameters $\boldsymbol{\theta} \Rightarrow$ Appears on the step **1**.

Adaptive parameter estimation is a dynamic estimation procedure that produce updated parameter estimates on-line $\Rightarrow$ Appears on the step **2&3**.

Adaptive parameter estimation is crucial for **indirect adaptive control** design where controller parameters $\boldsymbol{\theta}_c(t)$ are some continuous functions of the estimates $\hat{\boldsymbol{\theta}}(t)$:



The general scheme of adaptive control.   The scheme of indirect adaptive control.

Key issues in the classical adaptive parameter estimation:

– linear parameterization of system models,

– linear representation of parametric error models,

– stable design of adaptive estimation algorithms,

– analytical proof of system stability,

– parameter convergence,

– robustness of adaptive estimation.

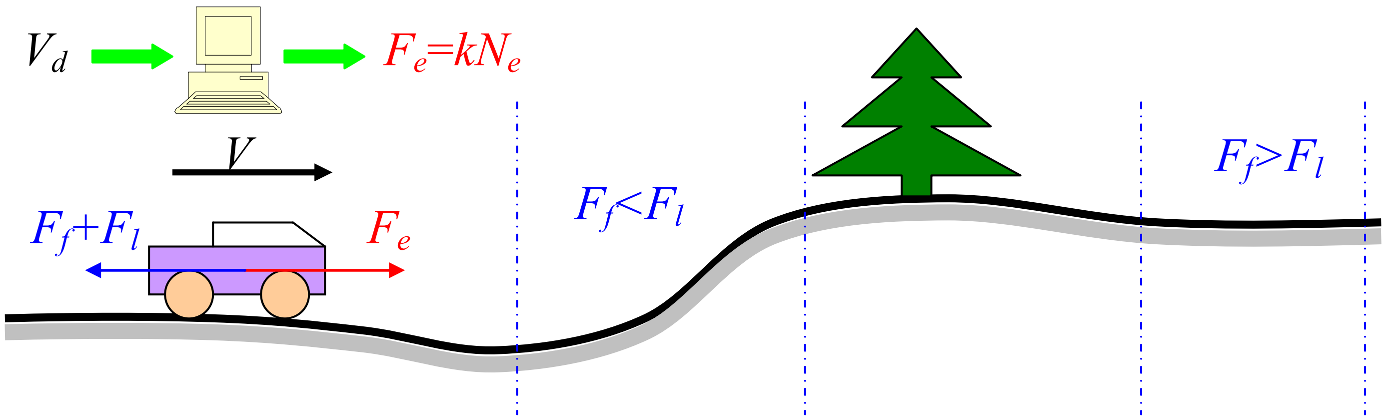
Realization:

– continuous-time,

– discrete-time.

b. Running example

Moving vehicle:



V is velocity (regulating variable), $\dot{V} = dV/dt$ is acceleration, m is *unknown* vehicle mass,

F_e is engine force, $F_e = kN_e$, where N_e is torque, k is *unknown* conversion coefficient,

F_f is friction force, $F_f = \rho V$, where ρ is *unknown* friction coefficient,

F_l is load force (*unknown*, dependent on the road profile).

The first order dynamics (Newton's Second Law):

$$m\dot{V} = F_e - (F_f + F_l) = kN_e - \rho V - F_l.$$

Define the state variable $x = V$, the control input $u = N_e$, the disturbance $d = -F_l / m$:

$$\dot{x} = -ax + bu + d, \qquad (1)$$

$$y = x + v,$$

where y is the output, v is the measurement noise, $a = \rho / m$, $b = k / m$.

Note: the engine from the introduction lecture has the same model $I\dot{\omega} = -f\omega + Ku$.

Features:

− the constant parameters $a > 0$ and $b > 0$ are unknown $\Rightarrow$ (1) is a variant of Σ_l;

− the time-varying signals d and v are unknown, but bounded;

− the unperturbed noise-free case: $d = v = 0$,

− the reference signal $r = V_d$, where V_d − desired velocity.

Control problem (the asymptotic tracking):

$$x(t) \to r(t) \text{ with } t \to +\infty.$$

<u>A variant of the solution</u>:

$$u = b^{-1}[ay - a_m y + b_m r],$$

where $a_m > 0$ and b_m are parameters of the reference model:

$$\dot{x}_m = -a_m x_m + b_m r .$$

The closed loop system has form:

$$\dot{x} = -a_m x + b_m r + \tilde{d} , \; \tilde{d} = d + (a - a_m) v .$$

In the noise-free case ($d = v = 0 \Rightarrow \tilde{d} = 0$) the variable x has the desired dynamics!

⇓

To design the control u we have to estimate the unknown parameters a and b!

Let us try to solve this problem for the noise-free case. We will analyze the robustness issue later. In this case the model (1) can be rewritten as follows:

$$\dot{y} = -ay + bu . \qquad (1')$$

2. ADAPTIVE PARAMETER ESTIMATION

a. Parameterized system model

Consider a **linear time-invariant SISO system** described by the differential equation:

$$P(s)[y](t) = Z(s)[u](t), \qquad (2)$$

$y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ are the measured output and input as before;

$$P(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_1 s + p_0,$$
$$Z(s) = z_m s^m + z_{m-1}s^{m-1} + \ldots + z_1 s + z_0,$$

are polynomials in s, with s being the differentiation operator

$$s[x](t) = \dot{x}(t);$$

p_i, $i = \overline{0, n-1}$ and z_j, $j = \overline{0, m}$ are the unknown but constant parameters to be estimated.

Note: $n = 1$, $m = 0 \Rightarrow$ (1’) with $p_0 = a$ and $z_0 = b$.

The objective: estimate the values p_i, $i = \overline{0, n-1}$ and z_j, $j = \overline{0, m}$ using available for **on-line** measurements signals $y(t)$ and $u(t)$ (no *a priori* accessible datasets).

Parameterization:

let $\Lambda(s) = s^n + \lambda_{n-1}s^{n-1} + \ldots + \lambda_1 s + \lambda_0$ be a stable polynomial (all zeros are in $\mathrm{Re}[s] < 0$).

Then (2) can be represented as follows:

$$\frac{P(s)}{\Lambda(s)}[y](t) = \frac{Z(s)}{\Lambda(s)}[u](t) \Rightarrow (1 - \frac{\Lambda(s)}{\Lambda(s)})[y](t) + \frac{P(s)}{\Lambda(s)}[y](t) = \frac{Z(s)}{\Lambda(s)}[u](t) \Rightarrow$$

$$y(t) = \frac{Z(s)}{\Lambda(s)}[u](t) + \frac{\Lambda(s) - P(s)}{\Lambda(s)}[y](t). \quad (3)$$

Define **parameter vector**

$$\boldsymbol{\theta}^* = [z_0, z_1, \ldots, z_{m-1}, z_m, \lambda_0 - p_0, \lambda_1 - p_1, \ldots, \lambda_{n-2} - p_{n-2}, \lambda_{n-1} - p_{n-1}]^T \in \mathbb{R}^{n+m+1}$$

and **regressor function**

$$\phi(t) = \left[\frac{1}{\Lambda(s)}[u](t), \ldots \frac{s^m}{\Lambda(s)}[u](t), \frac{1}{\Lambda(s)}[y](t), \ldots \frac{s^{n-1}}{\Lambda(s)}[y](t)\right]^T \in \mathbb{R}^{n+m+1}.$$

Then (3) can be expressed in the equivalent form

$$y(t) = \boldsymbol{\theta}^{*T}\phi(t). \quad (4)$$

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In (4):

– the vector $\boldsymbol{\theta}^*$ contains all unknown parameters of the system (2);

– the regressor $\phi(t)$ can be computed using the filters $\dfrac{s^i}{\Lambda(s)}$, $i=\overline{0,n-1}$.

Another variant of **<u>implementation</u>**:

$$\dot{\omega}_1(t) = \mathbf{A}_\lambda \omega_1(t) + \mathbf{b}u(t),$$
$$\dot{\omega}_2(t) = \mathbf{A}_\lambda \omega_2(t) + \mathbf{b}y(t),$$

where $\omega_1(t)\in\mathbb{R}^n$, $\omega_2(t)\in\mathbb{R}^n$ and

$$\mathbf{A}_\lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -\lambda_0 & -\lambda_1 & \cdots & \cdots & -\lambda_{n-2} & -\lambda_{n-1} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Then, we generate the regressor $\phi(t)$ from

$$\phi(t) = [\{\mathbf{C}_m \omega_1(t)\}^T, \omega_2(t)^T]^T,$$

where $\mathbf{C}_m = [\mathbf{I}_{m+1}, \mathbf{0}_{(m+1)\times(n-m-1)}] \in \mathbb{R}^{(m+1)\times n}$ ($\phi(t) = [\omega_1(t)^T, \omega_2(t)^T]^T$ for $m = n-1$).

b. Linear parametric model

Linear parametric model has the form

$$y(t) = \boldsymbol{\theta}^{*T}\phi(t),\ t \geq t_0, \tag{4}$$

where $\boldsymbol{\theta}^* \in \mathbb{R}^{n_\theta}$ is an unknown parameter vector, $y(t) \in \mathbb{R}$ is a known (measured) signal, $\phi(t) \in \mathbb{R}^{n_\theta}$ is a known vector signal (regressor), $n_\theta = n + m + 1$ is the dimension of the model.

Features:

1) The model (4) is commonly seen in system modeling when unknown system parameters can be separated from known signals.
2) The components of $\phi(t)$ may contain nonlinear and/or filtered functions of $y(t)$ and $u(t)$ (or some other system signals).
3) **Adaptive parameter estimation** based on $y(t)$, $u(t)$ $\Leftrightarrow$ **Linear parametric model**.

Let $\boldsymbol{\theta}(t)$ be **the estimate** of $\boldsymbol{\theta}^*$ obtained from an adaptive update law, $\tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}^*$ is **the parametric error**, then define **the estimation error**

$$\varepsilon(t) = \boldsymbol{\theta}(t)^T\phi(t) - y(t) = \boldsymbol{\theta}(t)^T\phi(t) - \boldsymbol{\theta}^{*T}\phi(t) = \tilde{\boldsymbol{\theta}}(t)^T\phi(t). \tag{5}$$

Example 1

$$\dot{y} = -ay + bu. \qquad (1')$$

It has the form (2) for $P(s) = s + p_0$, $Z(s) = z_0$ with $p_0 = a$, $z_0 = b$, $m = n-1$, $n = 1$.

The filter
$$\frac{1}{\Lambda(s)} = \frac{1}{s+1}.$$

The parameter vector $\boldsymbol{\theta}^* = [\theta_1^*, \theta_2^*]^T = [b, 1-a]^T \in \mathbb{R}^2$, $n_\theta = 2$.

The regressor
$$\phi(t) = \left[\frac{1}{s+1}[u](t), \frac{1}{s+1}[y](t)\right]^T \in \mathbb{R}^2.$$

The fast implementation $\phi(t) = [\omega_1(t), \omega_2(t)]^T$ for

$$\begin{aligned} \dot{\omega}_1(t) &= -\omega_1(t) + u(t), \\ \dot{\omega}_2(t) &= -\omega_2(t) + y(t), \end{aligned} \quad \mathbf{A}_\lambda = -1,\ \mathbf{b} = 1.$$

The estimation error for the estimate $\boldsymbol{\theta}(t) = [\theta_1(t), \theta_2(t)]^T \in \mathbb{R}^2$:

$$\varepsilon(t) = \boldsymbol{\theta}(t)^T \phi(t) - y(t) = \boxed{\omega_1(t)\theta_1(t) + \omega_2(t)\theta_2(t) - y(t)} =$$
$$= \omega_1(t)(\theta_1(t) - b) + \omega_2(t)(\theta_2(t) - 1 + a) = \tilde{\boldsymbol{\theta}}(t)^T \phi(t).$$

c. Normalized gradient algorithm

How to update $\boldsymbol{\theta}(t)$? How to minimize the error $\varepsilon(t)=\boldsymbol{\theta}(t)^T\phi(t)-y(t)=\tilde{\boldsymbol{\theta}}(t)^T\phi(t)$?

The idea is to choose the derivative of $\boldsymbol{\theta}(t)$ in a steepest descent direction in order to minimize a normalized quadratic cost functional

$$J(t,\boldsymbol{\theta})=\frac{\varepsilon(t)^2}{2m(t)^2}=\frac{\tilde{\boldsymbol{\theta}}(t)^T\phi(t)\phi(t)^T\tilde{\boldsymbol{\theta}}(t)}{2m(t)^2}=\frac{(\boldsymbol{\theta}(t)-\boldsymbol{\theta}^*)^T\phi(t)\phi(t)^T(\boldsymbol{\theta}(t)-\boldsymbol{\theta}^*)}{2m(t)^2},$$

where $m(t)$ is a normalizing signal not depending (explicitly) on $\boldsymbol{\theta}(t)$.

The idea of $m(t)$ choice: $\phi(t)\phi(t)^T/m(t)^2$ has to be bounded (return later to this issue).

The steepest descent direction of $J(t,\boldsymbol{\theta})$ is $-\frac{\partial J(t,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}=-\frac{\varepsilon(t)}{m(t)^2}\frac{\partial\varepsilon}{\partial\boldsymbol{\theta}}=-\varepsilon(t)\frac{\phi(t)}{m(t)^2}$, therefore:

$$\dot{\boldsymbol{\theta}}(t)=-\varepsilon(t)\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2},\ \boldsymbol{\theta}(t_0)=\boldsymbol{\theta}_0,\ t\geq t_0, \qquad (6)$$

where $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^T>0$ is a design matrix gain, $\boldsymbol{\theta}_0$ is an initial estimate of $\boldsymbol{\theta}^*$.

For (6) an admissible choice of the normalizing function $m(t)$ is

$$m(t) = \sqrt{1 + \kappa\phi(t)^T \phi(t)},$$

where $\kappa > 0$ is a design parameter.

Example 1

The estimation error and the regressor:

$$\varepsilon(t) = \omega_1(t)\theta_1(t) + \omega_2(t)\theta_2(t) - y(t),\ \phi(t) = [\omega_1(t), \omega_2(t)]^T.$$

The cost functional and derivative:

$$J(t,\boldsymbol{\theta}) = \frac{\varepsilon(t)^2}{2m(t)^2} = \frac{[\omega_1(t)\theta_1(t) + \omega_2(t)\theta_2(t) - y(t)]^2}{2m(t)^2},\quad \frac{\partial J(t,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{\varepsilon(t)}{m(t)^2}\begin{bmatrix}\omega_1(t)\\ \omega_2(t)\end{bmatrix}.$$

The normalized gradient algorithm for $\boldsymbol{\Gamma} = \gamma \mathbf{I}_2$, $\gamma > 0$ and $\kappa = 1$:

$$\dot{\boldsymbol{\theta}}(t) = -\gamma \frac{\varepsilon(t)}{1 + \omega_1^2(t) + \omega_2^2(t)}\begin{bmatrix}\omega_1(t)\\ \omega_2(t)\end{bmatrix}.$$

Lemma 1. *The adaptive algorithm (6) guarantees*:

(i) $\boldsymbol{\theta}(t)$, $\dot{\boldsymbol{\theta}}(t)$ *and* $\varepsilon(t)/m(t)$ *are bounded (belong to* L_∞*)*;

(ii) $\varepsilon(t)/m(t)$ *and* $\dot{\boldsymbol{\theta}}(t)$ *belong to* L_2.

Proof. Introduce the positive definite (Lyapunov) function $V(\tilde{\boldsymbol{\theta}}) = \tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} \tilde{\boldsymbol{\theta}}$, then ($\dot{\tilde{\boldsymbol{\theta}}} = \dot{\boldsymbol{\theta}}$)

$$\dot{V} = 2\tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} \dot{\tilde{\boldsymbol{\theta}}} = 2\tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} \left[-\varepsilon(t) \boldsymbol{\Gamma} \frac{\phi(t)}{m(t)^2} \right] = -2\varepsilon(t) \frac{\tilde{\boldsymbol{\theta}}^T \phi(t)}{m(t)^2} = -2 \frac{\varepsilon(t)^2}{m(t)^2},\ t \geq t_0. \tag{7}$$

Since $\dot{V} \leq 0$ we have: $V(t) \in L_\infty \Rightarrow \tilde{\boldsymbol{\theta}}(t) \in L_\infty \Rightarrow \boldsymbol{\theta}(t) \in L_\infty$ = all these signals are bounded.

The boundedness of $\varepsilon(t)/m(t)$ follows the boundedness of $\tilde{\boldsymbol{\theta}}(t)$ and the inequality

$$\frac{|\varepsilon(t)|}{m(t)} = \frac{|\tilde{\boldsymbol{\theta}}(t)^T \phi(t)|}{\sqrt{1 + \kappa \phi(t)^T \phi(t)}} \leq \frac{\|\phi(t)\|}{\sqrt{1 + \kappa \|\phi(t)\|^2}} \|\tilde{\boldsymbol{\theta}}(t)\|.$$

Then boundedness of $\dot{\boldsymbol{\theta}}(t)$ follows from the inequality

$$\|\dot{\boldsymbol{\theta}}\| = \left\| \varepsilon(t) \boldsymbol{\Gamma} \frac{\phi(t)}{m(t)^2} \right\| \leq \|\boldsymbol{\Gamma}\| \frac{|\varepsilon(t)|}{m(t)} \frac{\|\phi(t)\|}{m(t)} \leq \|\boldsymbol{\Gamma}\| \frac{\|\phi(t)\|}{\sqrt{1 + \kappa \|\phi(t)\|^2}} \frac{|\varepsilon(t)|}{m(t)}. \Rightarrow \textbf{(i)}$$

Lemma 1. *The adaptive algorithm (6) guarantees*:

(i) $\boldsymbol{\theta}(t)$, $\dot{\boldsymbol{\theta}}(t)$ *and* $\varepsilon(t)/m(t)$ *are bounded (belong to* L_∞*)*;

(ii) $\varepsilon(t)/m(t)$ *and* $\dot{\boldsymbol{\theta}}(t)$ *belong to* L_2.

Proof. Let us rewrite the equality (7) in the form $2\dfrac{\varepsilon(t)^2}{m(t)^2} = -\dot{V}(t)$ and integrate it:

$$-2\int_{t_0}^{t} \frac{\varepsilon(t)^2}{m(t)^2} dt = \int_{t_0}^{t} \dot{V}(t)dt = V(t_0) - V(t) \le V(t_0) = (\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*)^T \boldsymbol{\Gamma}^{-1} (\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*) < \infty,\ t \ge t_0,$$

therefore $\dfrac{\varepsilon(t)}{m(t)} \in L_2$. From the inequality

$$\| \dot{\boldsymbol{\theta}} \| \le \| \boldsymbol{\Gamma} \| \frac{\| \phi(t) \|}{\sqrt{1 + \kappa \| \phi(t) \|^2}} \frac{| \varepsilon(t) |}{m(t)}$$

we obtain that $\dot{\boldsymbol{\theta}}(t)$ belongs to L_2. $\Rightarrow$ **(ii)** $\Rightarrow$ The Lemma 1 is proven. ■

Note:

We did not prove that $\lim_{t\to\infty} \boldsymbol{\theta}(t) = \boldsymbol{\theta}^*$!

Discussion:

1) The algorithm has equilibriums when $\|\dot{\boldsymbol{\theta}}(t)\|=0$, from (6) we have $\|\dot{\boldsymbol{\theta}}(t)\|=\|\varepsilon(t)\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2}\|$:

$$\|\phi(t)\|=0 \Rightarrow \boxed{\|\dot{\boldsymbol{\theta}}(t)\|=0} \Leftarrow \varepsilon(t)=0 \Leftarrow \boldsymbol{\theta}(t)=\boldsymbol{\theta}^*!$$

$\boldsymbol{\theta}(t)=\boldsymbol{\theta}^*$ is not unique equilibrium of (6) (the usual drawback of any gradient algorithm)!

2) $V(t)=\tilde{\boldsymbol{\theta}}(t)^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t)$ is a measure of deviation of $\boldsymbol{\theta}(t)$ from $\boldsymbol{\theta}^*$, and from (7)

$$\dot{V}(t)\le 0 \Rightarrow [\boldsymbol{\theta}(t)-\boldsymbol{\theta}^*]^T\boldsymbol{\Gamma}^{-1}[\boldsymbol{\theta}(t)-\boldsymbol{\theta}^*]=V(t)\le V(t_0)=[\boldsymbol{\theta}_0-\boldsymbol{\theta}^*]^T\boldsymbol{\Gamma}^{-1}[\boldsymbol{\theta}_0-\boldsymbol{\theta}^*].$$

3) From Lemma 1 we have that $\varepsilon(t)/m(t)\in L_\infty\cap L_2$ and $\lim_{t\to\infty}\varepsilon(t)/m(t)=0$.

4) From (7) we have that the function is nonincreasing ($\dot{V}(t)\le 0$) and bounded from below ($V(t)\ge 0$), thus there exists $\lim_{t\to\infty}V(t)=V_\infty$ for some constant $V_\infty\ge 0$:

- $V_\infty=0 \Rightarrow \lim_{t\to\infty}\boldsymbol{\theta}(t)=\boldsymbol{\theta}^*$;
- $V_\infty>0 \not\Rightarrow \lim_{t\to\infty}\boldsymbol{\theta}(t)=\boldsymbol{\theta}_\infty$ for some constant vector $\boldsymbol{\theta}_\infty\in\mathbb{R}^{n_\theta}$.

5) if $\ddot{\boldsymbol{\theta}}(t)\in L_\infty \Rightarrow \dot{\boldsymbol{\theta}}(t)\in L_\infty\cap L_2$ (Lemma 1) $\Rightarrow \lim_{t\to\infty}\dot{\boldsymbol{\theta}}(t)=0 \not\Rightarrow \lim_{t\to\infty}\boldsymbol{\theta}(t)=\boldsymbol{\theta}_\infty$.

$$\theta(t)=\sin(\sqrt{t+1}),\ \dot{\theta}(t)=0.5\frac{\cos(\sqrt{t+1})}{\sqrt{t+1}} \Rightarrow \dot{\theta}(t)\in L_\infty\cap L_2,\ \lim_{t\to\infty}\dot{\theta}(t)=0,\ \lim_{t\to\infty}\theta(t)=?$$

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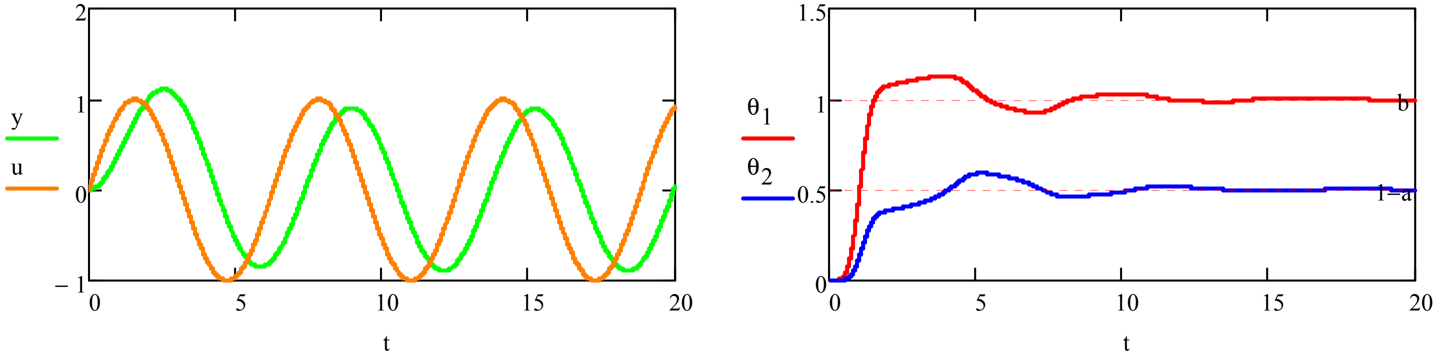
Example 1

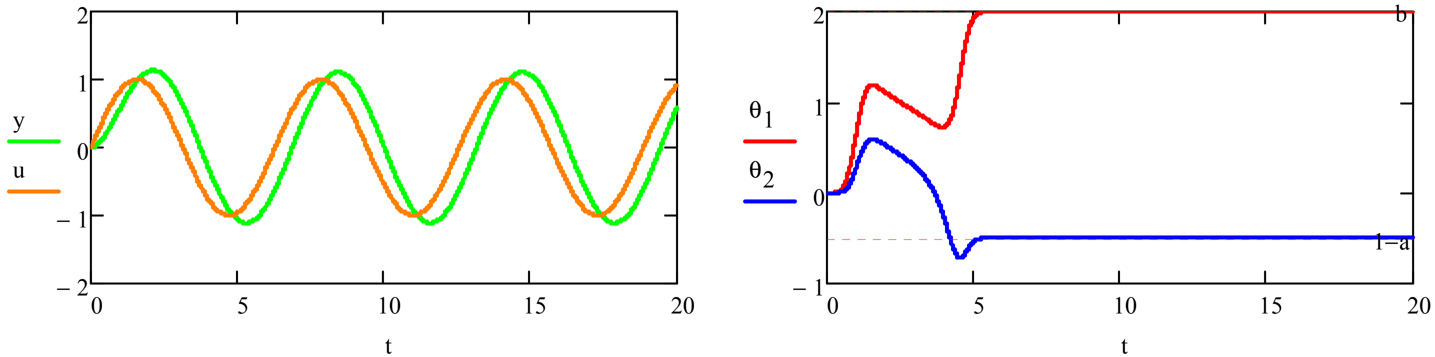
Plant: $\dot{y} = -ay + bu\,.$

Adaptive estimator: $$\dot{\boldsymbol{\theta}}(t) = -\gamma \frac{\omega_1(t)\theta_1(t) + \omega_2(t)\theta_2(t) - y(t)}{1 + \omega_1^2(t) + \omega_2^2(t)} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}, \quad \begin{matrix} \dot{\omega}_1(t) = -\omega_1(t) + u(t), \\ \dot{\omega}_2(t) = -\omega_2(t) + y(t). \end{matrix}$$

Simulation 1: $a = 0.5,\ b = 1,\ \gamma = 20$ and $u(t) = \sin(t)$,



Simulation 2: $a = 1.5,\ b = 2,\ \gamma = 20$ and $u(t) = \sin(t)$,



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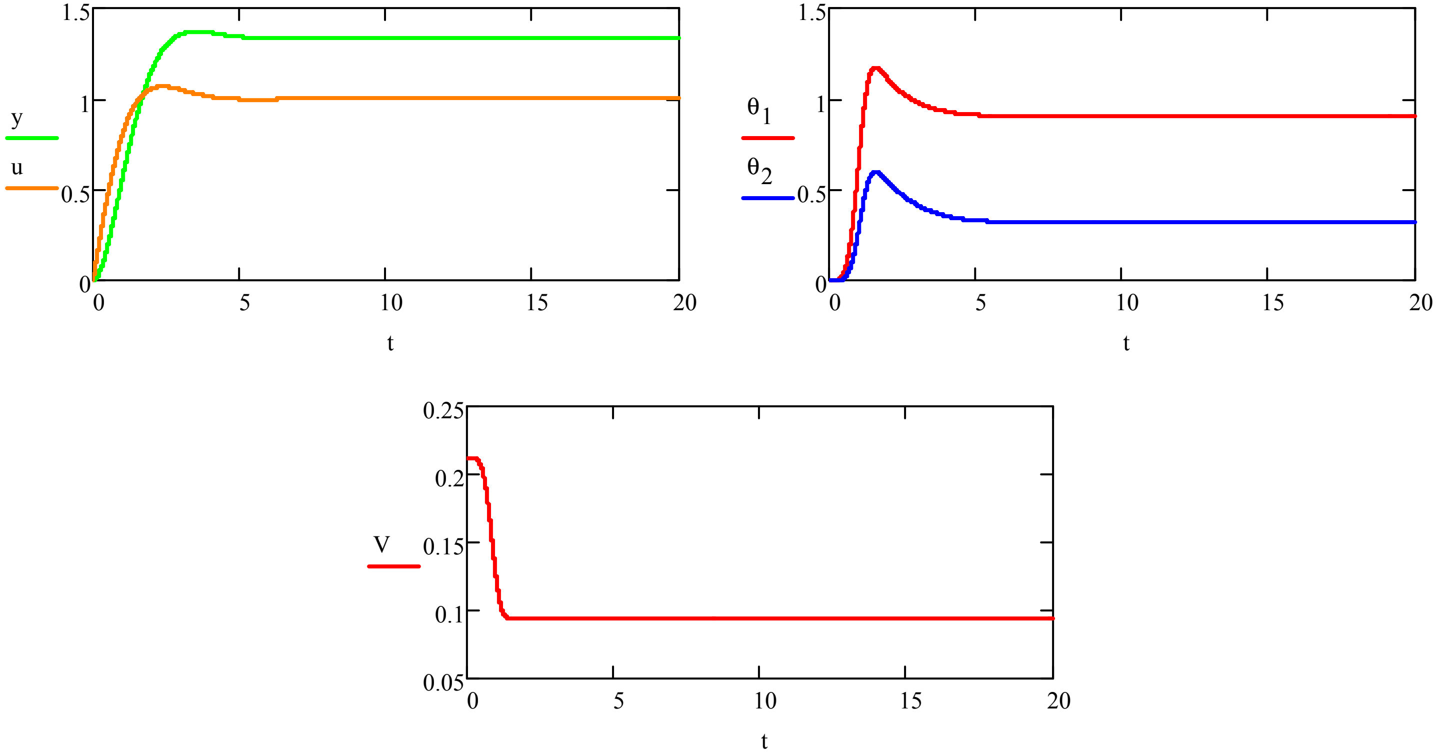
Simulation 3: $a = 1.5$, $b = 2$, $\gamma = 20$ and $u(t) = 1 - e^{-t}\cos(t)$,



<u>**Conclusions**</u>:

– the convergence of adjusted estimates $\boldsymbol{\theta}(t)$ to their ideal values $\boldsymbol{\theta}^*$ depends on the input u;

– y, u are oscillating $\Rightarrow \boldsymbol{\theta}(t) \to \boldsymbol{\theta}^*$; $y \to const$, $u \to const$ (set-point) $\Rightarrow \boldsymbol{\theta}(t) \not\to \boldsymbol{\theta}^*$.

d. Normalized lease-squares algorithm

$$\dot{\boldsymbol{\theta}}(t) = -\varepsilon(t)\frac{\mathbf{P}(t)\phi(t)}{m(t)^2},\ \boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0,\ t \geq t_0, \tag{8}$$

$$\dot{\mathbf{P}}(t) = -\frac{\mathbf{P}(t)\phi(t)\phi(t)^T\mathbf{P}(t)}{m(t)^2},\ \mathbf{P}(t_0) = \mathbf{P}_0 = \mathbf{P}_0^T > 0,\ t \geq t_0, \tag{9}$$

$$m(t) = \sqrt{1 + \kappa\phi(t)^T\mathbf{P}(t)\phi(t)},$$

where $\kappa > 0$ is a design parameter, $\boldsymbol{\theta}_0$ is the initial estimate of $\boldsymbol{\theta}^*$ and $\mathbf{P}_0$ is the initial value of the gain matrix $\mathbf{P}(t) \in \mathbb{R}^{n_\theta \times n_\theta}$.

<u>**Note**</u>:

− if $\mathbf{P}(t) = \boldsymbol{\Gamma}$ for all $t \geq t_0$, then (8) $\Rightarrow$ (6);

− the dimension of (6) is $n_\theta = n + m + 1$, as far as the dimension of (8), (9) is $n_\theta + n_\theta^2$.

Example 1

$$\dot{\boldsymbol{\theta}} = -\frac{\varepsilon}{m^2}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix},\ \dot{\mathbf{P}} = -\frac{1}{m^2}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix}^T$$

$$m^2 = 1 + P_{1,1}\omega_1^2 + P_{1,2}\omega_1\omega_1 + P_{2,1}\omega_1\omega_1 + P_{2,2}\omega_2^2.$$

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Lemma 2. *The adaptive algorithm (8),(9) guarantees*:

(i) $\mathbf{P}(t) = \mathbf{P}^T(t) > 0$ *for all* $t \geq t_0$, $\mathbf{P}(t)$, $\dot{\mathbf{P}}(t)$ *are bounded*;

(ii) $\boldsymbol{\theta}(t)$ *and* $\varepsilon(t)/\bar{m}(t)$ *are bounded (belong to* L_∞*), where* $\bar{m}(t) = \sqrt{1 + \phi(t)^T \phi(t)}$;

(iii) $\varepsilon(t)/m(t)$, $\varepsilon(t)/\bar{m}(t)$ *and* $\dot{\boldsymbol{\theta}}(t)$ *belong to* L_2;

(iv) *there exist a constant matrix* $\mathbf{P}_\infty \in \mathbb{R}^{n_\theta \times n_\theta}$, and a constant vector $\boldsymbol{\theta}_\infty \in \mathbb{R}^{n_\theta}$ such that $\lim_{t\to\infty} \mathbf{P}(t) = \mathbf{P}_\infty$, $\lim_{t\to\infty} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_\infty$.

Proof. First, $\mathbf{P}(t) = \mathbf{P}^T(t)$ and $\dot{\mathbf{P}}(t)$ is bounded by the algorithm (9) construction:

$$\dot{\mathbf{P}}(t) = -\frac{\mathbf{P}(t)\phi(t)\phi(t)^T}{1 + \kappa\phi(t)^T \mathbf{P}(t)\phi(t)}\mathbf{P}(t).$$

Second, the identity $\mathbf{P}(t)\mathbf{P}(t)^{-1} = \mathbf{I}_{n_\theta}$ implies

$$\frac{d}{dt}\left(\mathbf{P}(t)^{-1}\right) = -\mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t)\mathbf{P}(t)^{-1} = m(t)^{-2}\phi(t)\phi(t)^T,$$

then integrating this equality we obtain:

$$\mathbf{P}(t)^{-1} = \mathbf{P}(t_0)^{-1} + \int_{t_0}^{t} m(\tau)^{-2}\phi(\tau)\phi(\tau)^T d\tau,\ t \geq t_0. \tag{10}$$

$\mathbf{P}(t_0)^{-1} > 0 \Rightarrow \mathbf{P}(t)^{-1} \geq \mathbf{P}(t_0)^{-1} > 0 \Rightarrow \mathbf{P}(t) > 0$ and $\mathbf{P}(t)$ is bounded. $\Rightarrow$ **(i)**

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Consider the positive definite function $V(t,\tilde{\boldsymbol{\theta}}) = \tilde{\boldsymbol{\theta}}^T \mathbf{P}(t)^{-1}\tilde{\boldsymbol{\theta}}$, then $(\varepsilon(t) = \tilde{\boldsymbol{\theta}}(t)^T \phi(t))$

$$\dot{V} = \dot{\tilde{\boldsymbol{\theta}}}(t)^T \mathbf{P}(t)^{-1}\tilde{\boldsymbol{\theta}}(t) + \tilde{\boldsymbol{\theta}}(t)^T \mathbf{P}(t)^{-1}\dot{\tilde{\boldsymbol{\theta}}}(t) + \tilde{\boldsymbol{\theta}}(t)^T \frac{d}{dt}\left(\mathbf{P}(t)^{-1}\right)\tilde{\boldsymbol{\theta}}(t) =$$

$$= -\varepsilon(t)\frac{\phi(t)^T \mathbf{P}(t)}{m(t)^2}\mathbf{P}(t)^{-1}\tilde{\boldsymbol{\theta}}(t) - \tilde{\boldsymbol{\theta}}(t)^T \mathbf{P}(t)^{-1}\varepsilon(t)\frac{\mathbf{P}(t)\phi(t)}{m(t)^2} + \tilde{\boldsymbol{\theta}}(t)^T \frac{\phi(t)\phi(t)^T}{m(t)^2}\tilde{\boldsymbol{\theta}}(t) = \quad (11)$$

$$= -\varepsilon(t)\frac{\phi(t)^T \tilde{\boldsymbol{\theta}}(t)}{m(t)^2} - \varepsilon(t)\frac{\tilde{\boldsymbol{\theta}}(t)^T \phi(t)}{m(t)^2} + \frac{\tilde{\boldsymbol{\theta}}(t)^T \phi(t)\phi(t)^T \tilde{\boldsymbol{\theta}}(t)}{m(t)^2} = -\frac{\varepsilon(t)^2}{m(t)^2}, t \geq t_0.$$

Hence, $V(t) = V[t,\tilde{\boldsymbol{\theta}}(t)]$ is bounded, and using (10) we obtain:

$$V(t) = \tilde{\boldsymbol{\theta}}(t)^T \mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t) + \tilde{\boldsymbol{\theta}}(t)^T \left(\int_{t_0}^{t} m(\tau)^{-2}\phi(\tau)\phi(\tau)^T d\tau\right)\tilde{\boldsymbol{\theta}}(t) < \infty,\ t \geq t_0.$$

Therefore

$$\tilde{\boldsymbol{\theta}}(t)^T \mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t) \text{ is bounded} \Rightarrow \tilde{\boldsymbol{\theta}}(t) \text{ and } \boldsymbol{\theta}(t) \text{ are bounded.}$$

Boundedness of $\varepsilon(t)/\overline{m}(t)$ follows the proven property $\tilde{\boldsymbol{\theta}}(t) \in L_\infty$ and the inequality

$$\frac{|\varepsilon(t)|}{\overline{m}(t)} = \frac{|\tilde{\boldsymbol{\theta}}(t)^T \phi(t)|}{\sqrt{1+\phi(t)^T \phi(t)}} \leq \frac{\|\phi(t)\|}{\sqrt{1+\phi(t)^T \phi(t)}}\|\tilde{\boldsymbol{\theta}}(t)\|. \Rightarrow \textbf{(ii)}$$

Rewriting the equality (11) in the form $2\varepsilon(t)^2 / m(t)^2 = -\dot{V}(t)$ and integrating it, we obtain:

$$-2\int_{t_0}^{t} \frac{\varepsilon(t)^2}{m(t)^2} dt = \int_{t_0}^{t} \dot{V}(t) dt = V(t_0) - V(t) \le V(t_0) = (\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*)^T \mathbf{P}_0^{-1} (\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*) < \infty, \ t \ge t_0,$$

therefore $\dfrac{\varepsilon(t)}{m(t)} \in L_2$ and

$$\frac{\varepsilon(t)}{\overline{m}(t)} = \frac{\varepsilon(t)}{m(t)} \frac{m(t)}{\overline{m}(t)} + \frac{\varepsilon(t)}{m(t)} \in L_2, \ \frac{m(t)}{\overline{m}(t)} \in L_\infty \Rightarrow \frac{\varepsilon(t)}{\overline{m}(t)} \in L_2.$$

Since $\mathbf{P}(t) = \mathbf{P}(t)^T$ is bounded and $\mathbf{P}(t) = \mathbf{P}_s(t)\mathbf{P}_s(t)$ ($\mathbf{P}_s(t)$ is also bounded) we have

$$\| \dot{\boldsymbol{\theta}}(t) \| = \left\| \varepsilon(t) \frac{\mathbf{P}(t)\phi(t)}{m(t)^2} \right\| = \left\| \frac{\mathbf{P}(t)\phi(t)}{\sqrt{1 + \kappa\phi(t)^T \mathbf{P}(t)\phi(t)}} \right\| \frac{|\varepsilon(t)|}{m(t)} =$$

$$= \left\| \frac{\mathbf{P}_s(t)\mathbf{P}_s(t)\phi(t)}{\sqrt{1 + \kappa\phi(t)^T \mathbf{P}_s(t)\mathbf{P}_s(t)\phi(t)}} \right\| \frac{|\varepsilon(t)|}{m(t)} = \| \mathbf{P}_s(t) \| \frac{\| \mathbf{P}_s(t)\phi(t) \|}{\sqrt{1 + \kappa \| \mathbf{P}_s(t)\phi(t) \|^2}} \frac{|\varepsilon(t)|}{m(t)},$$

therefore, $\dot{\boldsymbol{\theta}}(t) \in L_2$. $\Rightarrow$ **(iii)**

The integration of the differential equation (9) gives for $t \geq t_0$:

$$\mathbf{P}(t) = \mathbf{P}(t_0) - \int_{t_0}^{t} \frac{\mathbf{P}(\tau)\phi(\tau)\phi(\tau)^T \mathbf{P}(\tau)}{m(\tau)^2} d\tau > 0 \Rightarrow \mathbf{P}(t_0) > \int_{t_0}^{t} \frac{\mathbf{P}(\tau)\phi(\tau)\phi(\tau)^T \mathbf{P}(\tau)}{m(\tau)^2} d\tau .$$

For any $\mathbf{z} \in \mathbb{R}^{n_\theta}$ we have $\infty > \mathbf{z}^T \mathbf{P}(t_0)\mathbf{z} > \int_{t_0}^{t} \mathbf{z}^T \frac{\mathbf{P}(\tau)\phi(\tau)\phi(\tau)^T \mathbf{P}(\tau)}{m(\tau)^2} \mathbf{z} d\tau \geq 0$, consequently, the scalar function $f(t, \mathbf{z}) = \int_{t_0}^{t} \mathbf{z}^T \frac{\mathbf{P}(\tau)\phi(\tau)\phi(\tau)^T \mathbf{P}(\tau)}{m(\tau)^2} \mathbf{z} d\tau$ has properties:

− it is a nondecreasing function of $t \geq t_0$;
− it is upper and lower bounded,

then there exists $f_{\mathbf{z}} \in \mathbb{R}$ such that $\lim_{t \to \infty} f(t, \mathbf{z}) = f_{\mathbf{z}}$. $\Rightarrow \lim_{t \to \infty} \mathbf{P}(t) = \mathbf{P}_\infty$, $\mathbf{P}_\infty \in \mathbb{R}^{n_\theta \times n_\theta}$.

Note that ($\varepsilon(t) = \phi(t)^T \tilde{\boldsymbol{\theta}}(t)$)

$$\dot{\tilde{\boldsymbol{\theta}}} = \dot{\boldsymbol{\theta}} = -\frac{\mathbf{P}\phi}{m^2}\varepsilon = -\frac{\mathbf{P}\phi}{m^2}\phi^T \mathbf{P}\mathbf{P}^{-1}\tilde{\boldsymbol{\theta}} = \dot{\mathbf{P}}\mathbf{P}^{-1}\tilde{\boldsymbol{\theta}} \Rightarrow \tilde{\boldsymbol{\theta}}(t) = \mathbf{P}(t)\mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0) \Rightarrow$$

$$\lim_{t \to \infty} \boldsymbol{\theta}(t) = \boldsymbol{\theta}^* + \lim_{t \to \infty} \mathbf{P}(t)\mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0) = \boldsymbol{\theta}^* + \mathbf{P}_\infty \mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0) = \boldsymbol{\theta}_\infty \in \mathbb{R}^{n_\theta} . \quad \blacksquare$$

Discussion:

1) The algorithm (8)−(9) can be presented in the form

$$\dot{\tilde{\boldsymbol{\theta}}}(t) = \dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t)\tilde{\boldsymbol{\theta}}(t) = \mathbf{A}(t_0,t)\tilde{\boldsymbol{\theta}}(t),\ \tilde{\boldsymbol{\theta}}(t) = \mathbf{P}(t)\mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0),$$

thus it is a linear time-varying system!!! The same as the algorithm (6):

$$\dot{\tilde{\boldsymbol{\theta}}}(t) = \dot{\boldsymbol{\theta}}(t) = -\varepsilon(t)\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2} = -\boldsymbol{\Gamma}\frac{\phi(t)\phi(t)^T}{m(t)^2}\tilde{\boldsymbol{\theta}}(t) = \mathbf{B}(t)\tilde{\boldsymbol{\theta}}(t).$$

2) Uniform stability: $\|\tilde{\boldsymbol{\theta}}(t)\| = \|\mathbf{P}(t)\mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0)\| \le c_0 \|\tilde{\boldsymbol{\theta}}(t_0)\|$ for some $c_0 > 0$.

3) The least-squares algorithm (8), (9) minimizes a cost function which is an integral of squared errors at many time instants with a penalty on the initial estimate $\boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0$:

$$J(t,\boldsymbol{\theta}) = \frac{1}{2}\int_{t_0}^{t}\frac{(\boldsymbol{\theta}(\tau)^T\phi(\tau) - y(\tau))^2}{m(\tau)^2}d\tau + \frac{1}{2}[\boldsymbol{\theta}(t) - \boldsymbol{\theta}_0]^T\mathbf{P}_0^{-1}[\boldsymbol{\theta}(t) - \boldsymbol{\theta}_0] =$$

$$= \frac{1}{2}\int_{t_0}^{t}\frac{\varepsilon(\tau)^2}{m(\tau)^2}d\tau + \frac{1}{2}\tilde{\boldsymbol{\theta}}(t_0)^T\mathbf{P}_0^{-1}\tilde{\boldsymbol{\theta}}(t_0).$$

Compare with the gradient descent algorithm (6): $J(t,\boldsymbol{\theta}) = \frac{1}{2}\frac{\varepsilon(t)^2}{m(t)^2}$.

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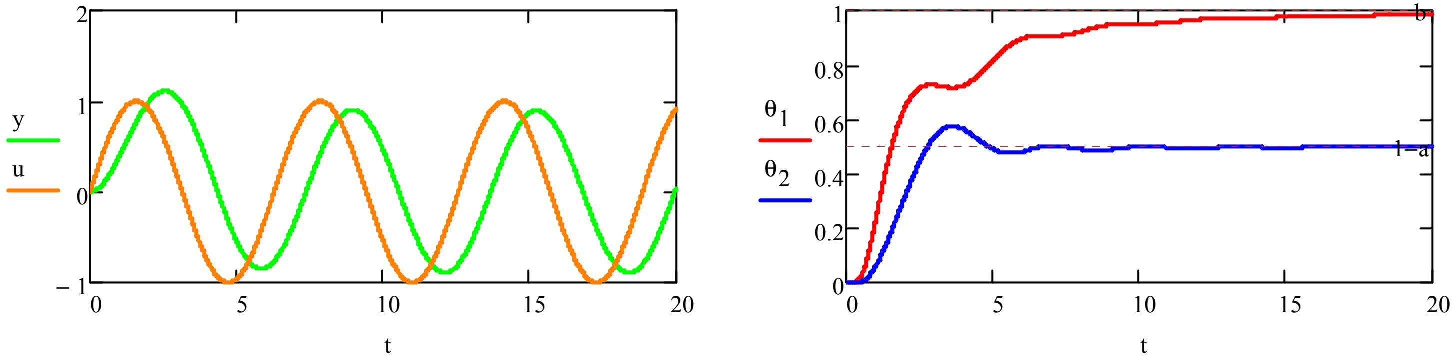
Example 1

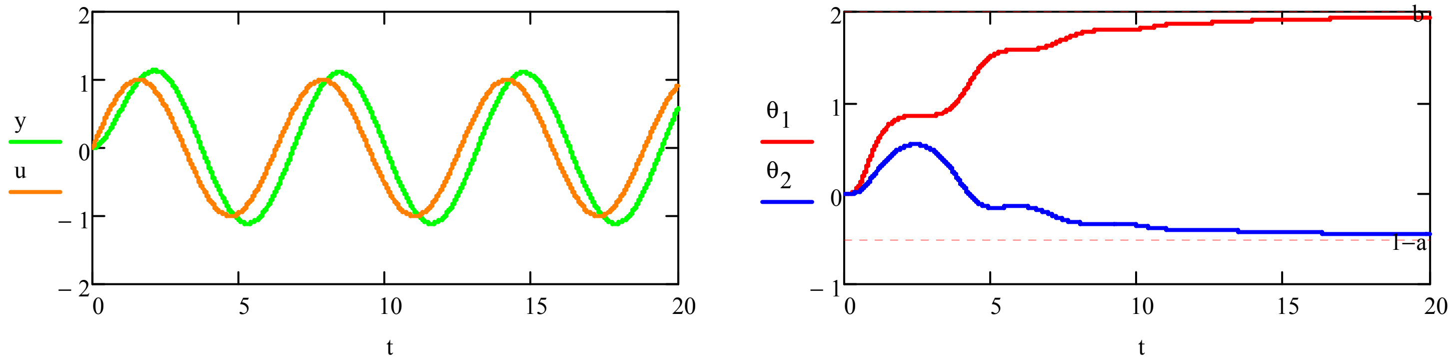
Plant: $\dot{y} = -ay + bu$.

Estimator: $\dot{\boldsymbol{\theta}} = -\frac{\varepsilon}{m^2}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix}$, $\dot{\mathbf{P}} = -\frac{1}{m^2}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix}\begin{bmatrix} P_{1,1}\omega_1 + P_{1,2}\omega_2 \\ P_{2,1}\omega_1 + P_{2,2}\omega_2 \end{bmatrix}^T$

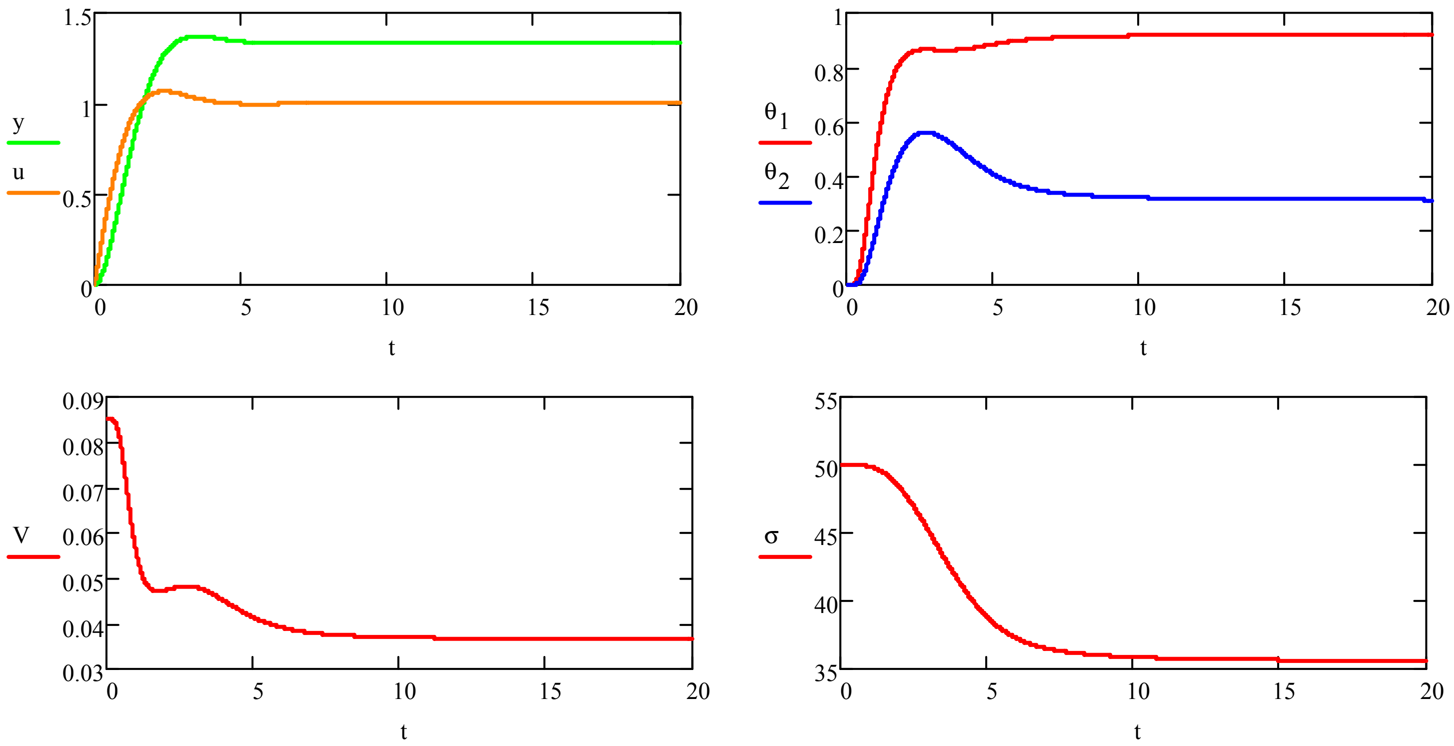
Simulation 1: $a = 0.5$, $b = 1$, $\mathbf{P}_0 = 20\mathbf{I}_2$ and $u(t) = \sin(t)$,



Simulation 2: $a = 1.5$, $b = 2$, $\mathbf{P}_0 = 50\mathbf{I}_2$ and $u(t) = \sin(t)$,



Simulation 3: $a = 1.5$, $b = 2$, $\mathbf{P}_0 = 50\mathbf{I}_2$ and $u(t) = 1 - e^{-t}\cos(t)$, $\sigma(t) = \| \mathbf{P}(t) \|$,



Conclusions:

– the rate of convergence in the algorithm (8), (9) is a more complex issue than in (6);

– the convergence of adjusted estimates $\boldsymbol{\theta}(t)$ to their ideal values $\boldsymbol{\theta}^*$ depends on the input u;

– y, u are oscillating $\Rightarrow \boldsymbol{\theta}(t) \to \boldsymbol{\theta}^*$; $y \to const$, $u \to const$ (set-point) $\Rightarrow \boldsymbol{\theta}(t) \not\to \boldsymbol{\theta}^*$.

e. Discrete-time version of adaptive algorithms

Continuous time $t \geq t_0 \;\Rightarrow\;$ Discrete time $t \in \{t_0, t_0 + T, t_0 + 2T \ldots\}$, $T > 0$ is the period.

<u>The normalized gradient algorithm</u>:

$$\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t) - \boldsymbol{\Gamma}\frac{\phi(t)\varepsilon(t)}{m(t)^2},\; \boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0,\; 2\mathbf{I}_{n_\theta} > \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > 0,$$

$$m(t) = \sqrt{\kappa + \phi(t)^T \phi(t)},\; \kappa > 0.$$

<u>The normalized least-squares algorithm</u>:

$$\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t) - \mathbf{P}(t-1)\frac{\phi(t)\varepsilon(t)}{m(t)^2},\; \boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0,$$

$$\mathbf{P}(t) = \mathbf{P}(t-1) - \mathbf{P}(t-1)\frac{\phi(t)\phi(t)^T}{m(t)^2}\mathbf{P}(t-1),\; \mathbf{P}(t_0 - 1) = \mathbf{P}_0 = \mathbf{P}_0^T > 0,$$

$$m(t) = \sqrt{\kappa + \phi(t)^T \mathbf{P}(t-1)\phi(t)},\; \kappa > 0.$$

<u>Proprieties</u>:

− $\boldsymbol{\theta}(t)$, $\varepsilon(t)/m(t)$, $\varepsilon(t)/\overline{m}(t)$ *and* $\mathbf{P}(t) = \mathbf{P}(t)^T > 0$ *are bounded*;
− $\varepsilon(t)/m(t)$, $\varepsilon(t)/\overline{m}(t)$ *and* $\boldsymbol{\theta}(t+1) - \boldsymbol{\theta}(t)$ *belong to* L_2.

3. IDENTIFICATION AND ROBUSTNESS

− identification ⇔ parameter convergence;
− robustness ⇔ $\|\mathbf{d}\| \neq 0$, $\|\mathbf{v}\| \neq 0$.

a. Parametric convergence and persistency of excitation

Lemma 3. *For the gradient algorithm (6) or least-squares algorithm (8)−(9), if* $m(t) \in L_\infty$ *and* $\dot{\phi}(t) \in L_\infty$, *then* $\lim_{t\to\infty} \varepsilon(t) = 0$.

Proof. $\frac{\varepsilon(t)}{m(t)} \in L_2 \cap L_\infty$ and $\tilde{\boldsymbol{\theta}}(t), \dot{\tilde{\boldsymbol{\theta}}}(t) \in L_\infty$ from lemmas 1, 2 . Since $\varepsilon(t) = \phi(t)^T \tilde{\boldsymbol{\theta}}(t)$ we have $\dot{\varepsilon}(t) = \dot{\phi}(t)^T \tilde{\boldsymbol{\theta}}(t) + \phi(t)^T \dot{\tilde{\boldsymbol{\theta}}}(t)$. Hence: $\dot{\phi}(t) \in L_\infty \Rightarrow \boxed{\dot{\varepsilon}(t) \in L_\infty}$, $m(t) \in L_\infty \Rightarrow \boxed{\varepsilon(t) \in L_2 \cap L_\infty}$. ■

Under conditions of lemma 3 asymptotically $\varepsilon(t) = \sum_i^{n_\theta} [\theta_i(t) - \theta_i^*] \phi_i(t) = 0$, $t \geq t_1$:

a) $\phi(t) = [1, 0, \ldots, 0]^T \Rightarrow \theta_1(t) - \theta_1^* = 0$, $\theta_i(t)$ for $2 \leq i \leq n_\theta$ −?

b) $\phi(t) = [1, 1, \ldots, 1]^T \Rightarrow \sum_i^{n_\theta} [\theta_i(t) - \theta_i^*] = 0$ −?

c) $\phi_i(t) = \sin(\omega i t)$, $i = \overline{1, n_\theta}$, $\omega > 0 \Rightarrow \sum_i^{n_\theta} [\theta_i(t) - \theta_i^*] \sin(\omega i t) = 0 \Rightarrow \theta_i(t) = \theta_i^*$, $i = \overline{1, n_\theta}$.

Definition 1. *A bounded vector signal* $\varphi(t) \in \mathbb{R}^q$, $q \geq 1$, *is exciting over the finite time interval* $[\sigma_0, \sigma_0 + \delta_0]$, $\delta_0 > 0$, $\sigma_0 \geq t_0$, *if for some* $\alpha_0 > 0$

$$\int_{\sigma_0}^{\sigma_0+\delta_0} \varphi(\tau)\varphi(\tau)^T d\tau \geq \alpha_0 \mathbf{I}_q . \qquad \square$$

Definition 2. *A bounded vector signal* $\varphi(t) \in \mathbb{R}^q$, $q \geq 1$, *is* ***Persistently Exciting*** (PE) *if there exist* $\delta > 0$ *and* $\alpha > 0$ *such that*

$$\int_{\sigma}^{\sigma+\delta} \varphi(\tau)\varphi(\tau)^T d\tau \geq \alpha \mathbf{I}_q , \ \forall \sigma \geq t_0 . \qquad \square$$

$$\varphi(t) \in \mathbb{R}^q \text{ is PE} \Leftrightarrow \exists \rho > 0,\ \delta > 0 : \int_{t_0}^{t} \varphi(\tau)\varphi(\tau)^T d\tau \geq \rho(t - t_0)\mathbf{I}_q , \ \forall t \geq t_0 + \delta$$

(positive definite in average).

<u>The idea</u>: $\operatorname{rank}[\varphi(t)\varphi(t)^T] = 1,\ t \geq t_0 \Rightarrow \operatorname{rank}[\int_{t_0}^{t_0+\delta} \varphi(\tau)\varphi(\tau)^T d\tau] = q$.

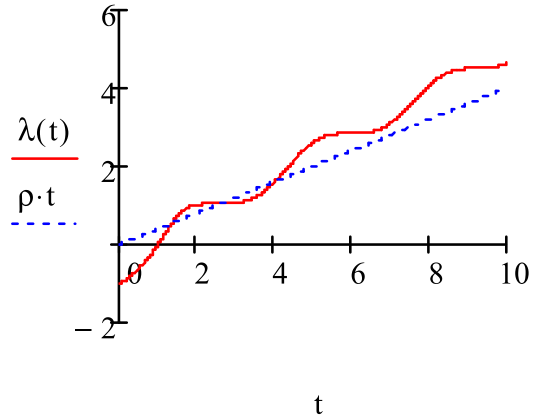
Example 2.

$$\varphi(t) = [1,1]^T \Rightarrow \varphi(t)\varphi(t)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \int_0^{\delta} \varphi(\tau)\varphi(\tau)^T d\tau = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \delta \geq 0 \Rightarrow \text{not PE.}$$

$$\varphi(t) = [1, e^{-t}]^T \Rightarrow \int_0^\delta \varphi(\tau)\varphi(\tau)^T d\tau = \begin{bmatrix} \delta & -e^{-\delta} \\ -e^{-\delta} & -0.5e^{-2\delta} \end{bmatrix} \Rightarrow \text{exciting over some finite intervals.}$$

$$\varphi(t) = [1, \sin(t)]^T \Rightarrow \int_0^\delta \varphi(\tau)\varphi(\tau)^T d\tau = \begin{bmatrix} \delta & -\cos(\delta) \\ -\cos(\delta) & -0.5\delta - 0.25\sin(2\delta) \end{bmatrix} \geq \lambda(\delta)\mathbf{I}_2,$$

$$\lambda(\delta) = \frac{6\delta - \sin(2\delta)}{8} - \sqrt{\frac{[2\delta + \sin(2\delta)]^2}{64} + \cos(\delta)^2} \geq \rho\delta,\ \rho = 0.4 \text{ for } \delta > 5:$$



$\Rightarrow$ PE!!!

$$\varphi(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \Rightarrow \int_0^\delta \varphi(\tau)\varphi(\tau)^T d\tau = \frac{1}{2}\begin{bmatrix} \delta + 0.5\sin(2\delta) & \sin(\delta)^2 \\ \sin(\delta)^2 & \delta - 0.5\sin(2\delta) \end{bmatrix} \geq \frac{1}{2}[\delta - \sin(\delta)]\mathbf{I}_2 \Rightarrow \text{PE.}$$

Normalized gradient algorithm (6) ($\tilde{\boldsymbol\theta}(t)=\boldsymbol\theta(t)-\boldsymbol\theta^*$, $\varepsilon(t)=\phi(t)^T\tilde{\boldsymbol\theta}(t)$):

$$\dot{\tilde{\boldsymbol\theta}}(t)=\boxed{\dot{\boldsymbol\theta}(t)=-\varepsilon(t)\boldsymbol\Gamma\frac{\phi(t)}{m(t)^2}}=-\boldsymbol\Gamma\frac{\phi(t)}{m(t)^2}\phi(t)^T\tilde{\boldsymbol\theta}(t)=\mathbf{B}(t)\tilde{\boldsymbol\theta}(t),\ \mathbf{B}(t)=-\boldsymbol\Gamma\frac{\phi(t)\phi(t)^T}{m(t)^2}.$$

Let $\Phi(t_0,t)$ be the state transition matrix of the linear time-varying system (6), then

– $\tilde{\boldsymbol\theta}(t)=\Phi(t_0,t)\tilde{\boldsymbol\theta}(t_0)$;

– $\boxed{\phi(t)\text{ is PE}}\Rightarrow\phi(t)/m(t),\ m(t)=\sqrt{1+\kappa\phi(t)^T\phi(t)}$ is PE $\Rightarrow\eta(t)=\Phi(t_0,t)^T\phi(t)/m(t)$ is PE:

$$\exists\rho>0,\ \delta>0:\ \int_{t_0}^{t}\eta(\tau)\eta(\tau)^T d\tau\geq\rho(t-t_0)\mathbf{I}_{n_\theta},\ \forall t\geq t_0+\delta.$$

Consider the Lyapunov function $V(\tilde{\boldsymbol\theta})=\tilde{\boldsymbol\theta}^T\boldsymbol\Gamma^{-1}\tilde{\boldsymbol\theta}$:

$$\dot V=-2\frac{\varepsilon(t)^2}{m(t)^2}=-2\tilde{\boldsymbol\theta}(t)^T\frac{\phi(t)\phi(t)^T}{m(t)^2}\tilde{\boldsymbol\theta}(t)=-2\tilde{\boldsymbol\theta}_0^T\eta(t)\eta(t)^T\tilde{\boldsymbol\theta}_0,$$

integrating this equality for $t\geq t_0+\delta$ we obtain ($V(t_0)=\tilde{\boldsymbol\theta}_0^T\boldsymbol\Gamma^{-1}\tilde{\boldsymbol\theta}_0$):

$$V(t)=V(t_0)-2\tilde{\boldsymbol\theta}_0^T\int_{t_0}^{t}\eta(\tau)\eta(\tau)^T d\tau\tilde{\boldsymbol\theta}_0\leq V(t_0)-2\rho(t-t_0)\tilde{\boldsymbol\theta}_0^T\tilde{\boldsymbol\theta}_0=\tilde{\boldsymbol\theta}_0^T[\boldsymbol\Gamma^{-1}-2\rho(t-t_0)]\tilde{\boldsymbol\theta}_0\Rightarrow$$

$$\lim_{t\to\infty}V(t)=0\Rightarrow\lim_{t\to\infty}\boldsymbol\theta(t)=\boldsymbol\theta^*.$$

Normalized least-squares algorithm (8)−(9):

$$\tilde{\boldsymbol{\theta}}(t) = \mathbf{P}(t)\mathbf{P}(t_0)^{-1}\tilde{\boldsymbol{\theta}}(t_0),\ t \ge t_0.$$

Properties:

− $\lim_{t\to\infty} \mathbf{P}(t) = 0 \Leftrightarrow \lim_{t\to\infty} \tilde{\boldsymbol{\theta}}(t) = 0$;

− $\mathbf{P}(t) = \mathbf{P}(t_0) - \int_{t_0}^{t} \frac{\mathbf{P}(\tau)\phi(\tau)\phi(\tau)^T\mathbf{P}(\tau)}{m(\tau)^2} d\tau,\ \mathbf{P}(t) = \mathbf{P}^T(t) > 0$ for all $t \ge t_0$;

− $\phi(t)$ is PE $\Rightarrow \phi(t)/m(t),\ m(t) = \sqrt{1+\kappa\phi(t)^T\phi(t)}$ is PE $\Rightarrow$ $\eta(t) = \mathbf{P}(t)\phi(t)/m(t)$ is PE:

$$\exists\rho > 0,\ \delta > 0:\ \int_{t_0}^{t} \eta(\tau)\eta(\tau)^T d\tau \ge \rho(t - t_0)\mathbf{I}_{n_\theta},\ \forall t \ge t_0 + \delta.$$

Then

$$0 < \mathbf{P}(t) = \mathbf{P}(t_0) - \int_{t_0}^{t} \eta(\tau)\eta(\tau)^T d\tau \le \mathbf{P}(t_0) - \rho(t - t_0)\mathbf{I}_{n_\theta} \le 0 \text{ for some } t \ge t_0 \Rightarrow$$

$$\lim_{t\to\infty} \boldsymbol{\theta}(t) = \boldsymbol{\theta}^*.$$

⇓

Lemma 4. *For the gradient algorithm (6) or least-squares algorithm (8)−(9), if* $\phi(t)$ *is* PE, *then* $\lim_{t\to\infty} \boldsymbol{\theta}(t) = \boldsymbol{\theta}^*$. ■

Discussion:

What is PE property of the regressor $\phi(t)$:

$$\phi(t) = [\{\mathbf{C}_m \omega_1(t)\}^T, \omega_2(t)^T]^T,$$

where $\omega_1(t) \in \mathbb{R}^n$, $\omega_2(t) \in \mathbb{R}^n$ and for a Hurwitz matrix $\mathbf{A}_\lambda$:

$$\dot{\omega}_1(t) = \mathbf{A}_\lambda \omega_1(t) + \mathbf{b}u(t),$$
$$\dot{\omega}_2(t) = \mathbf{A}_\lambda \omega_2(t) + \mathbf{b}y(t).$$

⇓

PE of $\phi(t) \Leftarrow$ PE of $\omega_1(t)$ and $\omega_2(t) \Leftarrow$ PE of $u(t)$ and $y(t)$.

⇓

(2) is a linear system $\Rightarrow$ PE of $y(t)$ is determined by the input $u(t)$!

⇓

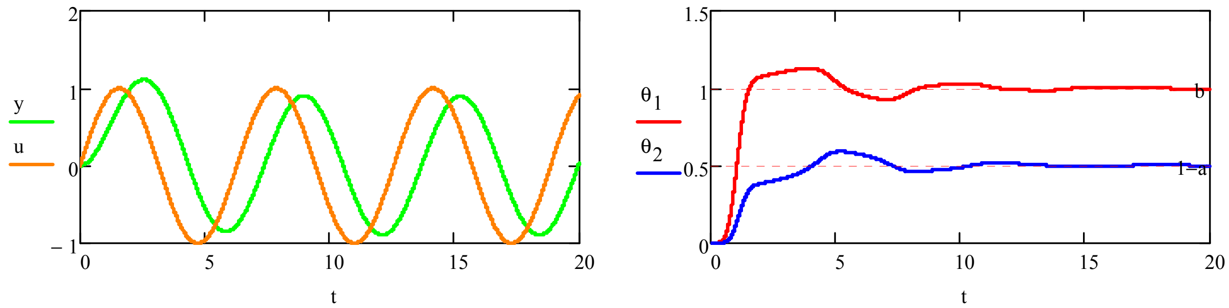
PE of $u(t) \Rightarrow$ PE of $\phi(t)$

(that we already observed in the example).

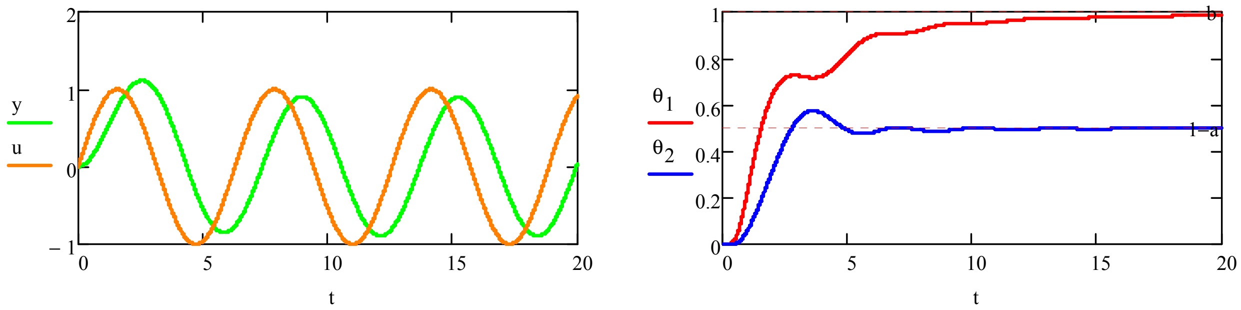
Example 1

Plant: $$\dot{y} = -ay + bu,$$
$$a = 0.5,\ b = 1 \text{ and } u(t) = \sin(t).$$

Gradient algorithm: $\gamma = 20$



Least-squares algorithm: $\mathbf{P}_0 = 20\mathbf{I}_2$



$$u(t) = \sin(t) \Rightarrow y(t) = \alpha \sin(t + \beta) \Rightarrow \omega_i(t) = \alpha_i \sin(t + \beta_i) \text{ due to } \begin{matrix} \dot{\omega}_1(t) = -\omega_1(t) + u(t), \\ \dot{\omega}_2(t) = -\omega_2(t) + y(t) \end{matrix} \Rightarrow$$

$$\phi(t) = [\omega_1(t), \omega_2(t)]^T \Rightarrow \varphi(t) = [\cos(t), \sin(t)]^T \Rightarrow \text{PE}.$$

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b. Robustness of adaptive algorithms

Before the noise free case with $\mathbf{d}(t)=0$ and $\mathbf{v}(t)=0$ has been considered for

$$\Sigma_l : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}(\boldsymbol{\theta})\mathbf{u} + \mathbf{d}; \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v}. \end{cases}$$

What happens if $\mathbf{d}(t)\neq 0$ or $\mathbf{v}(t)\neq 0$?

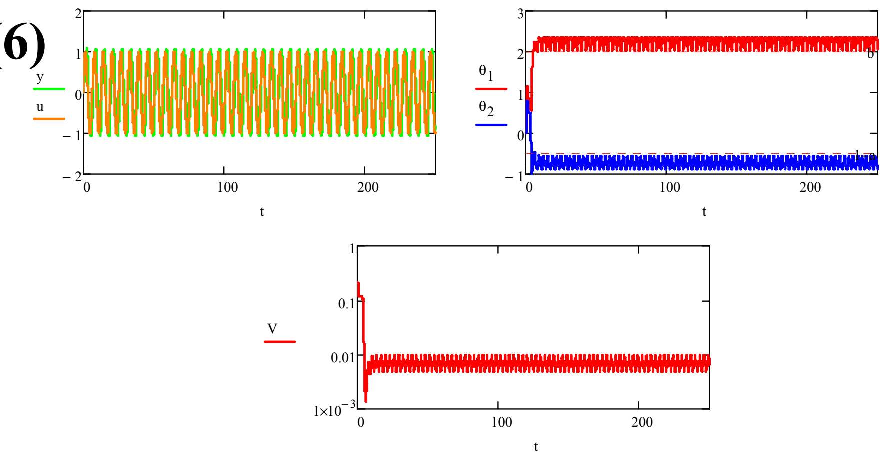
(only the case $\mathbf{d}(t)\neq 0$ will be considered)

Example 1

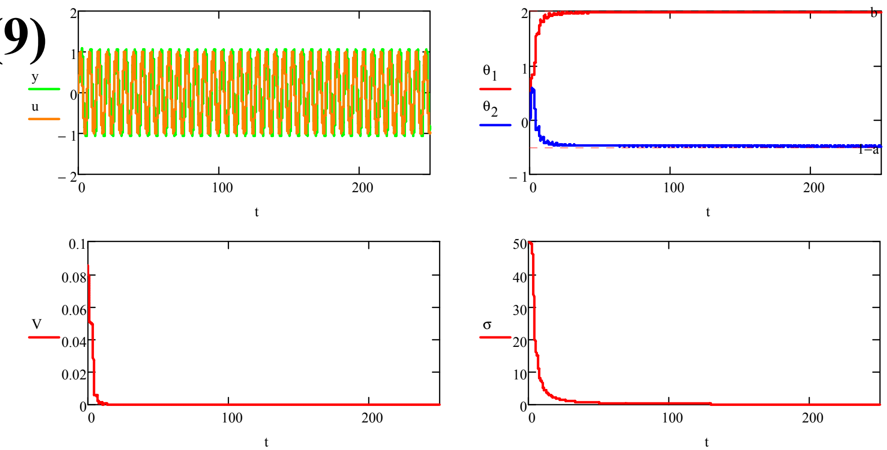
Plant: $$\dot{y} = -ay + bu + d(t),$$

$$a = 1.5,\ b = 2 \text{ and } u(t)=\sin(t),\ \boxed{d(t) = 0.5\sin(3t).}$$

(6)

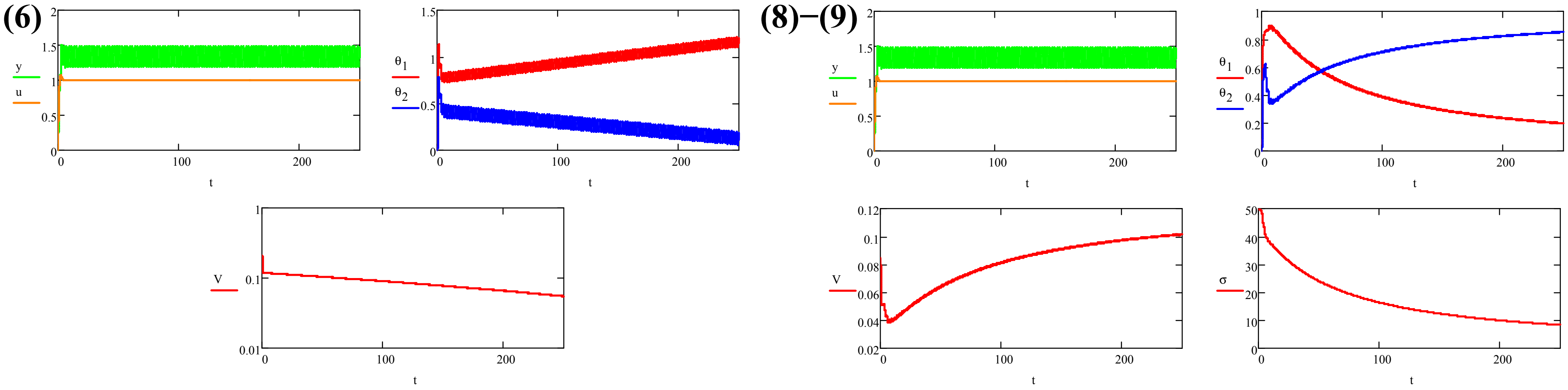


(8)−(9)

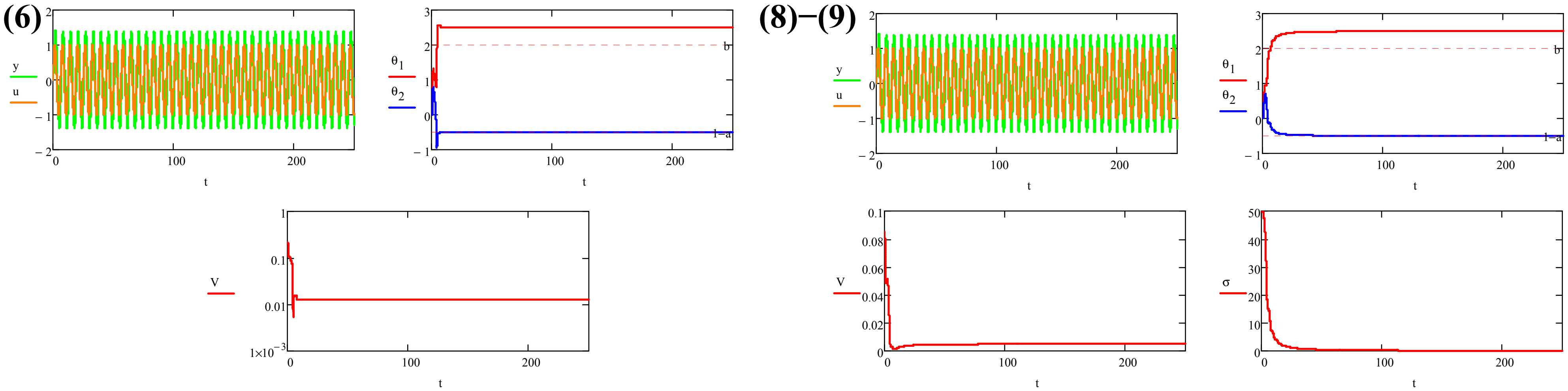


$\phi(t)$ **is PE $\Rightarrow$ Robustness!!!**

$$u(t) = 1 - e^{-t}\cos(t)$$



$$u(t) = \sin(t),\; d(t) = 0.5\sin(t)$$



<u>Conclusion</u>: the disturbance can seriously modify the system behavior.

Linear parametric model with modeling errors:

$$y(t) = \phi(t)^T \boldsymbol{\theta}^* + \delta(t),\ t \geq t_0,$$

where $\boldsymbol{\theta}^* \in \mathbb{R}^{n_\theta}$ is an unknown parameter vector, $\phi(t) \in \mathbb{R}^{n_\theta}$ is a known regressor, $y(t) \in \mathbb{R}$ is a measured output, $\delta(t) \in \mathbb{R}$ represents **system modeling errors**:

$$|\delta(t)| \leq c_1 \|\phi(t)\| + c_2,\ c_1 > 0,\ c_2 > 0.$$

Let $\boldsymbol{\theta}(t) \in \mathbb{R}^{n_\theta}$ be the estimate of $\boldsymbol{\theta}^*$ and define the **estimation error**

$$\varepsilon(t) = \phi(t)^T \boldsymbol{\theta}(t) - y(t) = \phi(t)^T \tilde{\boldsymbol{\theta}}(t) + \delta(t),\ t \geq t_0,$$

where $\tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}^*$ is the **parametric error**.

Modified gradient algorithm (6):

$$\dot{\boldsymbol{\theta}}(t) = -\varepsilon(t)\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2} + \boldsymbol{\Gamma}\mathbf{f}(t),\ \boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0,\ m(t) = \sqrt{1 + \kappa\phi(t)^T \phi(t)},\ \kappa > 0,\ t \geq t_0, \qquad (12)$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > 0$ is a design matrix gain, $\mathbf{f}(t) \in \mathbb{R}^{n_\theta}$ is the modification term for robustness.

Stability & robustness analysis for nonlinear systems ⇔ Lyapunov function theory

$$V(\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}},\ \dot{V} = -\frac{\varepsilon(t)^2}{m(t)^2} + \frac{\varepsilon(t)\delta(t)}{m(t)^2} + \tilde{\boldsymbol{\theta}}^T \mathbf{f}(t)$$

Note:

$$\frac{|\delta(t)|}{m(t)} \le \frac{c_1 \|\phi(t)\| + c_2}{\sqrt{1+\kappa\phi(t)^T\phi(t)}} \le \frac{c_1}{\sqrt{\kappa}} + \frac{c_2}{m(t)} \le \frac{c_1}{\sqrt{\kappa}} + c_2.$$

Then

$$\dot{V} \le -\frac{\varepsilon(t)^2}{m(t)^2} + \left[\frac{c_1}{\sqrt{\kappa}} + \frac{c_2}{m(t)}\right]\frac{|\varepsilon(t)|}{m(t)} + \tilde{\boldsymbol{\theta}}^T \mathbf{f}(t),$$

and

$$\frac{|\varepsilon(t)|}{m(t)} \ge \frac{c_1}{\sqrt{\kappa}} + \frac{c_2}{m(t)} \Rightarrow -\frac{\varepsilon(t)^2}{m(t)^2} \le -\left[\frac{c_1}{\sqrt{\kappa}} + \frac{c_2}{m(t)}\right]\frac{|\varepsilon(t)|}{m(t)} \Rightarrow \dot{V} \le \tilde{\boldsymbol{\theta}}^T \mathbf{f}(t).$$

The simplest modification:

$$\mathbf{f}(t) = \frac{\phi(t)}{m(t)^2} f_s(t),\ f_s(t) = \begin{cases} 0 & \text{if } |\varepsilon(t)|/m(t) \ge c_1/\sqrt{\kappa} + c_2/m(t), \\ \varepsilon(t) & \text{otherwise.} \end{cases} \Rightarrow \dot{V} \le 0.$$

$\Rightarrow \dot{\boldsymbol{\theta}}(t) = 0!$

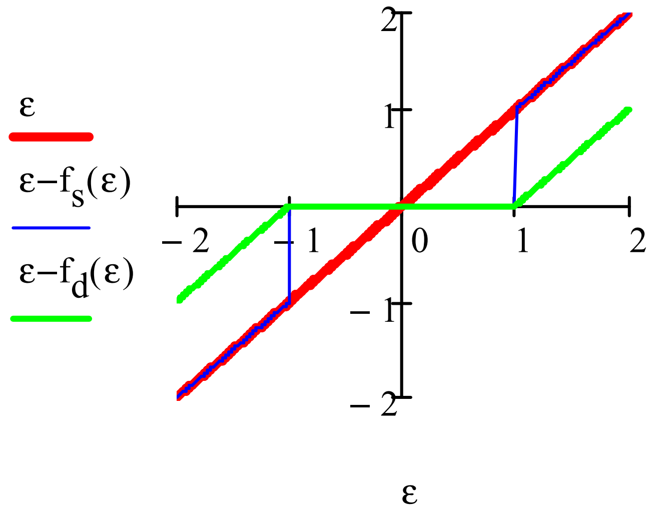
The simplest modification:

$$\mathbf{f}(t)=\frac{\phi(t)}{m(t)^2}f_s(t),\ f_s(t)=\begin{cases}\varepsilon(t) & \text{if}\ |\varepsilon(t)|/m(t)<c_1/\sqrt{\kappa}+c_2/m(t),\\ 0 & \text{otherwise.}\end{cases}\Rightarrow \dot{V}\le 0.$$

A dead zone modification:

$$\mathbf{f}(t)=\frac{\phi(t)}{m(t)^2}f_d(t),\ f_d(t)=\begin{cases}\varepsilon(t) & \text{if}\ |\varepsilon(t)|/m(t)<c_1/\sqrt{\kappa}+c_2/m(t),\\ [c_1m(t)/\sqrt{\kappa}+c_2]sign[\varepsilon(t)] & \text{otherwise.}\end{cases}\Rightarrow \dot{V}\le 0.$$

$$\dot{\boldsymbol{\theta}}(t)=-\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2}[\varepsilon(t)-f_d(t)]\ \Rightarrow$$



σ-Modification:

$$\mathbf{f}(t)=-\sigma\boldsymbol{\theta}(t)\Rightarrow \boxed{\dot{\boldsymbol{\theta}}(t)=-\sigma\boldsymbol{\Gamma}\boldsymbol{\theta}(t)}-\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2}\varepsilon(t)\Rightarrow \boldsymbol{\theta}(t)\in L_\infty.$$

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Projection: assume that the set of admissible values for $\boldsymbol{\theta}^*$ is given, i.e.

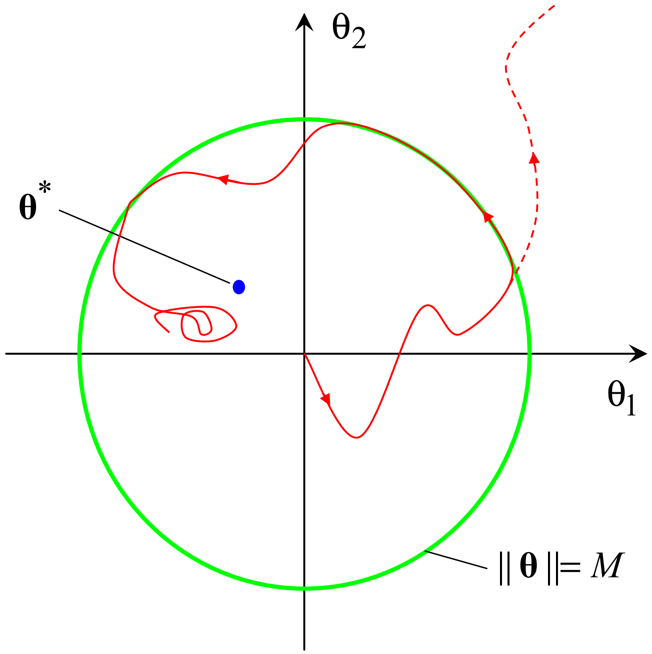
$$\boldsymbol{\theta}^* \in \Omega = \{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta} : \|\boldsymbol{\theta}\| \le M\},\ M > 0.$$

Projection has to ensure that $\boldsymbol{\theta}(t) \in \Omega$ for all $t \ge t_0$, therefore

$$\mathbf{f}(t) = \begin{cases} 0 \quad \text{if } \|\boldsymbol{\theta}(t)\| < M \text{ or } \|\boldsymbol{\theta}(t)\| = M \text{ and } \boldsymbol{\theta}(t)^T \boldsymbol{\Gamma} \dfrac{\phi(t)}{m(t)^2} \varepsilon(t) \le 0, \\ \dfrac{\boldsymbol{\Gamma}\boldsymbol{\theta}(t)\boldsymbol{\theta}(t)^T}{\boldsymbol{\theta}(t)^T \boldsymbol{\Gamma}\boldsymbol{\theta}(t)} \boldsymbol{\Gamma} \dfrac{\phi(t)}{m(t)^2} \varepsilon(t) \quad \text{otherwise.} \end{cases}$$

Inside the circle doing nothing.

On an attempt to exit the circle.



<u>The properties</u>:

− boundedness of $\boldsymbol{\theta}(t)$, $\dot{\boldsymbol{\theta}}(t)$ and $\varepsilon(t)\,/\,m(t)$ (belong to L_∞);

− $\varepsilon(t)\,/\,m(t)$ and $\dot{\boldsymbol{\theta}}(t)$ belong to L_2;

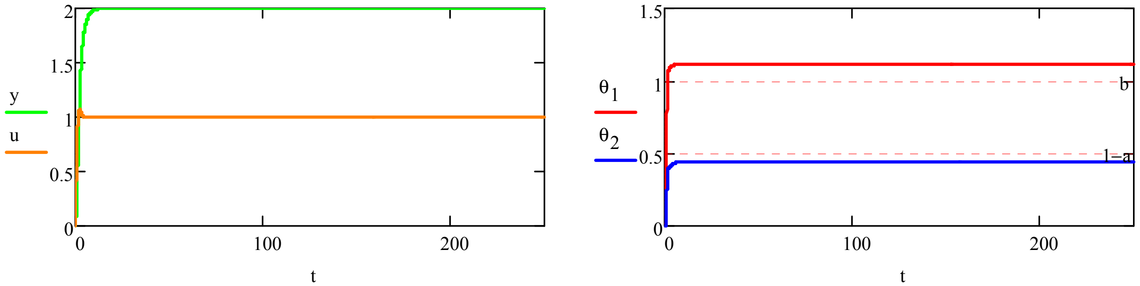
} **ROBUSTNESS!**

− in the noise-free case ($\mathbf{d}(t)=0$) the quality is preserved? ⇒ **ESTIMATION?**

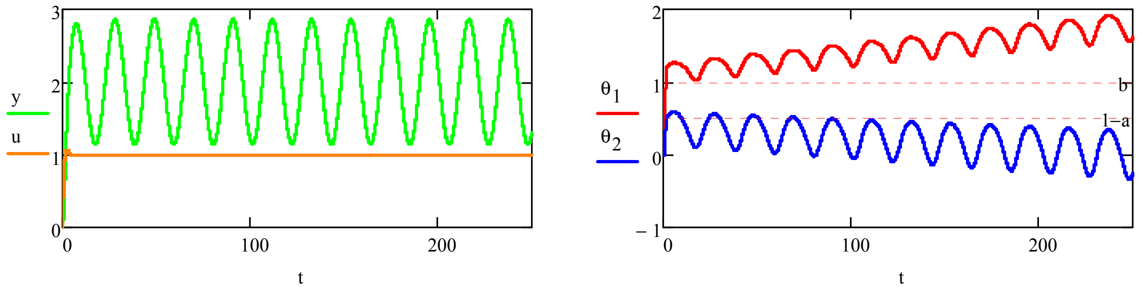
Example 1

Plant: $$\dot{y} = -ay + bu + d(t),$$

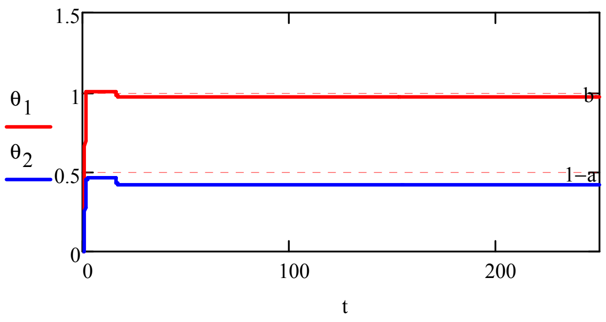
$$a = 0.5,\ b = 1 \text{ and } u(t) = u(t) = 1 - e^{-t}\cos(t);\ (6) \text{ with } \gamma = 20.$$

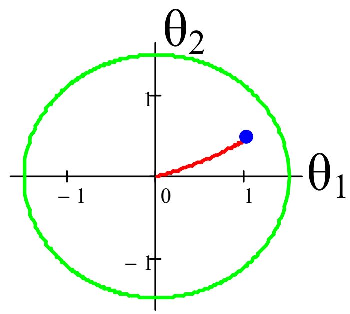


$$d(t) = 0.5\sin(0.3t)$$

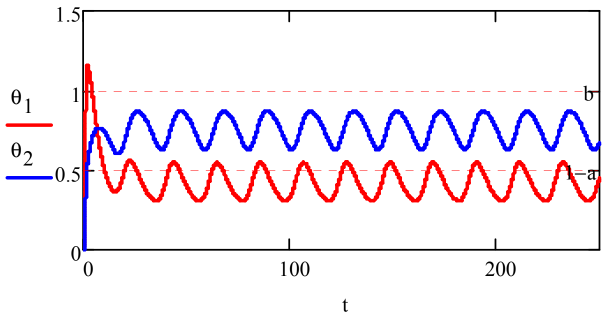


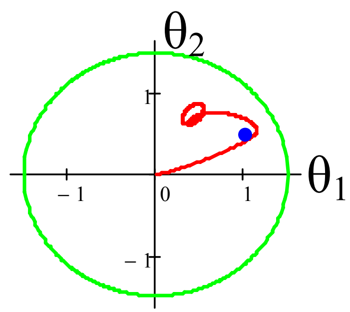
Dead zone algorithm:



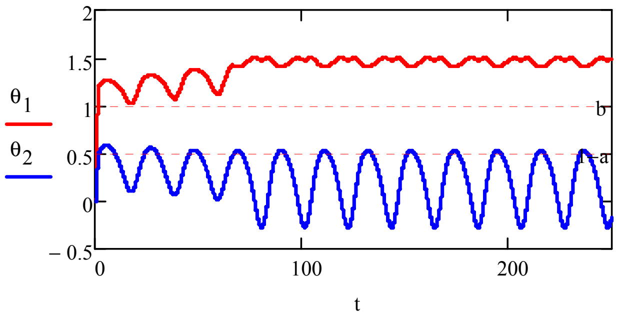


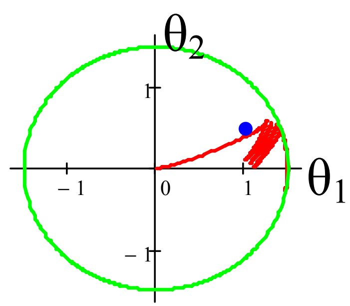
σ-Modification (σ = 0.01):





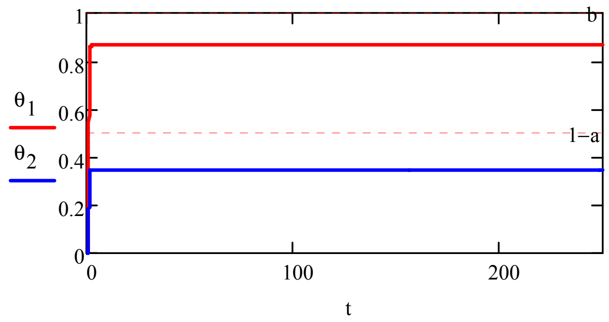
Projection (M = 1.5):

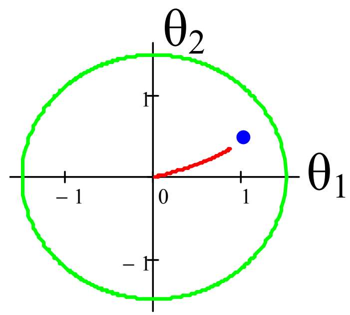




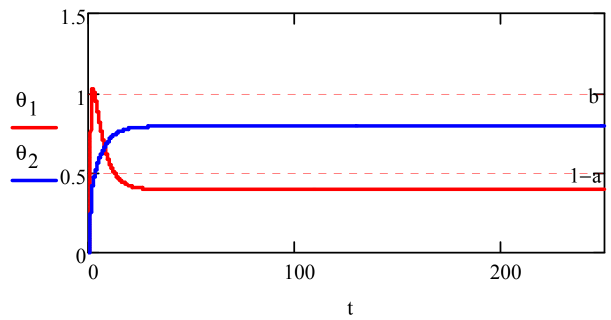
$d(t) = 0.5 \sin(0.3t)$

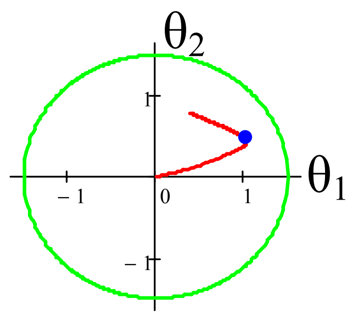
Dead zone algorithm:





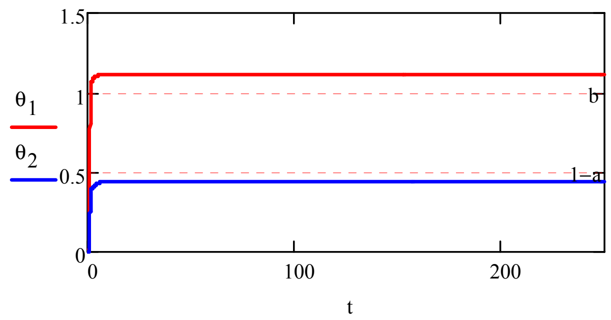
σ-Modification (σ = 0.01):

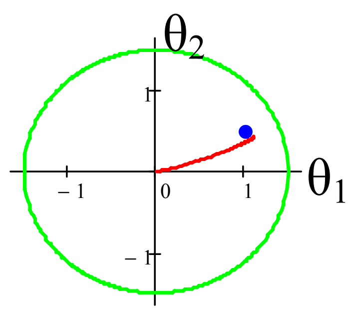




$d(t)=0$

Projection (M = 1.5):





SUMMARY

1. Adaptive parameter estimation:

   a. Parameterized system model $y(t) = \phi(t)^T \boldsymbol{\theta}^*$.

   b. Linear parametric model $\varepsilon(t) = \phi(t)^T \boldsymbol{\theta}(t) - y(t) = \phi(t)^T \tilde{\boldsymbol{\theta}}(t),\ \tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}^*$.

   c. Normalized gradient algorithm $\dot{\boldsymbol{\theta}}(t) = -\varepsilon(t)\boldsymbol{\Gamma}\dfrac{\phi(t)}{m(t)^2}$.

   d. Normalized least-squares algorithm $\dot{\boldsymbol{\theta}}(t) = -\varepsilon(t)\dfrac{\mathbf{P}(t)\phi(t)}{m(t)^2},\ \dot{\mathbf{P}}(t) = -\dfrac{\mathbf{P}(t)\phi(t)\phi(t)^T \mathbf{P}(t)}{m(t)^2}$.

   e. Discrete-time version of adaptive algorithms.

2. Identification and robustness:

   f. Parametric convergence and PE (PE $\Rightarrow$ convergence/estimation $\Rightarrow$ robustness).

   g. Robustness of adaptive algorithms (robustness $\not\Leftrightarrow$ estimation).

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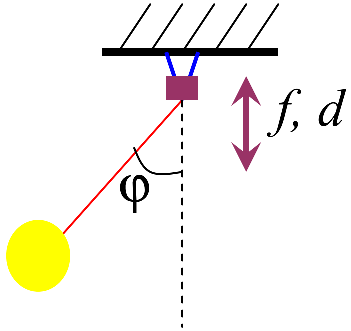
Example 2

Oscillating pendulum:



$\varphi \in [-\pi, \pi)$ is the pendulum angle, $f \in \mathbb{R}$ is the (controlling or exciting) input applied to the support, $d \in \mathbb{R}$ is the disturbance influencing the support also.

Nonlinear model:

$$\ddot{y} = -\omega^2 \sin(y) - \rho \dot{y} + b\cos(y) f(t) + d(t), \tag{13}$$

$y = \varphi \in [-\pi, \pi)$ is the measured angle, $\dot{y} \in \mathbb{R}$ and $\ddot{y} \in \mathbb{R}$ are the angle velocity and acceleration; $\rho > 0$ is an unknown friction coefficient, $\omega > 0$ is an unknown natural frequency, $b > 0$ is an unknown control gain.

3 unknown parameters + nonlinearity. $\Rightarrow$ Define $u_1 = \sin(y)$ and $u_2 = \cos(y)u$:

$$\ddot{y} + \rho \dot{y} = -\omega^2 u_1(t) + b u_2(t) + d(t) \Rightarrow \text{(2) for } n = 2,\ m = 1 \text{ and a vector } \mathbf{u} = [u_1, u_2]^T .$$

Define the polynomials:

$$P(s) = s^2 + p_1 s,\ p_1 = \rho;\ Z_1(s) = z_{1,0} = -\omega^2;\ Z_2(s) = z_{2,0} = b,$$

then the noise-free model (13) has the form $P(s)[y](t) = Z_1(s)[u_1](t) + Z_2(s)[u_2](t)$.

Parameterization for $\Lambda(s) = s^2 + \lambda_1 s + \lambda_0$:

$$\frac{P(s)}{\Lambda(s)}[y](t) = \frac{Z_1(s)}{\Lambda(s)}[u_1](t) + \frac{Z_2(s)}{\Lambda(s)}[u_2](t) \Rightarrow$$

$$(1 - \frac{\Lambda(s)}{\Lambda(s)})[y](t) + \frac{P(s)}{\Lambda(s)}[y](t) = \frac{Z_1(s)}{\Lambda(s)}[u_1](t) + \frac{Z_2(s)}{\Lambda(s)}[u_2](t) \Rightarrow$$

$$y(t) = \frac{\Lambda(s) - P(s)}{\Lambda(s)}[y](t) + \frac{Z_1(s)}{\Lambda(s)}[u_1](t) + \frac{Z_2(s)}{\Lambda(s)}[u_2](t) \Rightarrow \boxed{\tilde{y}(t) = \phi(t)^T \boldsymbol{\theta}^*},$$

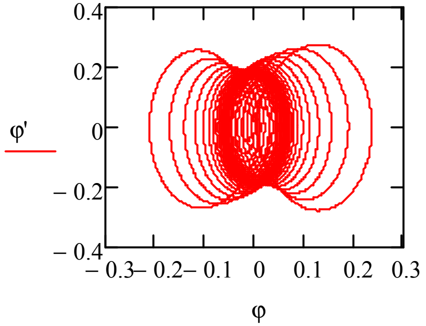
the parameterized system model for $\tilde{y}(t) = y(t) - \lambda_0 \Lambda^{-1}(s)[y](t)$, $\boldsymbol{\theta}^* = [\lambda_1 - \rho, -\omega^2, b]^T$ and

$$\phi(t) = \left[\frac{s}{\Lambda(s)}[y](t), \frac{1}{\Lambda(s)}[u_1](t), \frac{1}{\Lambda(s)}[u_2](t) \right]^T = [\omega_{0,2}, \omega_{1,1}, \omega_{2,1}]^T,\ \tilde{y}(t) = y(t) - \lambda_0 \omega_{0,1}(t),$$

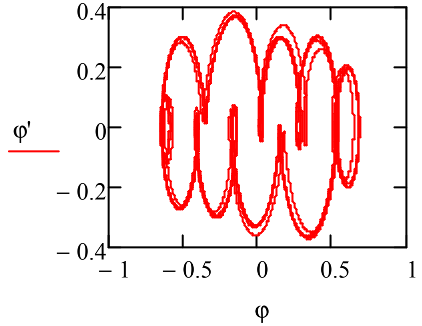
$$\begin{aligned} \dot{\omega}_0(t) &= \mathbf{A}_\lambda \omega_0(t) + \mathbf{b} y(t), \\ \dot{\omega}_1(t) &= \mathbf{A}_\lambda \omega_1(t) + \mathbf{b} u_1(t), \\ \dot{\omega}_2(t) &= \mathbf{A}_\lambda \omega_2(t) + \mathbf{b} u_2(t), \end{aligned} \qquad \mathbf{A}_\lambda = \begin{bmatrix} 0 & 1 \\ -\lambda_0 & -\lambda_1 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\omega = 1,\ \rho = 0.1,\ b = 0.5,\ f(t) = \sin(3t),\ \lambda_0 = 1,\ \lambda_1 = 2,\ \gamma = 100.$$

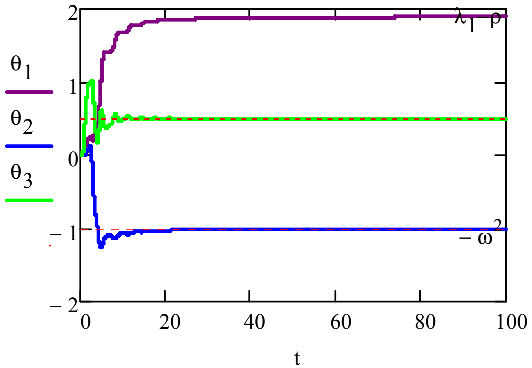
$d(t) = 0$

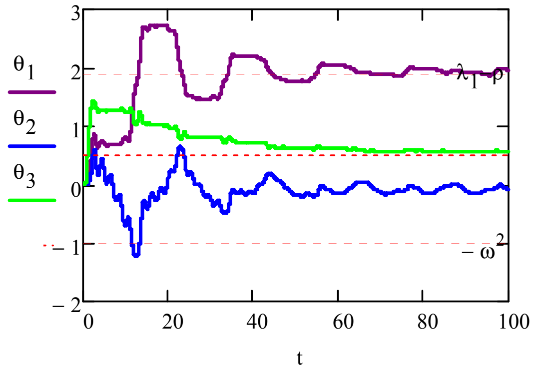


$d(t) = 0.5\sin(0.3t)$

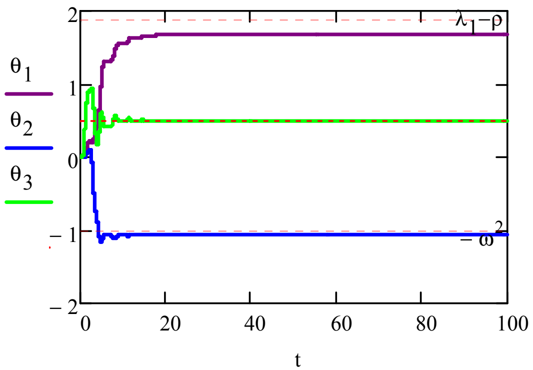


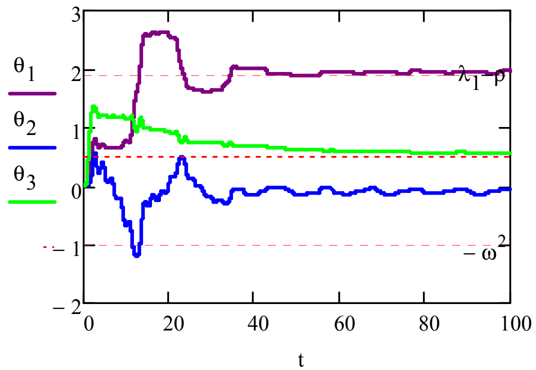
Normalized gradient algorithm



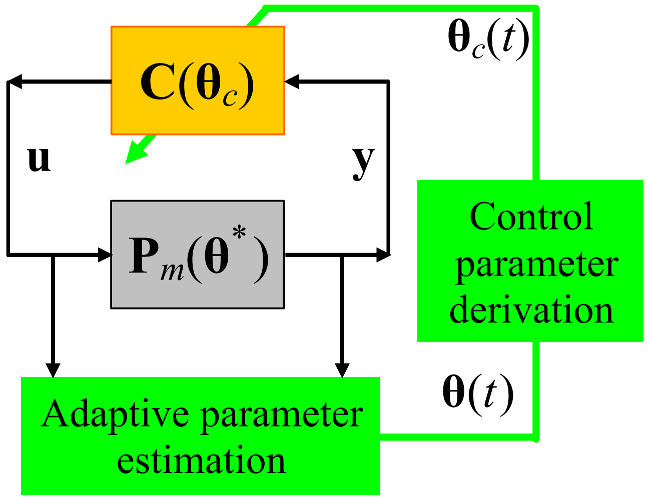


Dead zone modification





4. INDIRECT ADAPTIVE CONTROL



<u>Adjustment of control parameters</u>:

− *direct* (from an adaptive control law/Lyapunov analysis);
− *indirect* (from adaptive estimates of the system parameters).

<u>Indirect adaptive control design</u>:

1) adaptive estimation of the plant parameters;
2) calculation of control parameters.

a. Model reference control

<u>The main steps</u>:

1) adaptive estimation algorithm design;

2) reference model selection;

3) controller structure construction;

4) controller parameter calculation;

5) stability and robustness analysis.

Example 1

Plant: $\dot{y} = -ay + bu + d$.

Adaptive estimation algorithm ($\boldsymbol{\theta}^* = [\theta_1^*, \theta_2^*]^T = [b, 1-a]^T$):

$$\dot{\boldsymbol{\theta}}(t) = -\gamma \frac{\varepsilon(t)}{m(t)^2} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}, \; m(t) = \sqrt{1 + \omega_1^2(t) + \omega_2^2(t)}, \; \begin{matrix} \dot{\omega}_1(t) = -\omega_1(t) + u(t), \\ \dot{\omega}_2(t) = -\omega_2(t) + y(t), \end{matrix} \; \mathbf{A}_\lambda = -1, \; \mathbf{b} = 1.$$

Reference model: $\dot{y}_m = -a_m y_m + b_m r(t)$

where $r(t) \in \mathbb{R}$ is the reference signal to be tracked, $a_m > 0$ (the reference model is stable).

Controller structure: $u = b^{-1}[(a - a_m)y + b_m r] \Rightarrow \dot{y} = -a_m y + b_m r + d$.

Controller parameter calculation:

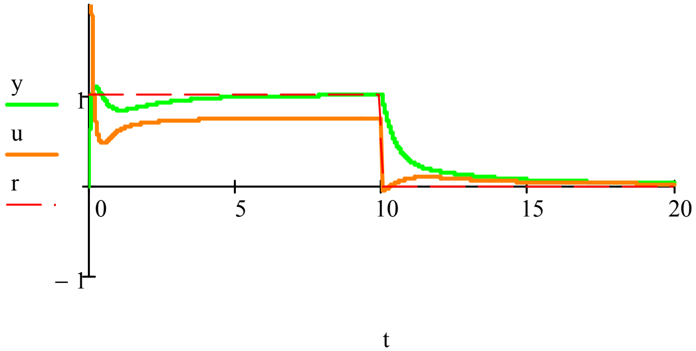
$$u = \theta_1^c y + \theta_2^c r, \; \theta_1^c = \theta_1^{-1}(1 - \theta_2 - a_m), \; \theta_2^c = \theta_1^{-1} b_m.$$

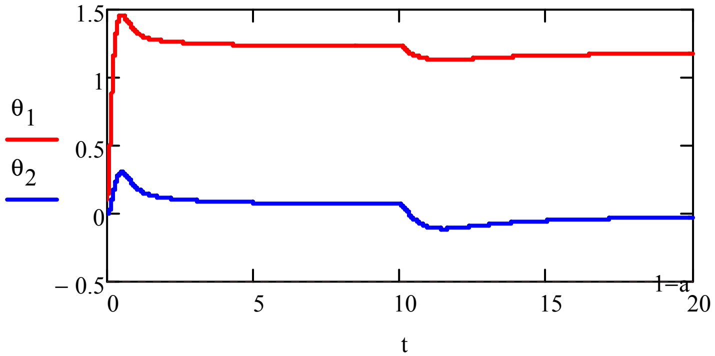
Division on $\theta_1 \Rightarrow$ projection modification of the adaptation algorithm:

$$\dot{\boldsymbol{\theta}}(t) = -\gamma \frac{\varepsilon(t)}{m(t)^2} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}, \; f_1(t) = \begin{cases} 0 & \text{if } \theta_1(t) > b_{\min} \text{ or } \theta_1(t) = b_{\min} \text{ and } \varepsilon(t)\omega_1(t) \geq 0, \\ \gamma \varepsilon(t) \omega_1(t) m(t)^{-2} & \text{otherwise.} \end{cases}$$

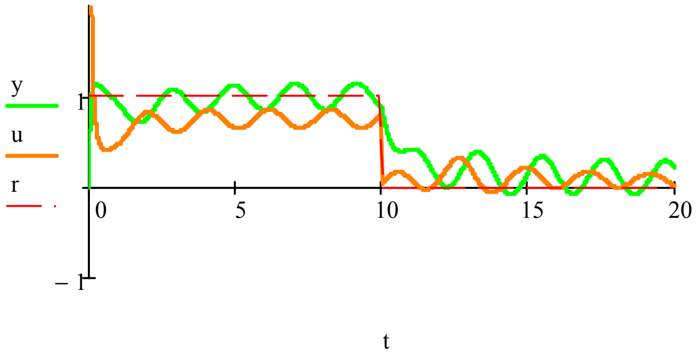
$b_{\min} > 0$ is the low bound for b, i.e. $b \geq b_{\min}$.

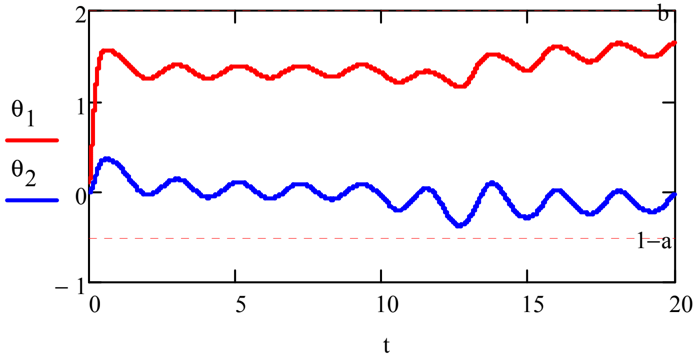
$$a = 1.5,\ b = 2,\ a_m = 1,\ b_m = 1,\ b_{\min} = 0.1$$



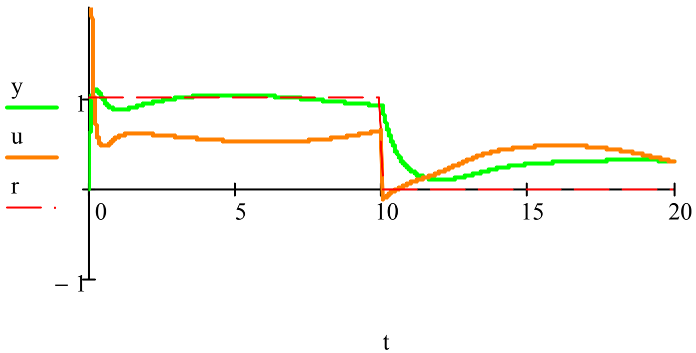


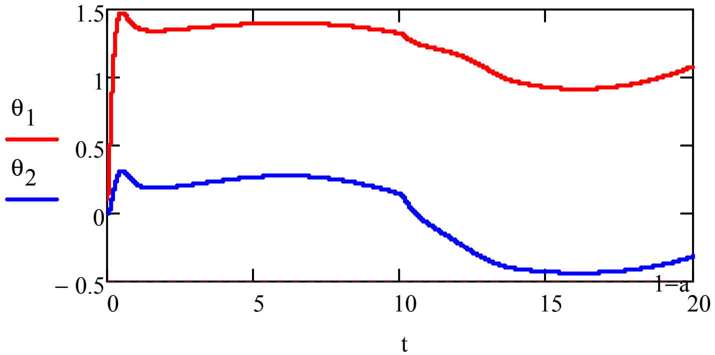
$d(t) = 0$





$d(t) = 0.5\sin(3t)$





$d(t) = 0.5\sin(0.3t)$

The general procedure:

$$P(s)[y](t) = k_p Z(s)[u](t) + d(t),\ t \ge 0, \tag{14}$$

$y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ are the measured output and input as before;

$$P(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_1 s + p_0,\ Z(s) = s^m + z_{m-1}s^{m-1} + \ldots + z_1 s + z_0,$$

k_p, p_i, $i = \overline{0, n-1}$ and z_j, $j = \overline{0, m-1}$ are the unknown but constant parameters.

Assumption 1. *The constant* $|k_p| \ge k_{\min} > 0$ *and* $sign(k_p)$ *are given.* $\Rightarrow$ Necessary.

Assumption 2. $\underline{k} \le k_p \le \overline{k}$; $\underline{p_i} \le p_i \le \overline{p}_i$, $i = \overline{0, n-1}$; $\underline{z_j} \le z_j \le \overline{z}_j$, $j = \overline{0, m-1}$. $\Rightarrow$ Desired.

1) Adaptive estimation algorithm design:

$$y(t) = k_p \frac{Z(s)}{\Lambda(s)}[u](t) + \frac{\Lambda(s) - P(s)}{\Lambda(s)}[y](t),\ \Lambda(s) = s^n + \lambda_{n-1}s^{n-1} + \ldots + \lambda_1 s + \lambda_0 \Rightarrow$$

$$y(t) = \boldsymbol{\theta}^{*T}\phi(t),\ \boldsymbol{\theta}^* = [k_p z_0, \ldots, k_p z_{m-1}, k_p, \lambda_0 - p_0, \ldots, \lambda_{n-1} - p_{n-1}]^T,$$

$$\phi(t) = [\{\mathbf{C}_m \omega_1(t)\}^T, \omega_2(t)^T]^T,\ \mathbf{C}_m = [\mathbf{I}_{m+1}, \mathbf{0}_{(m+1)\times(n-m-1)}],$$

$$\dot{\omega}_1(t) = \mathbf{A}_\lambda \omega_1(t) + \mathbf{b}u(t),$$

$$\dot{\omega}_2(t) = \mathbf{A}_\lambda \omega_2(t) + \mathbf{b}y(t). \tag{15}$$

Normalized gradient algorithm with projection (assumption 2):

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{g}(t) + \mathbf{f}(t),\ \boldsymbol{\theta}(0) = \boldsymbol{\theta}_0,\ t \geq 0, \tag{16}$$

$$\mathbf{g}(t) = -\varepsilon(t)\boldsymbol{\Gamma}\frac{\phi(t)}{m(t)^2},\ \varepsilon(t) = \boldsymbol{\theta}(t)^T\phi(t) - y(t),\ m(t) = \sqrt{1 + \kappa\phi(t)^T\phi(t)},$$

$$f_k(t) = \begin{cases} 0 \text{ if } \underline{\theta}_k < \theta_k(t) < \overline{\theta}_k \text{ or} \\ \qquad \theta_k(t) = \underline{\theta}_k \text{ and } g_k(t) \geq 0 \text{ or} \\ \qquad \theta_k(t) = \overline{\theta}_k \text{ and } g_k(t) \leq 0, \\ -g_k(t) \text{ otherwise,} \end{cases},\ k = \overline{1, n_\theta}.$$

Properties:

$$\boldsymbol{\theta}(t),\ \dot{\boldsymbol{\theta}}(t),\ \varepsilon(t)/m(t) \in L_\infty \text{ and } \dot{\boldsymbol{\theta}}(t),\ \varepsilon(t)/m(t) \in L_2.$$

2) Reference model selection:

$$P_m(s)[y_m](t) = r(t), \tag{17}$$

where $P_m(s)$ is a stable polynomial of degree $n - m$ and $r(t)$ is a bounded and piecewise continuous reference input signal.

3) Controller structure construction:

$$u(t) = \omega_1^c(t)^T \theta_1^c + \omega_2^c(t)^T \theta_2^c + \theta_3^c y(t) + \theta_4^c r(t), \tag{18}$$

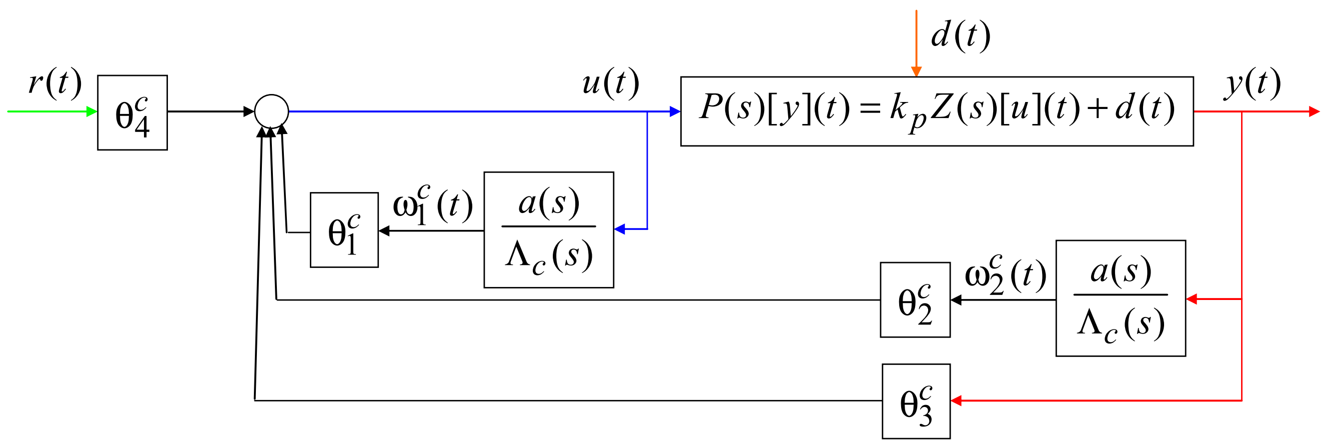
where $\theta_1^c \in \mathbb{R}^n$, $\theta_2^c \in \mathbb{R}^n$, $\theta_3^c \in \mathbb{R}$, $\theta_4^c \in \mathbb{R}$ are the controller parameters,

$$\omega_1^c(t) = \frac{a(s)}{\Lambda_c(s)}[u](t),\ \omega_2^c(t) = \frac{a(s)}{\Lambda_c(s)}[y](t),\ a(s) = [1, s, \ldots, s^{n-2}]^T,$$

and $\Lambda_c(s) = s^{n-1} + \lambda_{n-2}^c s^{n-2} + \ldots + \lambda_1^c s + \lambda_0^c$ is a stable polynomial. A variant of realization:

$$\dot{\omega}_1^c(t) = \mathbf{A}_\lambda^c \dot{\omega}_1^c(t) + \mathbf{b}^c u(t),$$
$$\dot{\omega}_2^c(t) = \mathbf{A}_\lambda^c \dot{\omega}_2^c(t) + \mathbf{b}^c y(t),$$

$$\mathbf{A}_\lambda^c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -\lambda_0^c & -\lambda_1^c & \cdots & \cdots & -\lambda_{n-3}^c & -\lambda_{n-2}^c \end{bmatrix},\ \mathbf{b}^c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$



$\approx (15)$

The controller parameter equation:

$$a(s)^T\theta_1^c P(s)+[a(s)^T\theta_2^c+\theta_3^c\Lambda_c(s)]k_pZ(s)=\Lambda_c(s)[P(s)-k_p\theta_4^cZ(s)P_m(s)]. \qquad (19)$$

Multiply (19) on $y(t)$ and substitute (14) for the case $d(t)=0$:

$$a(s)^T\theta_1^c P(s)[y](t)+[a(s)^T\theta_2^c+\theta_3^c\Lambda_c(s)]k_pZ(s)[y](t)= \Rightarrow$$
$$=\Lambda_c(s)P(s)[y](t)-k_p\theta_4^c\Lambda_c(s)Z(s)P_m(s)[y](t),$$

$$a(s)^T\theta_1^c k_pZ(s)[u](t)+[a(s)^T\theta_2^c+\theta_3^c\Lambda_c(s)]k_pZ(s)[y](t)=$$
$$=\Lambda_c(s)k_pZ(s)[u](t)-k_p\theta_4^c\Lambda_c(s)Z(s)P_m(s)[y](t).$$

Now divide both sides on $\Lambda_c(s)k_pZ(s)$ ($Z(s)$ and $\Lambda_c(s)$ are stable polynomials):

$$\frac{a(s)^T\theta_1^c}{\Lambda_c(s)}[u](t)+\frac{a(s)^T\theta_2^c+\theta_3^c\Lambda_c(s)}{\Lambda_c(s)}[y](t)=u(t)-\theta_4^cP_m(s)[y](t), \Rightarrow$$

$$\omega_1^c(t)^T\theta_1^c+\omega_4^c(t)^T\theta_2^c+\theta_3^cy(t)=u(t)-\theta_4^cP_m(s)[y](t).$$

Substitution of the control (18) gives

$$\theta_4^cP_m(s)[y](t)=\theta_4^cr(t) \Rightarrow \theta_4^cP_m(s)[y](t)=\theta_4^cP_m(s)[y_m](t) \Rightarrow P_m(s)\{[y](t)-[y_m](t)\}=0.$$

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4) Controller parameter calculation:

$\theta_4^c = k_p^{-1} \Rightarrow B(s) = \Lambda_c(s)[P(s) - Z(s)P_m(s)]$, then (19) takes the form:

$$a(s)^T \theta_1^c P(s) + [a(s)^T \theta_2^c + \theta_3^c \Lambda_c(s)] k_p Z(s) = B(s).$$

The right hand side is a polynomial of degree $2n-2$ with coefficients linearly dependent on θ_1^c, θ_2^c and θ_3^c. The left hand side is a polynomial of degree $2n-2$ with constant coefficients.

Equating the coefficients with the same powers of s we obtain the solution:

$$\theta_1^c = \Theta_1(p_{n-1},\ldots,p_0; z_{m-1},\ldots,z_0; \lambda_{n-2},\ldots,\lambda_0),\ \theta_2^c = \Theta_1(p_{n-1},\ldots,\lambda_0),\ \theta_3^c = \Theta_1(p_{n-1},\ldots,\lambda_0)$$

$$\theta_1^c = \Theta_1(\boldsymbol{\theta}^*; \lambda_{n-2},\ldots,\lambda_0),\ \theta_2^c = \Theta_1(\boldsymbol{\theta}^*; \lambda_{n-2},\ldots,\lambda_0),\ \theta_3^c = \Theta_1(\boldsymbol{\theta}^*; \lambda_{n-2},\ldots,\lambda_0).$$

Example 1: $\theta_1^c = \theta_1^{-1}(1 - \theta_2 - a_m),\ \theta_2^c = \theta_1^{-1} b_m.$

Theorem 1. *Under assumption 2 and that all zeros of $Z(s)$ are stable*:

(i) $y(t)$, $\boldsymbol{\theta}(t)$, $\dot{\boldsymbol{\theta}}(t)$, $\omega_1(t)$, $\omega_2(t) \in L_\infty$;

(ii) $y(t) - y_m(t) \in L_2$, $\lim_{t\to\infty}[y(t) - y_m(t)] = 0$. ■

b. Pole placement control

The pole placement equation:

$$A^*(s) = C(s)Q(s)P(s) + D(s)Z(s), \qquad (20)$$

where $A^*(s)$ is the desired polynomial of the closed loop system; $C(s)$ and $D(s)$ are polynomials of **the pole placement control**:

$$u(t) = \{\Lambda_c(s) - C(s)Q(s)\}\Lambda_c^{-1}(s)[u](t) + D(s)\Lambda_c^{-1}(s)[r-y](t), \qquad (21)$$

where $r(t)$ is a bounded and piecewise continuous reference input signal, $Q(s)[r](t) = 0 \Rightarrow$

(a) $r(t) = 0 \Rightarrow Q(s) = 1$;
(b) $r(t) = c \neq 0 \Rightarrow Q(s) = s$;
(c) $r(t) = ce^{-at} \Rightarrow Q(s) = s + a$, $a > 0$.

According to (21) the control is a dynamical system:

$$C(s)Q(s)[u](t) = D(s)[r-y](t). \qquad (22)$$

Controller structure ($a_1(s) = [1, s, \ldots, s^{n_\theta + n - 2}]^T$):

$$u(t) = \theta_1^{cT} a_1(s)\Lambda_c(s)[u](t) + \theta_2^{cT} a_1(s)\Lambda_c(s)[y-r](t) + \theta_3^c\{y(t) - r(t)\}.$$

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Properties:

1) multiplying both sides of (20) on $y(t)$ we obtain:

$$\begin{aligned} A^*(s)[y](t) &= C(s)Q(s)P(s)[y](t) + D(s)Z(s)[y](t) = \\ &= C(s)Q(s)P(s)[y](t) + Z(s)\{D(s)[r](t) - C(s)Q(s)[u](t)\} = \\ &= Z(s)D(s)[r](t). \end{aligned} \tag{23}$$

$$r(t) \in L_\infty \text{ and } A^* \text{ is stable} \Rightarrow y(t) \in L_\infty .$$

2) multiplying both sides of (20) on $u(t)$ we obtain:

$$\begin{aligned} A^*(s)[u](t) &= C(s)Q(s)P(s)[u](t) + D(s)Z(s)[u](t) = \\ &= P(s)D(s)[r-y](t) + D(s)Z(s)[u](t) = P(s)D(s)[r](t). \end{aligned}$$

$$r(t) \in L_\infty \text{ and } A^* \text{ is stable} \Rightarrow u(t) \in L_\infty .$$

3) using (20)−(23) we get:

$$A^*(s)[y-r](t) = 0 \Rightarrow \lim_{t\to\infty}\{y(t) - r(t)\} = 0 .$$

Assumption 3. $Q(s)P(s)$ and $Z(s)$ are coprime.

Theorem 2. *Under assumption 3 all signals are bounded and* $\lim_{t\to\infty}[y(t) - r(t)] = 0$. ■

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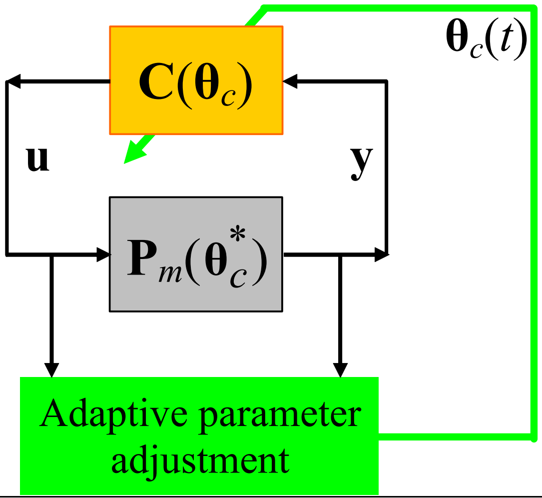
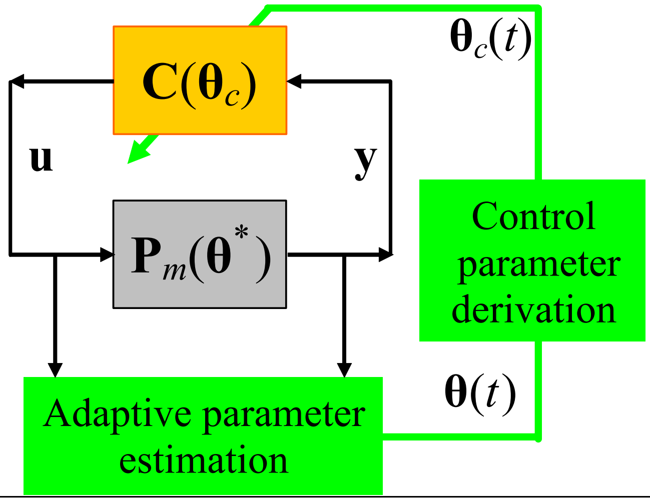
SUMMARY





	Direct adaptive control	**Indirect adaptive control**
Structure	+	–
Parameterization	–	+
Restrictions	(minimum phase)	

Certainty equivalence

Example 1

Indirect adaptive control ⇔ Robust control

Plant: $\dot{y} = -ay + bu + d\,.$

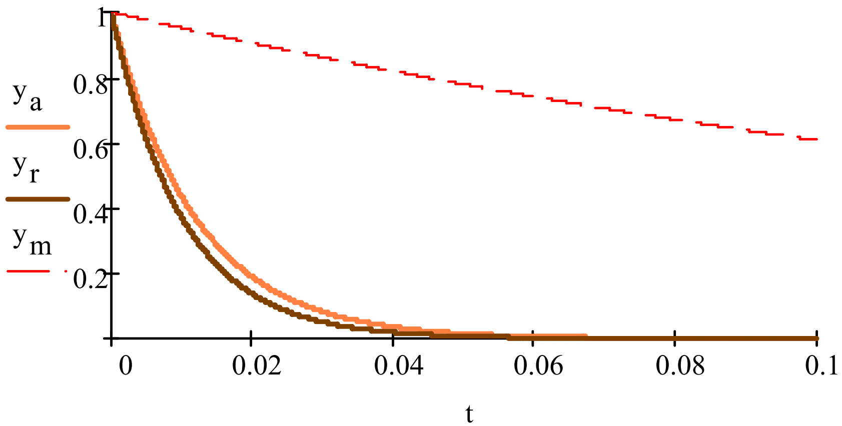
Assumption: $0 < \underline{a} \le a,\ 0 < \underline{b} \le b\ +\ a_m > 0,\ r(t) = 0\,.$

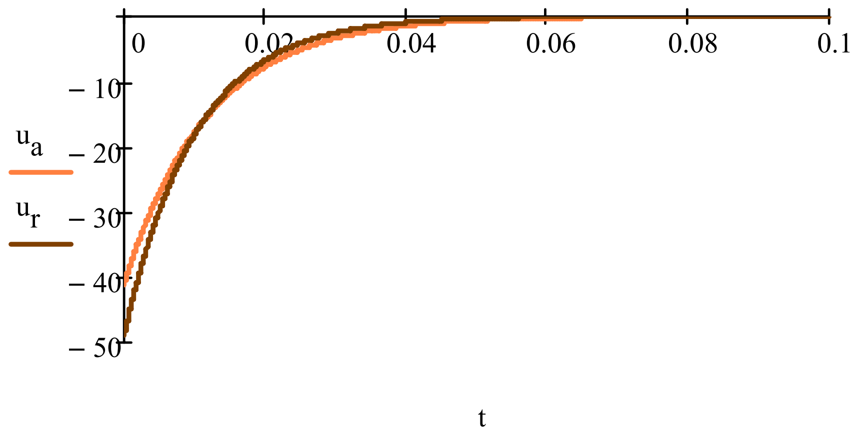
⇓

Normalized gradient descent algorithm with projection.

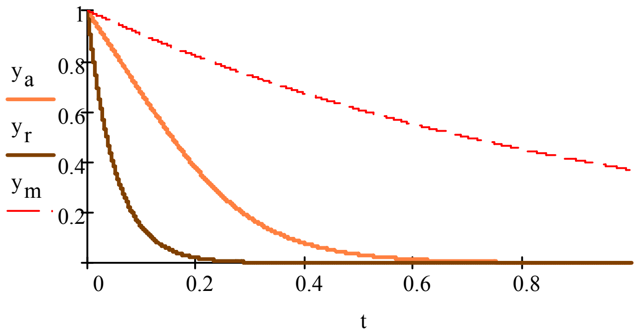
Robust control: $u = ky\,,\ k = \min\{\underline{b}^{-1}(\underline{a} - a_m), 0\}\,.$

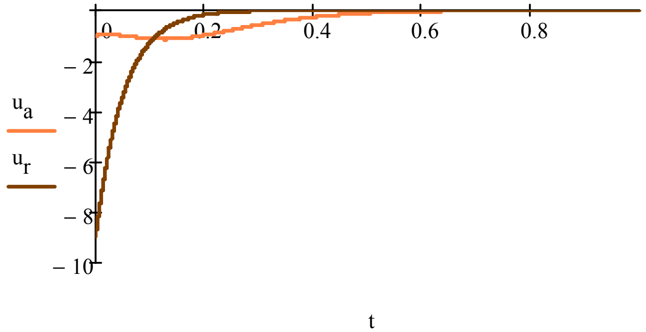
$$a = 1.5,\ b = 2,\ \underline{a} = 0.5,\ \underline{b} = 0.1,\ a_m = 5,\ d(t) = 0,\ v(t) = 0\,.$$



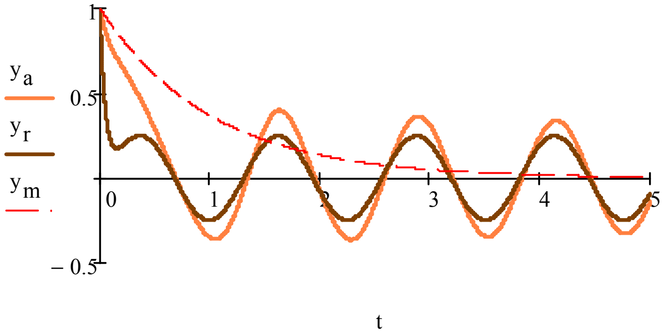


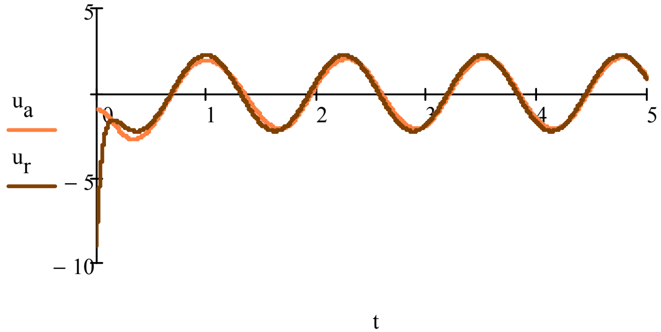
$a_m = 1,\ d(t) = 0,\ v(t) = 0.$



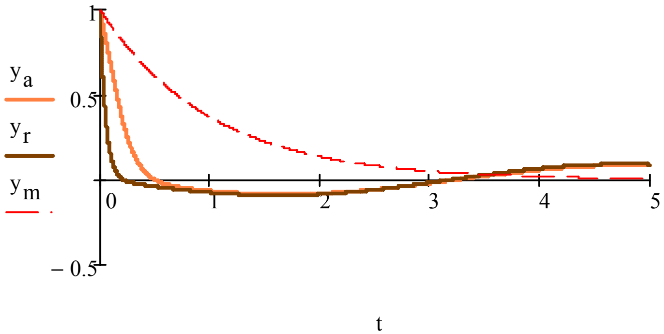


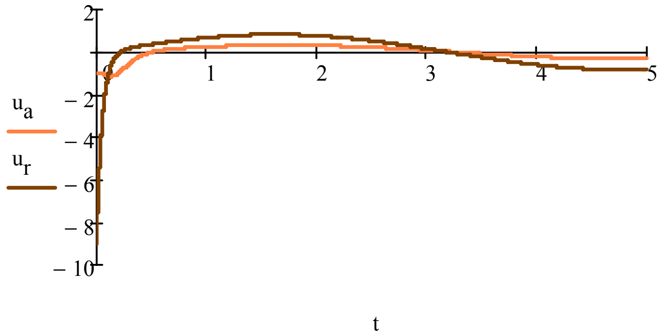
$a_m = 1,\ d(t) = 5\sin(5t),\ v(t) = 0.$





$a_m = 1,\ d(t) = 0,\ v(t) = 0.1\sin(t).$





5. ADAPTIVE OBSERVERS

A nonlinear system in state space presentation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y}), \ \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{24}$$

$\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ are the state, the input (control) and the measurable output;

$\mathbf{A}$, $\mathbf{C}$ are constant and known, the functions $\mathbf{B}(\mathbf{y})$ and $\boldsymbol{\varphi}(\mathbf{y})$ are continuous and known.

⇓

Everything is known except the state $\mathbf{x}$ (it is not measurable) ⇒ the **state observer** design:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y}) + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}],$$

$\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$; $\mathbf{L}$ is the observer matrix gain, $\mathbf{A} - \mathbf{LC}$ is Hurwitz.

Assumption 1. $\mathbf{x}(t) \in L_\infty$, $\mathbf{u}(t) \in L_\infty$ for all $t \geq 0$.

The **estimation error** $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$:

$$\dot{\mathbf{e}} = \boxed{\dot{\mathbf{x}}} - \boxed{\dot{\hat{\mathbf{x}}}} = \boxed{\{\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y})\}} - \boxed{\{\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y}) + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}]\}} = [\mathbf{A} - \mathbf{LC}]\mathbf{e}.$$

The matrix $\mathbf{A} - \mathbf{LC}$ is Hurwitz (design of $\mathbf{L}$) ⇒ $\hat{\mathbf{x}}(t) \in L_\infty$, $\lim_{t\to\infty}[\hat{\mathbf{x}}(t) - \mathbf{x}(t)] = 0$.

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A nonlinear system with parametric uncertainty:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y}) + \mathbf{G}(\mathbf{y},\mathbf{u})\boldsymbol{\theta}, \ \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{25}$$

$\boldsymbol{\theta} \in \mathbb{R}^q$ is the vector of unknown parameters, $\mathbf{G}(\mathbf{y},\mathbf{u})$ is a known continuous function.

The adaptive observer:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(\mathbf{y})\mathbf{u} + \boldsymbol{\varphi}(\mathbf{y}) + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}] + \mathbf{G}(\mathbf{y},\mathbf{u})\hat{\boldsymbol{\theta}} - \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}}, \tag{26}$$

$$\dot{\boldsymbol{\Omega}} = [\mathbf{A} - \mathbf{L}\mathbf{C}]\boldsymbol{\Omega} - \mathbf{G}(\mathbf{y},\mathbf{u}), \tag{27}$$

$$\dot{\hat{\boldsymbol{\theta}}} = -\gamma\boldsymbol{\Omega}^T\mathbf{C}^T[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}], \ \gamma > 0, \tag{28}$$

$\hat{\boldsymbol{\theta}} \in \mathbb{R}^q$ is the estimate of $\boldsymbol{\theta}$, $\boldsymbol{\Omega} \in \mathbb{R}^{n\times q}$ is an auxiliary filter variable.

The **state estimation error** $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$:

$$\dot{\mathbf{e}} = [\mathbf{A} - \mathbf{L}\mathbf{C}]\mathbf{e} + \mathbf{G}(\mathbf{y},\mathbf{u})[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] + \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}}.$$

$\mathbf{A} - \mathbf{L}\mathbf{C}$ is Hurwitz + Properties of $\hat{\boldsymbol{\theta}}(t)$ and $\dot{\hat{\boldsymbol{\theta}}}(t)$ $\Rightarrow$ Properties of $\mathbf{e}(t)$.

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The **auxiliary error** $\boldsymbol{\delta} = \mathbf{e} + \boldsymbol{\Omega}[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}]$:

$$\dot{\boldsymbol{\delta}} = \dot{\mathbf{e}} + \dot{\boldsymbol{\Omega}}[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] - \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}} =$$

$$= \{[\mathbf{A} - \mathbf{LC}]\mathbf{e} + \mathbf{G}(\mathbf{y},\mathbf{u})[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] + \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}}\} + \{[\mathbf{A} - \mathbf{LC}]\boldsymbol{\Omega} - \mathbf{G}(\mathbf{y},\mathbf{u})\}[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}] - \boldsymbol{\Omega}\dot{\hat{\boldsymbol{\theta}}} = [\mathbf{A} - \mathbf{LC}]\boldsymbol{\delta}.$$

$$\mathbf{A} - \mathbf{LC} \text{ is Hurwitz} \Rightarrow \boldsymbol{\delta}(t) \in L_\infty,\ \lim_{t\to\infty} \boldsymbol{\delta}(t) = 0.$$

$$\mathbf{A} - \mathbf{LC} \text{ is Hurwitz} + \mathbf{y}(t) \in L_\infty,\ \mathbf{u}(t) \in L_\infty \text{ (assumption 1)} \Rightarrow \boldsymbol{\Omega}(t) \in L_\infty.$$

The **parameter estimation error** $\tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)$:

$$\dot{\tilde{\boldsymbol{\theta}}} = -\dot{\hat{\boldsymbol{\theta}}} = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}] = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T\mathbf{C}\mathbf{e} = \gamma\boldsymbol{\Omega}^T\mathbf{C}^T\mathbf{C}[\boldsymbol{\delta} - \boldsymbol{\Omega}\tilde{\boldsymbol{\theta}}].$$

Intuition: $\lim_{t\to\infty} \boldsymbol{\delta}(t) = 0 \Rightarrow \dot{\tilde{\boldsymbol{\theta}}} = -\gamma\mathbf{h}(t)\mathbf{h}(t)^T\tilde{\boldsymbol{\theta}}$, $\mathbf{h}(t) = \boldsymbol{\Omega}^T(t)\mathbf{C}^T$ for $t \geq 0$ big enough.

Assumption 2. $\mathbf{h}(t)$ is PE: $\exists\rho > 0,\ \delta > 0$: $\int_0^t \mathbf{h}(\tau)\mathbf{h}(\tau)^T d\tau \geq \rho t \mathbf{I}_{n_q},\ \forall t \geq \delta$.

Assumption 2 $\Rightarrow \tilde{\boldsymbol{\theta}}(t) \in L_\infty,\ \lim_{t\to\infty} \tilde{\boldsymbol{\theta}}(t) = 0$ + properties of $\boldsymbol{\delta}(t) \Rightarrow \mathbf{e}(t) \in L_\infty,\ \lim_{t\to\infty} \mathbf{e}(t) = 0$.

Theorem 1. *Under assumptions 1 and 2 all signals in (25)−(28) are bounded and*

$$\lim_{t\to\infty}[\hat{\mathbf{x}}(t) - \mathbf{x}(t)] = 0,\ \lim_{t\to\infty}[\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}] = 0. \qquad \blacksquare$$

Example 2

Oscillating pendulum:

$$\ddot{y} = -\omega^2 \sin(y) - \rho\dot{y} + b\cos(y) f(t) + d(t), \tag{13}$$

$y = \varphi \in [-\pi, \pi)$ is the measured angle, $\dot{y} \in \mathbb{R}$ and $\ddot{y} \in \mathbb{R}$ are the angle velocity and acceleration; $\rho > 0$ is an known friction coefficient, $\omega > 0$ is an unknown natural frequency, $b > 0$ is an unknown control gain.

Presentation in the form (25) for $x_1 = y$, $x_2 = \dot{y}$, $u = f$ and $d(t) = 0$:

$$\begin{aligned} \dot{x}_1 &= x_2,\ y = x_1, \\ \dot{x}_2 &= -\rho x_2 - \omega^2 \sin(x_1) + b\cos(x_1) u(t). \end{aligned} \Rightarrow \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\rho \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T,\ \mathbf{B}(y) = 0,\ \varphi(y) = 0,$$

$$\mathbf{G}(y,u) = \begin{bmatrix} 0 & 0 \\ -\sin(y) & \cos(y)u \end{bmatrix},\ \boldsymbol{\theta} = \begin{bmatrix} \omega^2 \\ b \end{bmatrix}.$$

Both assumptions are satisfied for this example.

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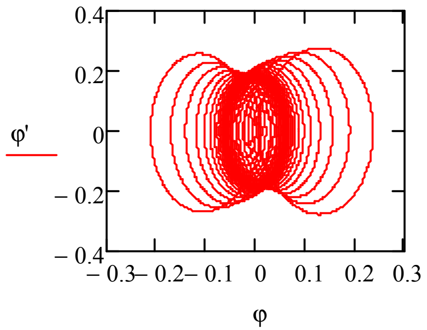
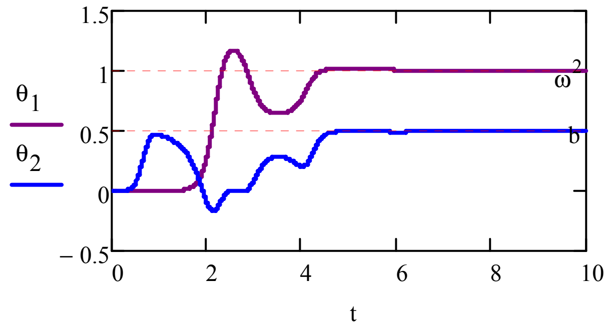
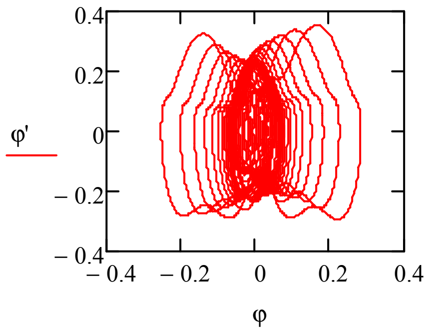
$$\omega = 1,\ \rho = 0.1,\ b = 0.5,\ f(t) = \sin(3t),\ \mathbf{L} = [2,1]^T,\ \gamma = 1000.$$

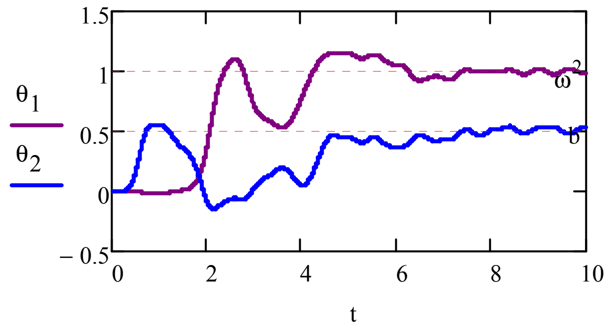
$d(t) = 0$



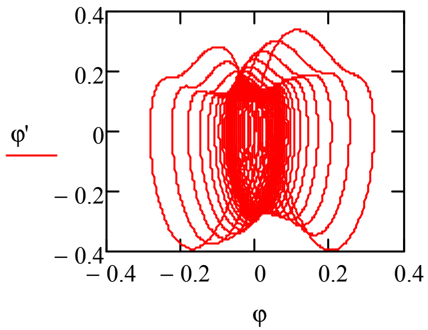


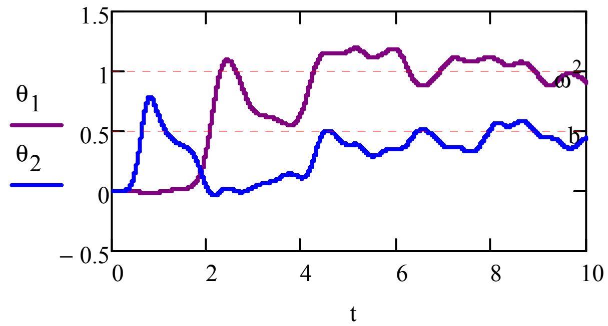
$d(t) = 0.5\sin(10t)$





$d(t) = 0.5\sin(6t)$





INDIRECT ADAPTIVE CONTROL

Denis EFIMOV - IMS - Université de Bordeaux 1

OUTLINE

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   a. *Main properties*
   b. *Running example*
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