Mioara Joldes

Journées TAMADI, 27-28 Oct. 2010
Based on:

- S. Chevillard, J. Harrison, M.J., Ch. Lauter, *Efficient and accurate computation of upper bounds of approximation errors*, accepted for TCS.

Outline

- Origin and Motivation of Supremum Norms Computations
  ⇝ Previous techniques
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  - Previous techniques
- Taylor Models and Enhancements
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- Chebyshev Models
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  - Previous techniques
- Taylor Models and Enhancements
- Chebyshev Models
- Comparison, Experimental Results
Origins

- Development of libms.
- Implementation of functions $f$ such as $f = \exp$, $f = \sin$, etc.
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  - Approximation: approximate $f$ by a polynomial $p$ on $[a, b]$. 

\[ \| \epsilon \|_\infty = \sup_{x \in [a, b]} |\epsilon(x)| \]
\[ \epsilon(x) = f(x) - p(x) \text{ or } \epsilon(x) = \frac{p(x)}{f(x)} - 1 \]
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  - Implementation: write a program that evaluates $p$. 

From a validation point of view:
Bound the errors occurring during the evaluation of $p$.
Bound the error between $p$ and $f$: compute $\|\varepsilon\|_\infty = \sup_{x \in [a, b]} \{|\varepsilon(x)|\}$
where $\varepsilon(x) = f(x) - p(x)$ or $\varepsilon(x) = p(x)/f(x) - 1$. 

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where $\varepsilon(x) = f(x) - p(x)$ or $\varepsilon(x) = p(x)/f(x) - 1$. 

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(easily obtained by numerical methods)

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Actual difficulty: certified upper bound $u$
Our approach

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- Correctly handles removable discontinuities e.g. $f(x) = \sin(x)$, $p(x) = x q(x)$ and $\varepsilon(x) = p(x)/f(x) - 1$.
- Could generate a complete formal proof without much effort.
It is a univariate global optimization problem.

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State of the art

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- Purely numerical algorithms: find the zeros of $\varepsilon'$ (e.g., Newton’s algorithm).

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  $\rightarrow$ Not rigorous: we might miss some of the zeros.

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State of the art

It is a univariate global optimization problem.

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- General-purpose rigorous global optimization algorithms\(^1\)
  - use Interval Arithmetic (IA)
  - Branch and bound: bisection, Interval Newton’s algorithm.

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Interval Arithmetic (IA)

- Each interval = pair of floating-point numbers
  (multiple precision IA libraries exist, e.g. MPFI\(^2\))

\(^2\text{http://gforge.inria.fr/projects/mpfi/}\)
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  Eg. \([1, 2] + [-3, 2] = [-2, 4]\)

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Range bounding for functions
- Eg. \( x \in [-1, 2] \), \( f(x) = x^2 - x + 1 \)
- \( F(X) = X^2 - X + 1 \)
- \( F([-1, 2]) = [-1, 2]^2 - [-1, 2] + [1, 1] \)
- \( F([-1, 2]) = [0, 4] - [-1, 2] + [1, 1] \)
- \( F([-1, 2]) = [-1, 6] \)

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    \(F([-1, 2]) = [0, 4] - [-1, 2] + [1, 1]\)
    \(F([-1, 2]) = [-1, 6]\)
  - \(x \in [-1, 2], f(x) \in [-1, 6], \text{but } \text{Im}(f) = [3/4, 3]\)

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Example (IA -Dependency phenomenon):

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Using IA, \( \varepsilon(x) \in [-233, 298] \), but \( \|\varepsilon(x)\|_\infty \approx 3.8325 \cdot 10^{-5} \)
Example (IA - Dependency phenomenon):

Overestimation can be reduced by using intervals of smaller width.

In this case, over \([0, 1]\) we need \(10^7\) intervals!
Ad-hoc techniques

- $f$ replaced with a rigorous polynomial approximation: $(T, \Delta)$
  - polynomial approximation $T$
  - interval $\Delta$ s. t. $f(x) - T(x) \in \Delta$, $\forall x \in [a, b]$
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- Makino and Berz: use Taylor Models\(^3\) for computing $(T, \Delta)$.

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- Idea used by (Kramer 1996), (Harrison 1997): functions manually handled, one by one.

- None of these techniques correctly handles removable discontinuities.

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Ingredients of our algorithm

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Note: Next, we’ll focus on computing $(T, \Delta)$.

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- $\|f - p\|_\infty \leq \|f - T\|_\infty + \|T - p\|_\infty \rightsquigarrow$ Relative error case
  slightly more technical

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Idea: Consider Taylor approximations
Taylor Models - How do we obtain them?

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Let $n \in \mathbb{N}$, $n + 1$ times differentiable function $f$ over $[a, b]$ around $x_0$.

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)(x - x_0)^i}{i!} + \Delta_n(x, \xi)$$

- $f(x)$
- $T(x)$ remainder
- $\Delta_n(x, \xi) = \frac{f^{(n+1)}(\xi)(x - x_0)^{n+1}}{(n + 1)!}$, $x \in [a, b]$, $\xi$ lies strictly between $x$ and $x_0$
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- \( T(x) \) remainsder
- \( \Delta_n(x, \xi) = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!} \), \( x \in [a, b] \), \( \xi \) lies strictly between \( x \) and \( x_0 \)

- How to compute the coefficients \( \frac{f^{(i)}(x_0)}{i!} \) of \( T(x) \)?
- How to compute an interval enclosure \( \Delta \) for \( \Delta_n(x, \xi) \)?
Automatic Differentiation - Point intervals

Compute \( f^{(i)}(x_0) \) - \( f \) represented as an expression tree
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Compute $f^{(i)}(x_0)$ - $f$ represented as an expression tree

Example:

Given $f(x) = \sin(x) \cos(x)$, compute $f^{(4)}(0)$
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- Simple formulas for derivatives of “basic functions”: exp, sin, etc.

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\sin(x) \rightarrow u = [\sin(0), \cos(0), -\sin(0), -\cos(0), \sin(0)]
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Compute $f^{(i)}([a, b])$ - $f$ represented as an expression tree
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$f(x) \rightarrow [u_0 v_0, u_0 v_1 + u_1 v_0, \ldots, [0, 13.5]]$ But $f^{(4)}([0, 1]) = [0, 8]$
What happens when $f$ is a composite function?

The interval bound $\Delta$ for $\Delta_n(x, \xi)$ can be largely overestimated.
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**Example:**

$$f(x) = e^{1/\cos x}, \text{ over } [0, 1], \ n = 13, \ x_0 = 0.5.$$  
$$f(x) - T(x) \in [0, 4.56 \cdot 10^{-3}]$$
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- Cauchy's Estimate
  $$\Delta = [-9.17 \cdot 10^{-2}, 9.17 \cdot 10^{-2}]$$
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  $$\Delta = [-1.93 \cdot 10^2, 1.35 \cdot 10^3]$$

- Cauchy’s Estimate
  $$\Delta = [-9.17 \cdot 10^{-2}, 9.17 \cdot 10^{-2}]$$

- Taylor Models
  $$\Delta = [-9.04 \cdot 10^{-3}, 9.06 \cdot 10^{-3}]$$
For bounding the remainders:

- For “basic functions” use Lagrange formula.

- For “composite functions” use a two-step procedure:
  - compute models \((T, I)\) for all basic functions;
  - apply algebraic rules with these models, instead of operations with the corresponding functions.
Taylor Models - Operations: Addition

Given two Taylor Models for $f_1$ and $f_2$, over $[a, b]$, degree $n$: $f_1(x) - P_1(x) \in \Delta_1$ and $f_2(x) - P_2(x) \in \Delta_2$, $\forall x \in [a, b]$.

Addition

$(P_1, \Delta_1) + (P_2, \Delta_2) = (P_1 + P_2, \Delta_1 + \Delta_2)$. 
Given two Taylor Models for $f_1$ and $f_2$, over $[a, b]$, degree $n$: $f_1(x) - P_1(x) \in \Delta_1$ and $f_2(x) - P_2(x) \in \Delta_2$, $\forall x \in [a, b]$.

Multiplication
We need algebraic rule for: $(P_1, \Delta_1) \cdot (P_2, \Delta_2) = (P, \Delta)$ s.t. $f_1(x) \cdot f_2(x) - P(x) \in \Delta$, $\forall x \in [a, b]$
Given two Taylor Models for \( f_1 \) and \( f_2 \), over \([a, b]\), degree \( n\):

\[ f_1(x) - P_1(x) \in \Delta_1 \text{ and } f_2(x) - P_2(x) \in \Delta_2, \forall x \in [a, b]. \]

**Multiplication**

We need algebraic rule for: \((P_1, \Delta_1) \cdot (P_2, \Delta_2) = (P, \Delta)\) s.t.

\[ f_1(x) \cdot f_2(x) - P(x) \in \Delta, \forall x \in [a, b] \]

\[
\begin{align*}
  f_1(x) \cdot f_2(x) & \in P_1 \cdot P_2 + P_2 \cdot \Delta_1 + P_1 \cdot \Delta_2 + \Delta_1 \cdot \Delta_2. \\
  (P_1 \cdot P_2)_{0...n} + (P_1 \cdot P_2)_{n+1...2n} & \in I_1 \quad I_2
\end{align*}
\]

\[
\Delta = I_1 + I_2
\]
Given two Taylor Models for $f_1$ and $f_2$, over $[a, b]$, degree $n$: $f_1(x) - P_1(x) \in \Delta_1$ and $f_2(x) - P_2(x) \in \Delta_2$, $\forall x \in [a, b]$.

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We need algebraic rule for: $(P_1, \Delta_1) \cdot (P_2, \Delta_2) = (P, \Delta)$ s.t. $f_1(x) \cdot f_2(x) - P(x) \in \Delta$, $\forall x \in [a, b]$

$$f_1(x) \cdot f_2(x) \in \underbrace{P_1 \cdot P_2}_{P} + \underbrace{P_2 \cdot \Delta_1}_{I_1} + \underbrace{P_1 \cdot \Delta_2}_{I_2} + \underbrace{\Delta_1 \cdot \Delta_2}_{I_2}.$$

$$(P_1 \cdot P_2)_{0\ldots n} + (P_1 \cdot P_2)_{n+1\ldots 2n}$$

$$\Delta = I_1 + I_2$$

In our case, for bounding “Ps”: IA evaluation.
Given TMs for $f_1$ over $[c, d]$, for $f_2$ over $[a, b]$, degree $n$:

$f_1(y) - P_1(y) \in \Delta_1$, $\forall y \in [c, d]$ and $f_2(x) - P_2(x) \in \Delta_2$, $\forall x \in [a, b]$. 

Given TMs for $f_1$ over $[c, d]$, for $f_2$ over $[a, b]$, degree $n$:

$$f_1(y) - P_1(y) \in \Delta_1, \forall y \in [c, d] \text{ and } f_2(x) - P_2(x) \in \Delta_2, \forall x \in [a, b].$$

Remark: $(f_1 \circ f_2)(x)$ is $f_1$ evaluated at $y = f_2(x)$.

We need: $f_2([a, b]) \subseteq [c, d]$, checked by $P_2 + \Delta_2 \subseteq [c, d]$
Given TMs for \( f_1 \) over \([c, d]\), for \( f_2 \) over \([a, b]\), degree \( n \):
\[
f_1(y) - P_1(y) \in \Delta_1, \quad \forall y \in [c, d] \quad \text{and} \quad f_2(x) - P_2(x) \in \Delta_2, \quad \forall x \in [a, b].
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Remark: \((f_1 \circ f_2)(x)\) is \( f_1 \) evaluated at \( y = f_2(x) \).
We need: \( f_2([a, b]) \subseteq [c, d] \), checked by \( P_2 + \Delta_2 \subseteq [c, d] \)

\[
f_1(y) \in P_1(y) + \Delta_1
\]
Taylor Models - Operations: Composition

Given TMs for $f_1$ over $[c, d]$, for $f_2$ over $[a, b]$, degree $n$: 
$f_1(y) - P_1(y) \in \Delta_1, \forall y \in [c, d]$ and $f_2(x) - P_2(x) \in \Delta_2, \forall x \in [a, b]$.

Remark: $(f_1 \circ f_2)(x)$ is $f_1$ evaluated at $y = f_2(x)$.
We need: $f_2([a, b]) \subseteq [c, d]$, checked by $P_2 + \Delta_2 \subseteq [c, d]$

$f_1(f_2(x)) \in P_1(f_2(x)) + \Delta_1$
Given TMs for \( f_1 \) over \([c, d]\), for \( f_2 \) over \([a, b]\), degree \( n \):
\[
f_1(y) - P_1(y) \in \Delta_1, \forall y \in [c, d] \quad \text{and} \quad f_2(x) - P_2(x) \in \Delta_2, \forall x \in [a, b].
\]

Remark: \((f_1 \circ f_2)(x)\) is \( f_1 \) evaluated at \( y = f_2(x) \).
We need: \( f_2([a, b]) \subseteq [c, d] \), checked by \( P_2 + \Delta_2 \subseteq [c, d] \)

\[
f_1(f_2(x)) \in P_1(P_2(x) + \Delta_2) + \Delta_1
\]

Extract polynomial and remainder: \( P_1 \) can be evaluated using only additions and multiplications: Horner’s algorithm
Example: \( F(x) = \frac{\sin(x)}{\log(1 + x)} \) over \([-\frac{1}{2}, \frac{1}{2}]\).

Taylor expansion exists, but using classical tm arithmetic:
we need a tm for \( \frac{1}{y} \) over \([\log(\frac{1}{2}), \log(\frac{3}{2})]\) \(\ni 0\) which cannot be computed.
TM for $f(x) = \sin(x)$ in 0, order $n + 1$, over $[-\frac{1}{2}, \frac{1}{2}]$. 
TM for $g(x) = \log(1 + x)$ in 0, order $n + 1$, over $[-\frac{1}{2}, \frac{1}{2}]$. 

$$\frac{f}{g} = \frac{x \times (T_f(x) + r_f x^n)}{x \times (T_g(x) + r_g x^n)}$$

Example:

$$\frac{f}{g} = \frac{x \times (1 - \frac{1}{6} x^2 + r_f x^3)}{x \times (1 - \frac{1}{2} x + \frac{1}{3} x^2 + r_g x^3)}$$
TMs for functions with removable discontinuities

Need: tm of order \( n \) for \( F = \frac{f}{g} \) over \( I \), knowing that \( x_0 \in I \) root of order \( k > 0 \) of \( f \) and \( g \).

Idea: Keep remainders in "relative" way: \( r = \Delta \times (x - x_0)^{n+k} \)

- Compute tms of order \( n + k \) for \( f \) and \( g \) in \( x_0 \).
- Formally simplify \( \frac{f}{g} \) by \( (x - x_0)^{k} \), then compute using regular tm arithmetic
- All operations can be easily extended to work with "relative remainders"
So far, some key points in our supnorm algorithm...

- $f$ replaced with a rigorous polynomial approximation: $(T, \Delta)$
- Use modified Taylor Models for computing $(T, \Delta)$.
- $\|f - p\|_{\infty} \leq \|f - T\|_{\infty} + \|T - p\|_{\infty}$
So far, some key points in our supnorm algorithm...

- \( f \) replaced with a rigorous polynomial approximation: \((T, \Delta)\)

- Use modified Taylor Models for computing \((T, \Delta)\).

\[
\|f - p\|_\infty \leq \|f - T\|_\infty + \|T - p\|_\infty
\]

Note: Our algorithm can fail if \( \Delta \) can not be made small enough.
Solution: Cut the interval into sub-intervals or use ChebModels.
So far, some key points in our supnorm algorithm...

- $f$ replaced with a rigorous polynomial approximation: $(T, \Delta)$

- use modified Taylor Models for computing $(T, \Delta)$.

$$\|f - p\|_{\infty} \leq \|f - T\|_{\infty} + \|T - p\|_{\infty}$$

Note: Our algorithm can fail if $\Delta$ can not be made small enough.

Solution: Cut the interval into sub-intervals or use ChebModels 4

Computing $\|T - p\|_{\infty}$ not discussed in this talk.

---

### Experimental results - Supnorm Algorithm with TM

<table>
<thead>
<tr>
<th>$f$</th>
<th>$[a, b]$</th>
<th>$\text{deg}(p)$</th>
<th>$\text{deg}(T)$</th>
<th>mode</th>
<th>quality $- \log_2 \eta$</th>
<th>time NR</th>
<th>time R</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exp(x) - 1$</td>
<td>$[-0.25, 0.25]$</td>
<td>5</td>
<td>13</td>
<td>rel.</td>
<td>37.6</td>
<td>14</td>
<td>42</td>
</tr>
<tr>
<td>$\log_2(1 + x)$</td>
<td>$[-2^{-9}, 2^{-9}]$</td>
<td>7</td>
<td>17</td>
<td>rel.</td>
<td>83.3</td>
<td>41</td>
<td>103</td>
</tr>
<tr>
<td>$\arcsin(x + m)$</td>
<td>$[a_3, b_3]$</td>
<td>22</td>
<td>32</td>
<td>rel.</td>
<td>15.9</td>
<td>270</td>
<td>364</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$[-0.5, 0.25]$</td>
<td>15</td>
<td>22</td>
<td>rel.</td>
<td>19.5</td>
<td>93</td>
<td>139</td>
</tr>
<tr>
<td>$\exp(x)$</td>
<td>$[-0.125, 0.125]$</td>
<td>25</td>
<td>34</td>
<td>rel.</td>
<td>42.3</td>
<td>337</td>
<td>443</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$[-0.5, 0.5]$</td>
<td>9</td>
<td>17</td>
<td>abs.</td>
<td>21.5</td>
<td>13</td>
<td>39</td>
</tr>
<tr>
<td>$\exp(\cos^2 x + 1)$</td>
<td>$[1, 2]$</td>
<td>15</td>
<td>44</td>
<td>rel.</td>
<td>25.5</td>
<td>180</td>
<td>747</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$[0.25, 0.5]$</td>
<td>10</td>
<td>22</td>
<td>rel.</td>
<td>26.0</td>
<td>47</td>
<td>94</td>
</tr>
<tr>
<td>$x^{2.5}$</td>
<td>$[1, 2]$</td>
<td>7</td>
<td>20</td>
<td>rel.</td>
<td>15.5</td>
<td>27</td>
<td>73</td>
</tr>
<tr>
<td>$\sin(x)/(\exp(x) - 1)$</td>
<td>$[-2^{-3}, 2^{-3}]$</td>
<td>15</td>
<td>27</td>
<td>abs.</td>
<td>15.5</td>
<td>43</td>
<td>168</td>
</tr>
</tbody>
</table>

Values for example #3:

$$m = \frac{770422123864867}{2^{50}}, \quad a_3 = \frac{-205674681606191}{2^{53}}, \quad b_3 = \frac{205674681606835}{2^{53}}.$$

In the last columns, “NR” stands for “not rigorous” and “R” stands for “rigorous”.

Timings are given in milliseconds on a Core i7-975.
Taylor Models Issues

Example:

\[ f(x) = \arctan(x) \text{ over } [-0.9, 0.9] \]
\[ p(x) - \text{ minimax, degree 15} \]
\[ \varepsilon(x) = p(x) - f(x) \]

\[ \|\varepsilon\|_{\infty} \approx 10^{-8} \]
Taylor Models Issues

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\[ f(x) = \arctan(x) \text{ over } [-0.9, 0.9] \]
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\[ \varepsilon(x) = p(x) - f(x) \]

\[ \|\varepsilon\|_\infty \approx 10^{-8} \]

In this case Taylor approximations are not good, we need theoretically a TM of degree 120.

Practically, the computed interval remainder can not be made sufficiently small due to overestimation.
Basic idea:

- Use a polynomial approximation better than Taylor:
  - Chebyshev interpolation polynomial.
  - Chebyshev truncated series.

- Use the two step approach as Taylor Models:
  - compute models \((P, I)\) for basic functions;
  - apply algebraic rules with these models, instead of operations with the corresponding functions (work in Chebyshev basis).
Quick Reminder: Chebyshev Polynomials

Over $[-1, 1]$, $T_n(x) = \cos(n \arccos x)$, $n \geq 0$.

"Chebyshev nodes": $n$ distinct real roots in $[-1, 1]$ of $T_n$:

$x_i = \cos\left(\frac{(i+1/2) \pi}{n}\right)$, $i = 0, \ldots, n - 1$. 
$P(x) = \sum_{i=0}^{n} p_i T_i(x)$ interpolates $f$ at $x_k \in [-1, 1]$, Chebyshev nodes of order $n + 1$. 
Chebyshev Models: using interpolation polynomial

\[ P(x) = \sum_{i=0}^{n} p_i T_i(x) \] interpolates \( f \) at \( x_k \in [-1, 1] \), Chebyshev nodes of order \( n + 1 \).

Computation of the coefficients

\[ p_i = \sum_{k=0}^{n} \frac{2}{n+1} f(x_k) T_i(x_k), \quad i = 0, \ldots, n \]
Chebyshev Models: using interpolation polynomial

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**Computation of the coefficients**

\[ p_i = \sum_{k=0}^{n} \frac{2}{n+1} f(x_k) T_i(x_k), \quad i = 0, \ldots, n \]

Remark: Currently, this step is more costly than in the case of TMs. We can use truncated Chebyshev series instead.
Chebyshev Models: using interpolation polynomial

\[ P(x) = \sum_{i=0}^{n} p_i T_i(x) \] interpolates \( f \) at \( x_k \in [-1, 1] \), Chebyshev nodes of order \( n + 1 \).

**Computation of the coefficients**

**Interpolation Error: Lagrange remainder**

\( \forall x \in [-1, 1], \ \exists \xi \in [-1, 1] \) s.t.

\[ f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \prod_{i=0}^{n} (x - x_i). \]
Chebyshev Models: using interpolation polynomial

\[ P(x) = \sum_{i=0}^{n} p_i T_i(x) \] interpolates \( f \) at \( x_k \in [-1, 1] \), Chebyshev nodes of order \( n + 1 \).

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\[ \forall x \in [-1, 1], \exists \xi \in [-1, 1] \text{ s.t.} \]
\[ f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \prod_{i=0}^{n} (x - x_i). \]

✓ We should have an improvement of \( 2^n \) in the width of the remainder, compared to Taylor remainder.
Chebyshev Models: using interpolation polynomial

\[ P(x) = \sum_{i=0}^{n} p_i T_i(x) \] interpolates \( f \) at \( x_k \in [-1, 1] \), Chebyshev nodes of order \( n + 1 \).

### Computation of the coefficients

#### Interpolation Error: Lagrange remainder

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✗ For composite functions, overestimation of \( f^{(n+1)} \)
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**Computation of the coefficients** (for “basic” functions)

**Interpolation Error: Lagrange remainder** (for “basic” functions)

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Computation of the coefficients (for “basic” functions)

Interpolation Error: Lagrange remainder (for “basic” functions)

- For composite functions, use algebraic rules (addition, multiplication, composition) with models
Chebyshev Models: using interpolation polynomial

\[ P(x) = \sum_{i=0}^{n} p_i T_i(x) \text{ interpolates } f \text{ at } x_k \in [-1, 1], \text{ Chebyshev nodes of order } n + 1. \]

**Computation of the coefficients (for “basic” functions)**

**Interpolation Error: Lagrange remainder (for “basic” functions)**

- For composite functions, use algebraic rules (**addition, multiplication, composition**) with models

- Note: Chebfun - ”Computing Numerically with Functions Instead of Numbers“ (N. Trefethen et al.): Chebyshev interpolation polynomials are already used, **but the approach is not rigorous**
Chebyshev Models: using truncated Chebyshev series

\[ P(x) = \sum_{k=0}^{n} a_k T_k(x), \text{ where } a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1 - x^2}} dx. \]
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\[ P(x) = \sum_{k=0}^{n} a_k T_k(x), \text{ where } a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \, dx. \]

Computation of the coefficients (for “basic” D-finite functions)

- recurrence formulae for computing \( a_k \)
Chebyshev Models: using truncated Chebyshev series

\[ P(x) = \sum_{k=0}^{n} a_k T_k(x), \ \text{where} \ a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \, dx. \]

Computation of the coefficients (for “basic” D-finite functions)
- recurrence formulae for computing \( a_k \)

Remark: As fast as TMs.
Chebyshev Models: using truncated Chebyshev series

\[ P(x) = \sum_{k=0}^{n} a_k T_k(x), \text{ where } a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \, dx. \]

Computation of the coefficients (for “basic” D-finite functions)

Truncation Error: Bernstein-like formula (for “basic” D-finite functions)

\[ \forall x \in [-1, 1], \exists \xi \in [-1, 1] \text{ s.t. } f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{2^n(n+1)!}. \]
Chebyshev Models: using truncated Chebyshev series

\[ P(x) = \sum_{k=0}^{n} a_k T_k(x), \text{ where } a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \, dx. \]

Computation of the coefficients (for “basic” D-finite functions)

Truncation Error: Bernstein-like formula (for “basic” D-finite functions)

- For composite functions, use algebraic rules (addition, multiplication, composition) with models
Chebyshev Models - Supremum norm example

Example: $\varepsilon(x) = f(x) - p(x)$

$f(x) = e^{1/\cos x}$, over $[0, 1]$, $p(x)$ - minimax, degree 10

$\|\varepsilon(x)\|_\infty \approx 3.8325 \cdot 10^{-5}$
Example: $\varepsilon(x) = f(x) - p(x)$

$f(x) = e^{1/\cos x}$, over $[0, 1]$, $p(x)$ - minimax, degree 10

$\|\varepsilon(x)\|_{\infty} \simeq 3.8325 \cdot 10^{-5}$

Need: TM of degree 30.
Chebyshev Models - Supremum norm example

Example: \( \varepsilon(x) = f(x) - p(x) \)

\[
f(x) = e^{1/\cos x}, \text{ over } [0, 1], \ p(x) - \text{minimax, degree 10}
\]

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CMs vs. TMs

Operations complexity:

- Addition ($O(n)$), Multiplication ($O(n^2)$) and Composition ($O(n^3)$) have similar complexity.
- Initial computation of coefficients for all basic D-nite functions is similar ($O(n)$).

Comparison between remainder bounds for several functions:

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<td>sin($x$), [3, 4], 10</td>
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<td>$3.24 \cdot 10^{-12}$</td>
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<tr>
<td>arctan($x$), [−0.9, 0.9], 15</td>
<td>$5.10 \cdot 10^{-3}$</td>
<td>$1.76 \cdot 10^{-8}$</td>
<td>$1.67 \cdot 10^2$</td>
<td>$5.70 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>exp($1 / \cos(x$), [0, 1], 14</td>
<td>$5.22 \cdot 10^{-7}$</td>
<td>$4.95 \cdot 10^{-7}$</td>
<td>$9.06 \cdot 10^{-3}$</td>
<td>$2.59 \cdot 10^{-3}$</td>
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<tr>
<td>$\exp(x) \log(2+x) \cos(x)$, [0, 1], 15</td>
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<td>$3.38 \cdot 10^{-5}$</td>
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<tr>
<td>sin($\exp(x)$),[−1, 1], 10</td>
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<td>$4.95 \cdot 10^{-7}$</td>
<td>$9.06 \cdot 10^{-3}$</td>
<td>$2.59 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\exp(x) / \log(2+x) \cos(x)$, $[0, 1]$, 15</td>
<td>$9.11 \cdot 10^{-9}$</td>
<td>$2.21 \cdot 10^{-9}$</td>
<td>$1.18 \cdot 10^{-3}$</td>
<td>$3.38 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$\sin(\exp(x))$, $[-1, 1]$, 10</td>
<td>$9.47 \cdot 10^{-5}$</td>
<td>$3.72 \cdot 10^{-6}$</td>
<td>$2.96 \cdot 10^{-2}$</td>
<td>$1.55 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>
Remark: It is known [Ehlich & Zeller, 1966] that Chebyshev interpolants are "near-best":

\[ \| \varepsilon \|_\infty \leq (2 + (2/\pi) \log(n)) \| \varepsilon_{\text{minimax}} \|_\infty \]

\[ \Lambda_n \]

- \[ \Lambda_{15} = 3.72... \rightarrow \text{we lose at most 2 bits} \]
- \[ \Lambda_{30} = 4.16... \rightarrow \text{we lose at most 3 bits} \]
- \[ \Lambda_{100} = 4.93... \rightarrow \text{we lose at most 3 bits} \]
- \[ \Lambda_{100000} = 9.32... \rightarrow \text{we lose at most 4 bits} \]
Quality of approximation compared to minimax

<table>
<thead>
<tr>
<th>No</th>
<th>$f(x)$, $I$, $n$</th>
<th>CM</th>
<th>Exact bound</th>
<th>Minimax</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sin(x)$, $[3, 4]$, 10</td>
<td>$1.19 \cdot 10^{-14}$</td>
<td>$1.13 \cdot 10^{-14}$</td>
<td>$1.12 \cdot 10^{-14}$</td>
</tr>
<tr>
<td>2</td>
<td>$\arctan(x)$, $[-0.25, 0.25]$, 15</td>
<td>$7.89 \cdot 10^{-15}$</td>
<td>$7.95 \cdot 10^{-17}$</td>
<td>$4.03 \cdot 10^{-17}$</td>
</tr>
<tr>
<td>3</td>
<td>$\arctan(x)$, $[-0.9, 0.9]$, 15</td>
<td>$5.10 \cdot 10^{-3}$</td>
<td>$1.76 \cdot 10^{-8}$</td>
<td>$1.01 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>4</td>
<td>$\exp(1/\cos(x))$, $[0, 1]$, 14</td>
<td>$5.22 \cdot 10^{-7}$</td>
<td>$4.95 \cdot 10^{-7}$</td>
<td>$3.57 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{\exp(x)}{\log(2+x) \cos(x)}$, $[0, 1]$, 15</td>
<td>$9.11 \cdot 10^{-9}$</td>
<td>$2.21 \cdot 10^{-9}$</td>
<td>$1.72 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>6</td>
<td>$\sin(\exp(x))$, $[-1, 1]$, 10</td>
<td>$9.47 \cdot 10^{-5}$</td>
<td>$3.72 \cdot 10^{-6}$</td>
<td>$1.78 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>
Supnorms computation is automatic, efficient, has \textit{a priori} control of result quality.
Supnorms computation is automatic, efficient, has *a priori* control of result quality.

Rigorous polynomial approximations are essential.
Conclusion

- Supnoms computation is automatic, efficient, has *a priori* control of result quality.
- Rigorous polynomial approximations are essential.
  \(\sim\) Work in progress: compute CMs for any D-finite function based on the given differential equation (direct application to rigorous ODE solving)
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Is an approximation of WORSE quality really BETTER?