Validated numerics for robust space mission design

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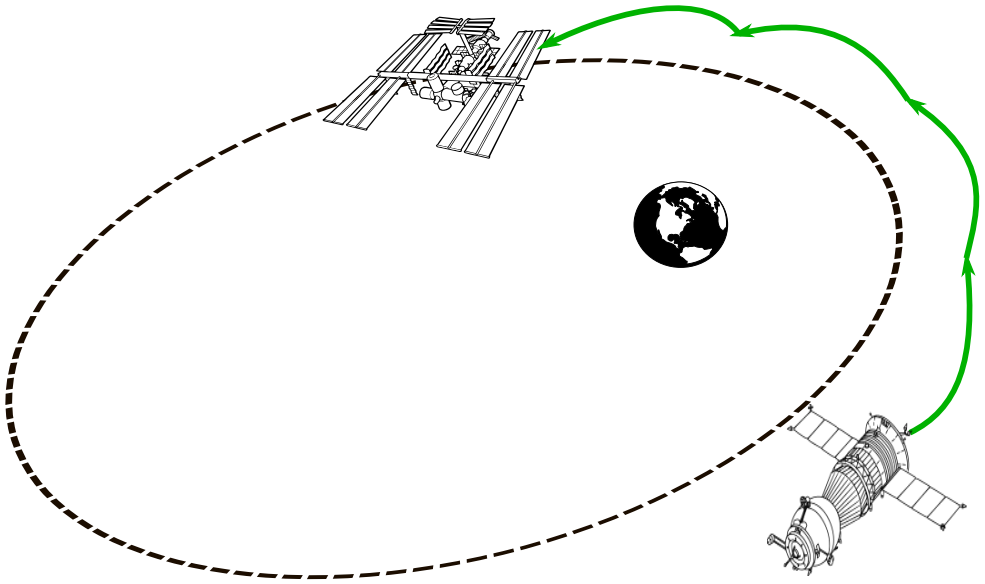
LAAS-CNRS
Joint Projects with CNES and Airbus Defence and Space



9èmes Rencontres Arithmétique de l'Informatique Mathématique
Lyon, October 24-26, 2017

Context

- Numerical Computing: floating-point arithmetic

$\rightarrow$ High Performance Computing (MultiCores, GPUs, FPGAs):

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A catastrophic cancellation example:
Evaluate

$$(333.75 - a^2)b^6 + a^2\left(11a^2b^2 - b^6 - 121b^4 - 2\right) + 5.5b^8 + \frac{a}{2b}$$

for $a = 77617.0$, $b = 33096.0$ (Rump '88)
Results of C program, gcc, Linux:
1.1726039400531787 in binary64;
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Example: $\exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}$

$$\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots$$

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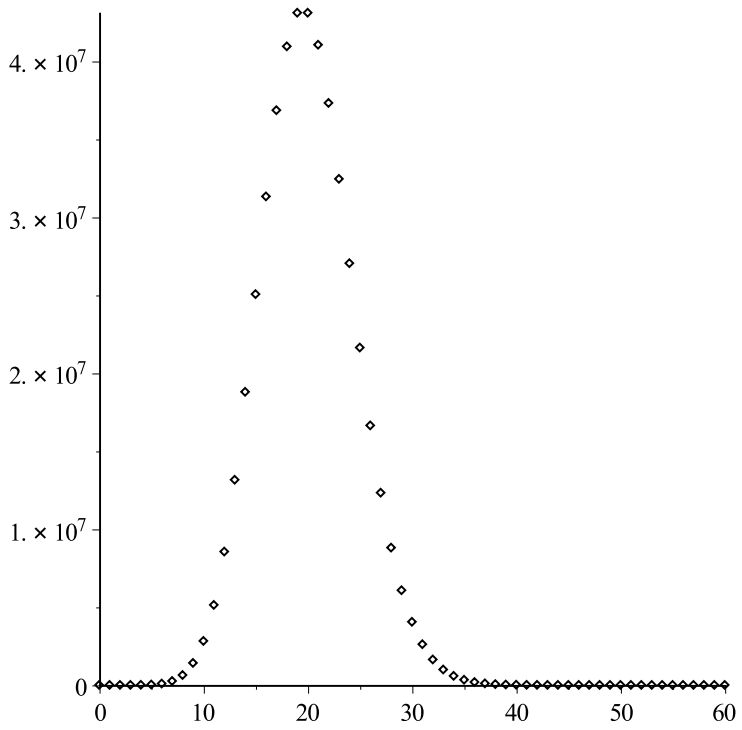

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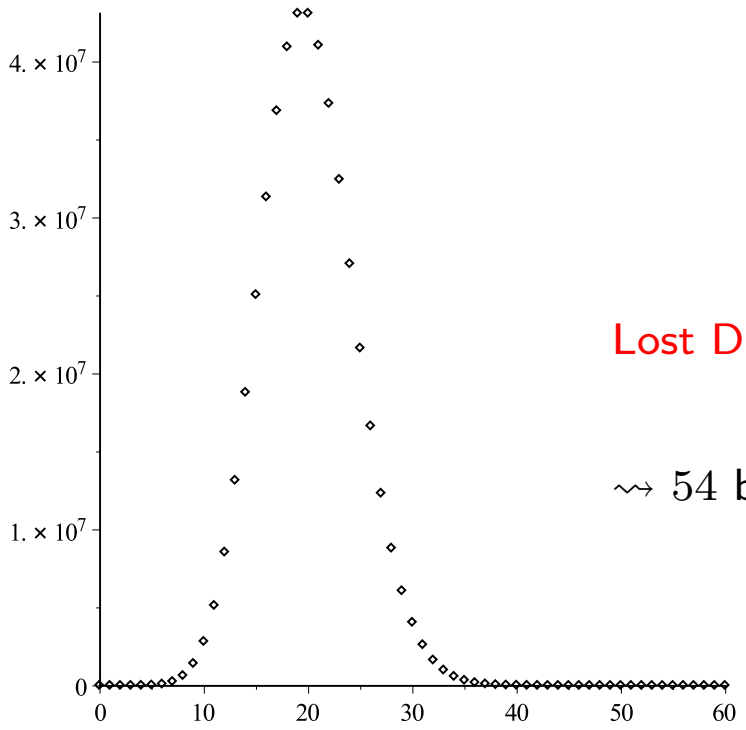

Lost Digits: $\simeq \log \frac{\max_i \frac{20^i}{i!}}{\exp(-20)}$

$\rightsquigarrow$ 54 bits lost, hence binary64 result: 0.01583705682...

Yet Another example: Chebyshev Series of D-finite Functions

Taylor series: $\exp = \sum \frac{1}{n!} x^n$

Recurrence for coefficients:

$$u(n+1) = \frac{u(n)}{n+1}$$

$u(0) = 1$	$1/0! = 1$
$u(1) = 1$	$1/1! = 1$
$u(2) = 0.5$	$1/2! = 0.5$
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$u(50) \approx 3.28 \cdot 10^{-65}$	$1/50! \approx 3.28 \cdot 10^{-65}$

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$$u(n+1) = -2nu(n) + u(n-1)$$

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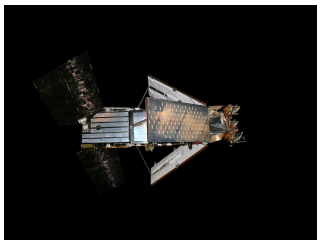
More subtle cause:

Convergent and Divergent Solutions of the Recurrence $u(n+1) = -2nu(n) + u(n-1)$:
If $u(n)$ is solution, then there exists another solution $v(n) \sim \frac{1}{u(n)}$

Safety-critical space applications

- 2009, Feb. 10: collision between Iridium 33 and Cosmos 2251, although predicted minimum distance of close approach was of 584m.



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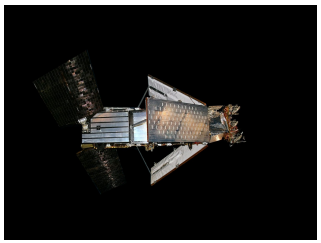

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Can/Should we trust numerics?

- they are fast...
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- Mathematica Book: *when Mathematica does a numerical integral, the only information it has about your integrand is a sequence of numerical values for it. If you give a sufficiently pathological integrand, Mathematica may simply give you the wrong answer*

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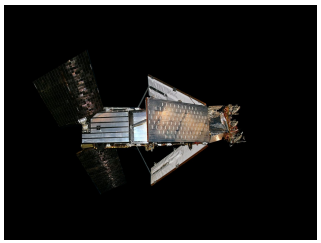

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First focus of this talk:

efficient and reliable numerical computation of on-orbit collision probability

Safety-critical space applications (2)

A goal of space missions

Achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption
e.g., rendezvous planning tools for formation flight (PRISMA), on-orbit satellite servicing, supply missions (ISS)...

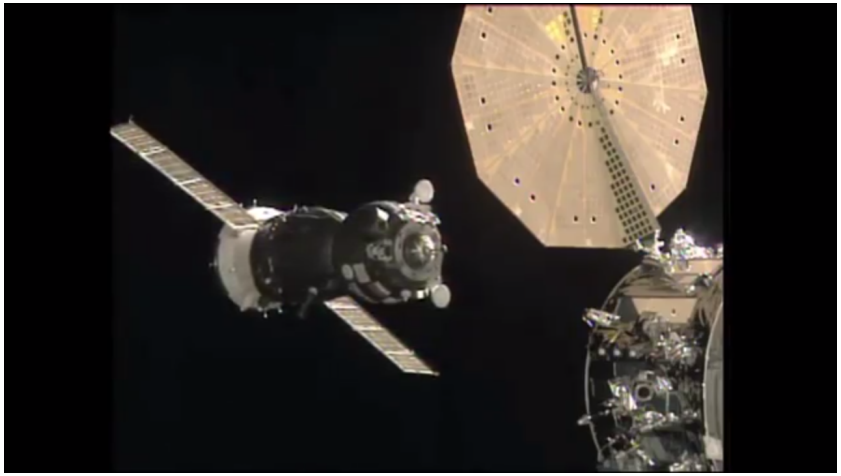

Figure: Soyuz TMA-19M docking to ISS (December 2015) - ISS resupply mission; NASA YouTube© Chanel

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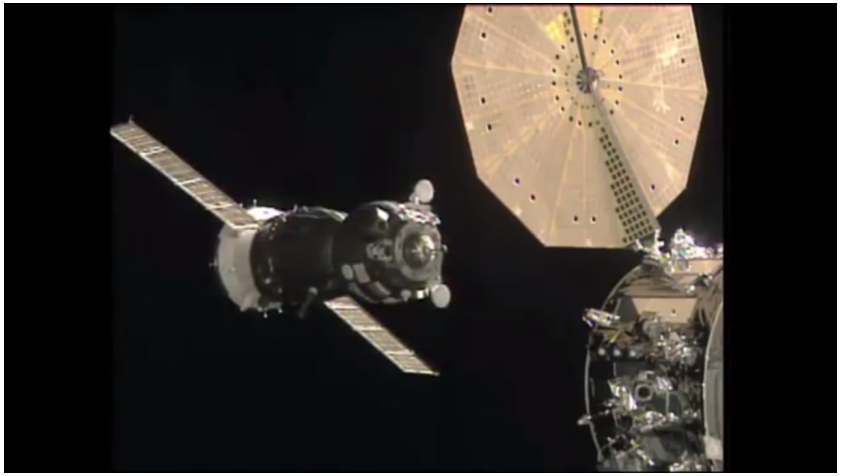

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On-orbit collision

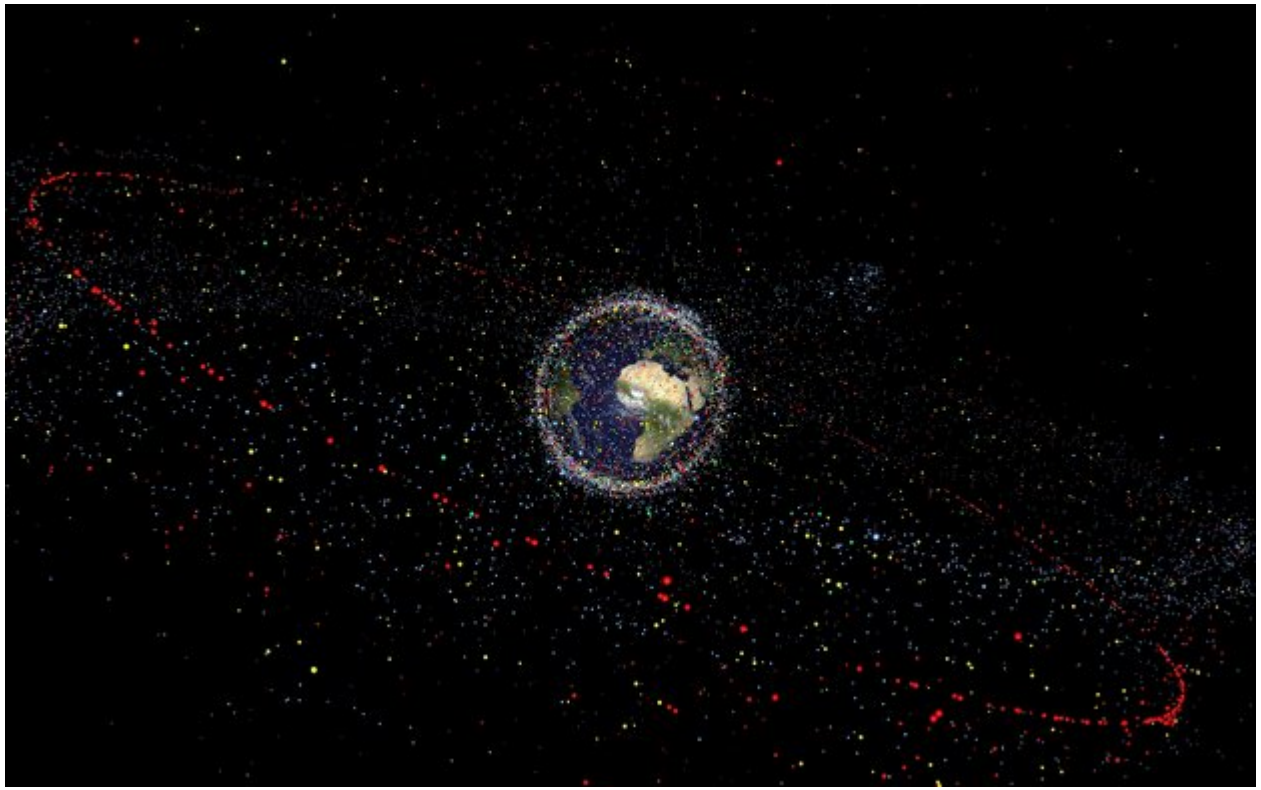

Figure: Space debris population model (source : ESA)

Thomas Pesquet, when a debris whizzes pas the ISS:**

"Climb into an escape shuttle, wait and hope. This happened four times"

** http://www.chron.com/news/science-environment/article/Thousands-of-tiny-satellites-are-about-to-go-into-11088984.php

On-orbit collision

Context

- Two objects: primary P (operational satellite) and secondary S (space debris)
- Information about their geometry, position, velocity at a given time
  $\rightsquigarrow$ Affected by uncertainty
- Needs:
  - Risk assessment
  - Design of a collision avoidance strategy

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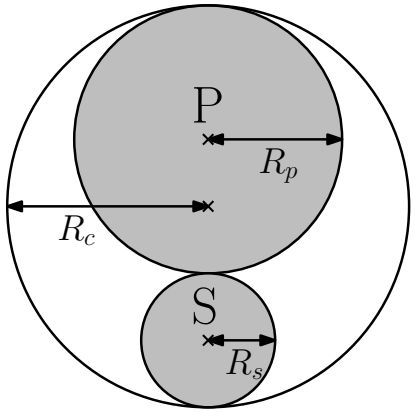
Classical assumptions

- Spherical objects
- Gaussian probability density functions
- Independent probability distribution laws



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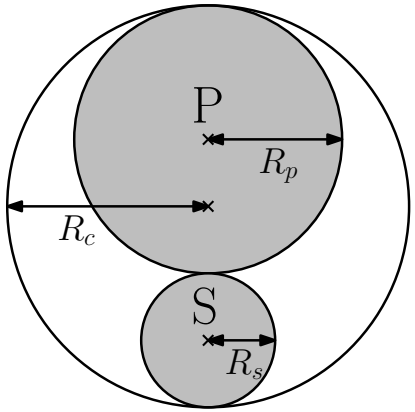

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Probability of collision

Generally: 12-dimensional, Gaussian integrand, Complex integration domain

Computation: Monte-Carlo trials and/or simplified models

Short-term encounter model and probability of collision

- Framework: High relative velocity
- Assumptions:
  - Rectilinear relative motion
  - No velocity uncertainty
  - Infinite encounter time horizon

  $\rightsquigarrow$ Probability of collision:
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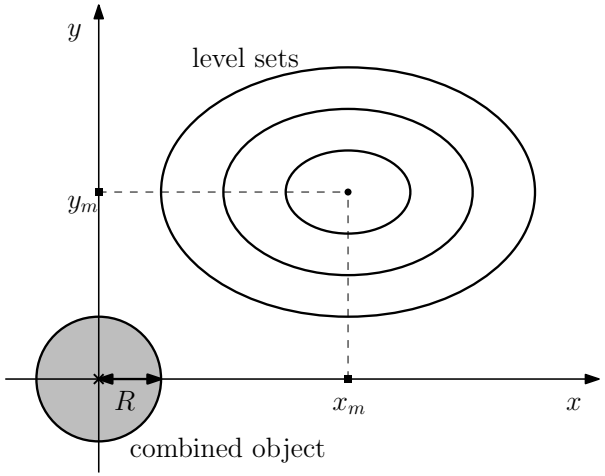

Figure: 2-D Gaussian integral over a disk

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where

- R: radius of combined object
- x_m, y_m: mean relative coordinates
- σ_x, σ_y: standard deviations of relative coordinates

Existing methods

- Methods based on numerical integration schemes:
  Foster '92, Patera '01, Alfano '05.
- Analytic methods: Chan '97 $\rightsquigarrow$ uses some simplifying assumptions ($\sigma_x = \sigma_y$)

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Our purpose

Give a "simple", "analytic" formula, suitable for double-precision evaluation and effective error bounds.

Our method - Underlying techniques

1. Laplace transform:
   $\rightsquigarrow$ *Lasserre and Zeron, Solving a Class of Multivariate Integration Problems via Laplace Techniques, Applicationes Mathematicae, 2001.*

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   $\rightsquigarrow$ power series coefficients satisfy a linear recurrence relation with polynomial coefficients

Example:

$$f(x) = \exp(x) = \sum_{n=0}^{\infty} f_n x^n \quad \leftrightarrow \quad \{f' - f = 0,\ f(0) = 1\} \leftrightarrow \{(n+1)f_{n+1} = f_n, f_0 = 1\}$$

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3. Finite-precision evaluation of power series prone to cancellation
   $\rightsquigarrow$ *Gawronski, Müller, Reinhard, Reduced Cancellation in the Evaluation of Entire Functions and Applications to the Error Function, SIAM Journal on Numerical Analysis, 2007.*
   $\rightsquigarrow$ *Chevillard, Mezzarobba, Multiple-Precision Evaluation of the Airy Ai Function with Reduced Cancellation, 21st IEEE SYMPOSIUM on Computer Arithmetic, 2013*

Sketch of the proof - Laplace Transform

$\forall z \in \mathbb{R}^+ :$

$$g(z) := \mathcal{P}(\sqrt{z}) = \frac{1}{2\pi\sigma_x\sigma_y} \int\limits_{\mathcal{B}((0,0),\sqrt{z})} \exp\left(-\frac{(x-x_m)^2}{2{\sigma_x}^2} - \frac{(y-y_m)^2}{2{\sigma_y}^2}\right) \mathrm{d}x\mathrm{d}y,$$

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$\mathcal{L}(g)$ is D-finite !

Sketch of the proof - Borel-Laplace

$g(z)$ $\xrightarrow{\text{Laplace Transform}}$ $\mathcal{L}(g)(t) = \dfrac{\exp\left(-\frac{{\sigma_x}^2 {y_m}^2 + {\sigma_y}^2 {x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2t{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2t{\sigma_x}^2+1)}\right)}{t\sqrt{(2t{\sigma_x}^2+1)(2t{\sigma_y}^2+1)}}$

$\mathcal{L}(g)(t)$ $\xrightarrow{\text{expansion at } \infty}$ $\hat{\mathcal{L}}(g)(t) := t^2\mathcal{L}(g)\left(\frac{1}{t}\right) = \sum_{i=0}^{\infty} \ell_i \left(\frac{1}{t}\right)^i$

$\hat{\mathcal{L}}(g)(t)$ $\xrightarrow{\text{Borel Transform}}$ $g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$

$g(z)$ is:

- D-finite
- entire function of exponential type
- type $\sigma = \frac{1}{2\sigma_y^2}$

$\hat{\mathcal{L}}(g)(t)$ is:

- D-finite
- Finite radius of convergence $2\sigma_y^2$

*B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.

Sketch of the proof - Borel-Laplace

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$\hat{\mathcal{L}}(g)(t)$ is:

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ℓ_i satisfy a linear recurrence with polynomial coefficients.
$\rightsquigarrow$ Compute everything with gfun*

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Sketch of the proof - Borel-Laplace

$g(z)$ —Laplace Transform→ $\mathcal{L}(g)(t) = \dfrac{\exp\left(-\frac{{\sigma_x}^2 {y_m}^2 + {\sigma_y}^2 {x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2t{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2t{\sigma_x}^2+1)}\right)}{t\sqrt{(2t{\sigma_x}^2+1)(2t{\sigma_y}^2+1)}}$

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$g(z)$ —→ $g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$

$g(z)$ is:

- D-finite
- entire function of exponential type
- type $\sigma = \frac{1}{2\sigma_y^2}$
- sum prone to cancellation

$\hat{\mathcal{L}}(g)(t)$ is:

- D-finite
- Finite radius of convergence $2\sigma_y^2$

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Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

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$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots$$

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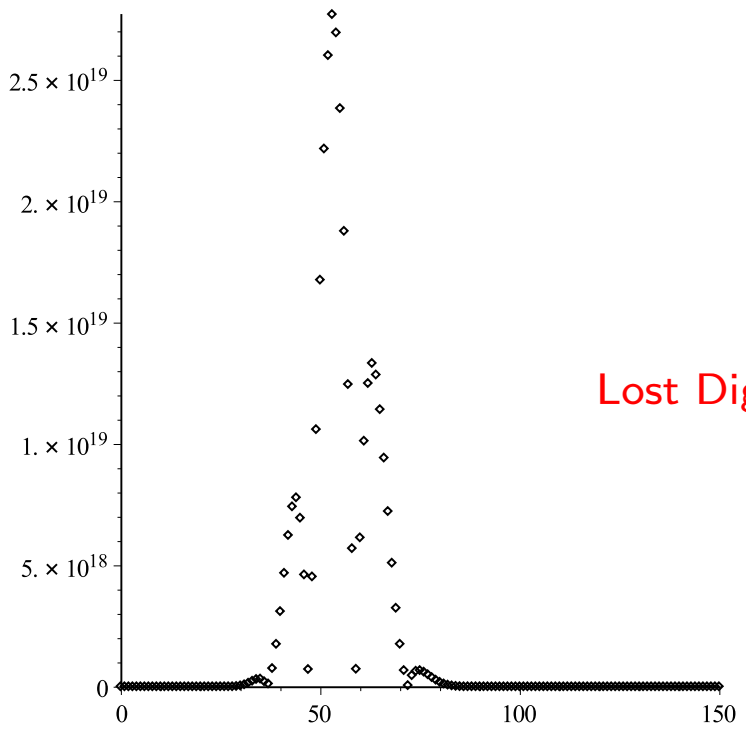
Values of $\left| \frac{\ell_i 225^{i+1}}{(i+1)!} \right|$, compared to $g(225) \simeq 0.1004$:



Lost Digits: $d_g(z) \simeq \log \dfrac{\max_i \left| g_i z^i \right|}{|g(z)|}$

Cancellation in finite precision power series evaluation

Example: $\exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}$

$$\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots$$

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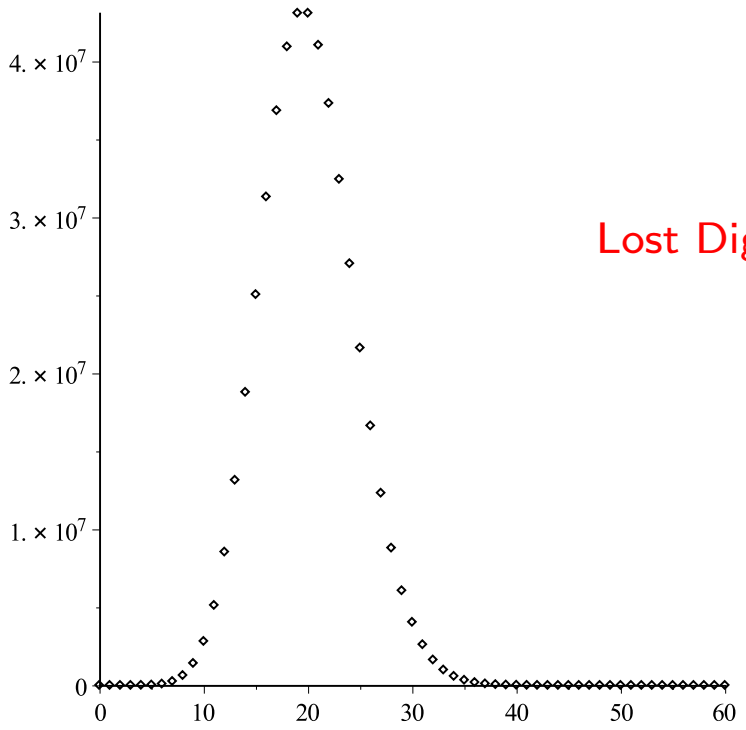

Lost Digits: $d_g(z) \simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

Cancellation in finite precision power series evaluation

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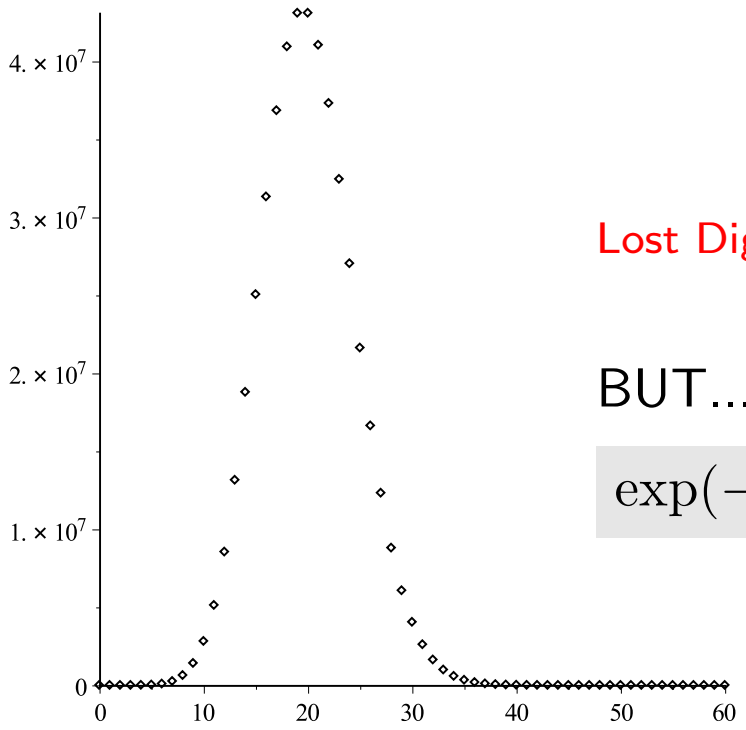

Lost Digits: $d_g(z) \simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

BUT...

$$\exp(-x) = \frac{1}{\exp(x)}$$

No cancellation!

Cancellation in finite precision power series evaluation

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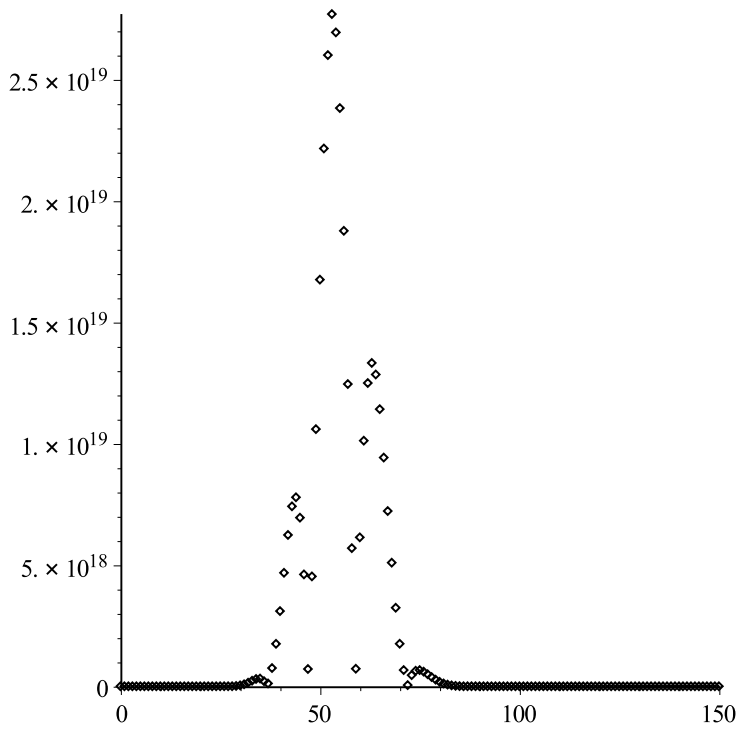

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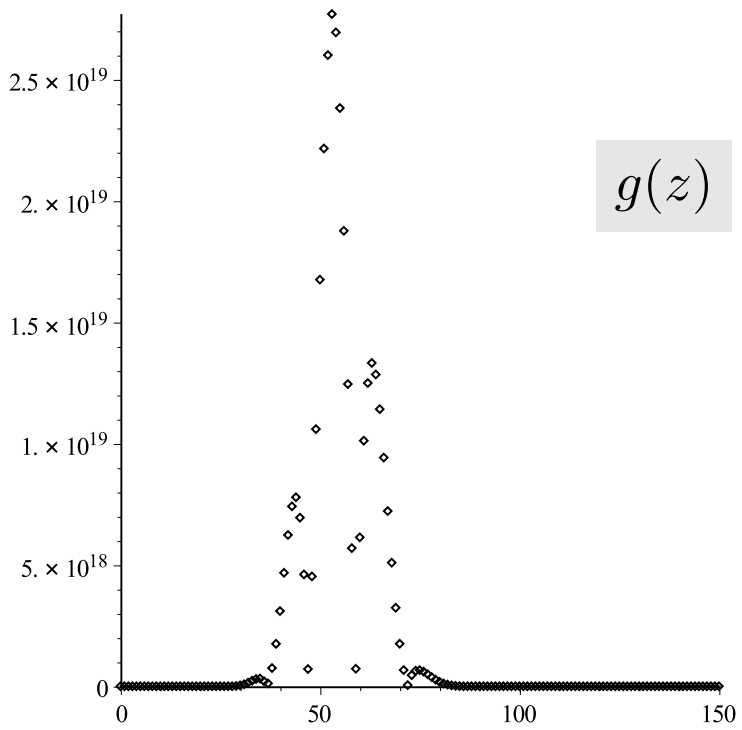

$$g(z) = \frac{F(z)}{G(z)}$$

No cancellation!

Cancellation in finite precision power series evaluation

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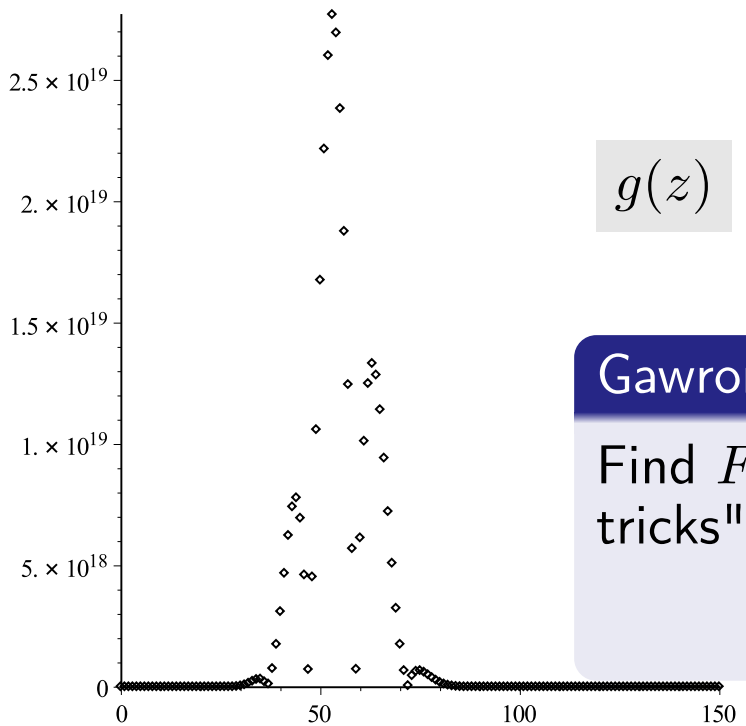

$$g(z) = \frac{F(z)}{G(z)}$$

No cancellation!

Gawronski, Müller, Reinhard (2007) Method:

Find $F(z)$ and $G(z)$ using "complex analysis tricks" $\rightsquigarrow$ indicator function more.

$$\rightsquigarrow G(z) = \exp(\sigma z).$$

Sketch of the proof - Borel-Laplace + GMR Method

$$\begin{tikzcd}
G(z)g(z) = F(z) \arrow[r, "\text{Laplace}"'] & \mathcal{L}(g)(t-\sigma) \arrow[dr, "\text{expansion at } \infty"] & \\
& & \sum_{k=0}^{\infty} \alpha_k \left(\frac{1}{t}\right)^k \arrow[dl, "\text{Borel Transform}"'] \\
G(z)g(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \arrow[u] &&
\end{tikzcd}$$

- $G(z) = \exp(\sigma z)$
- F is D-finite
- reduced cancellation for evaluating F, G on positive real line

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\end{tikzcd}$$

- $G(z) = \exp(\sigma z)$
- F is D-finite
- reduced cancellation for evaluating F, G on positive real line
- recurrence for α_k:

$$\begin{aligned}
-(32k\sigma_x^4\sigma_y^{10} + 128\sigma_x^4\sigma_y^{10})\alpha_{k+4} &= (\sigma_x^4 y_m^2 - 2\sigma_x^2\sigma_y^2 y_m^2 + \sigma_y^4 y_m^2)\alpha_k \\
+((-4\sigma_x^4\sigma_y^4 + 8\sigma_x^2\sigma_y^6 - 4\sigma_y^8)k - 10\sigma_x^4\sigma_y^4 - 6\sigma_x^4\sigma_y^2 y_m^2 + 20\sigma_x^2\sigma_y^6 + 8\sigma_x^2\sigma_y^4 y_m^2 - 10\sigma_y^8 - 2\sigma_y^6 y_m^2)\alpha_{k+1} \\
+((24\sigma_x^4\sigma_y^6 - 32\sigma_x^2\sigma_y^8 + 8\sigma_y^{10})k + 72\sigma_x^4\sigma_y^6 + 12\sigma_x^4\sigma_y^4 y_m^2 - 92\sigma_x^2\sigma_y^8 - 8\sigma_x^2\sigma_y^6 y_m^2 + 20\sigma_y^{10} + 4\sigma_y^8 x_m^2)\alpha_{k+2} \\
+((-48\sigma_x^4\sigma_y^8 + 32\sigma_x^2\sigma_y^{10})k - 168\sigma_x^4\sigma_y^8 - 8\sigma_x^4\sigma_y^6 y_m^2 + 104\sigma_x^2\sigma_y^{10} - 8\sigma_y^{10} x_m^2)\alpha_{k+3},
\end{aligned}$$

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- $G(z) = \exp(\sigma z)$
- F is D-finite
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- recurrence for α_k:

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+((-4\sigma_x^4\sigma_y^4 + 8\sigma_x^2\sigma_y^6 - 4\sigma_y^8)k - 10\sigma_x^4\sigma_y^4 - 6\sigma_x^4\sigma_y^2 y_m^2 + 20\sigma_x^2\sigma_y^6 + 8\sigma_x^2\sigma_y^4 y_m^2 - 10\sigma_y^8 - 2\sigma_y^6 y_m^2)\alpha_{k+1} \\
+((24\sigma_x^4\sigma_y^6 - 32\sigma_x^2\sigma_y^8 + 8\sigma_y^{10})k + 72\sigma_x^4\sigma_y^6 + 12\sigma_x^4\sigma_y^4 y_m^2 - 92\sigma_x^2\sigma_y^8 - 8\sigma_x^2\sigma_y^6 y_m^2 + 20\sigma_y^{10} + 4\sigma_y^8 x_m^2)\alpha_{k+2} \\
+((-48\sigma_x^4\sigma_y^8 + 32\sigma_x^2\sigma_y^{10})k - 168\sigma_x^4\sigma_y^8 - 8\sigma_x^4\sigma_y^6 y_m^2 + 104\sigma_x^2\sigma_y^{10} - 8\sigma_y^{10} x_m^2)\alpha_{k+3},
\end{aligned}$$

- We can prove that α_k are positive more
- Lower/upper bounds for α_k with majorant series

Bounds using majorant series

Let $\gamma := \dfrac{1 + \frac{1}{2}\left(1 - \dfrac{\sigma_y^2}{\sigma_x^2} + \dfrac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{{y_m}^2}{{\sigma_y}^2}\right)}{2{\sigma_y}^2}$, $\overline{\alpha}_k := \alpha_0 \gamma^k$ and $\underline{\alpha}_k := \alpha_0 \left(\dfrac{1}{2{\sigma_y}^2}\right)^k$.

Then $\underline{\alpha}_k \leq \alpha_k \leq \overline{\alpha}_k, \forall k \in \mathbb{N}$.

Bounds using majorant series

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Then $\underline{\alpha}_k \leq \alpha_k \leq \overline{\alpha}_k, \forall k \in \mathbb{N}$.

Let $\tilde{P}_N(z) := \sum_{k=0}^{N-1} \dfrac{\alpha_k z^k}{(k+1)!}$. Then we have the following error bounds:

$$\underline{\varepsilon}_N(z) \leq g(z) - \tilde{P}_N(z) \leq \overline{\varepsilon}_N(z), \text{ where}$$

$$\underline{\varepsilon}_N(z) := 2\alpha_0 \sigma_y^2 e^{-\frac{z}{2\sigma_y^2}} \frac{\left(\frac{z}{2\sigma_y^2}\right)^{N+1}}{(N+1)!},$$

$$\overline{\varepsilon}_N(z) := \frac{\alpha_0}{\gamma} e^{-\frac{z}{2\sigma_y^2}} \frac{(z\gamma)^{N+1}}{(N+1)!}.$$

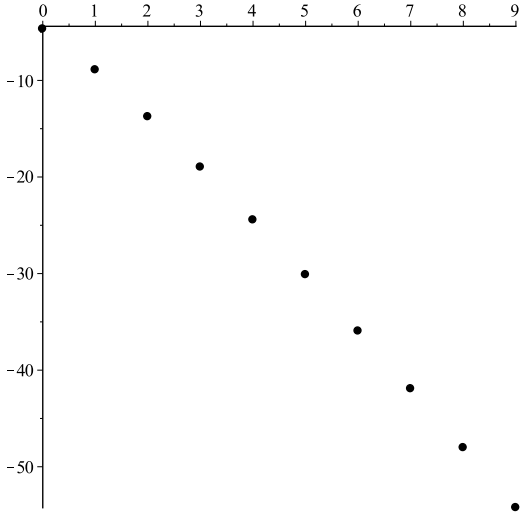
Examples

Sample 1

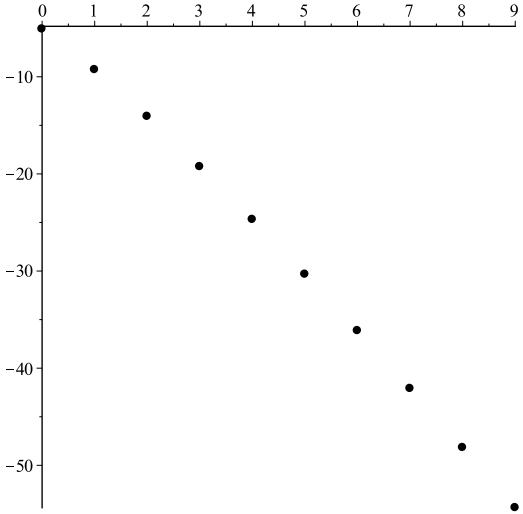
Case #	Input parameters (km)				
	σ_x	σ_y	R	x_m	y_m
1	0.05	0.025	0.005	0.01	0
2	0.05	0.025	0.005	0	0.01
3	0.075	0.025	0.005	0.01	0
4	0.075	0.025	0.005	0	0.01
5	3	1	0.01	1	0
6	3	1	0.01	0	1
7	3	1	0.01	10	0
8	3	1	0.01	0	10
9	10	1	0.01	10	0
10	10	1	0.01	0	10
11	3	1	0.05	5	0
12	3	1	0.05	0	5

Examples: quality $\eta = -\log \dfrac{\overline{\varepsilon}_{10}(z) - \underline{\varepsilon}_{10}(z)}{\underline{\varepsilon}_{10}(z) + \tilde{P}_{10}(z)}$ and plot $\log\left(\tilde{p}_i z^i\right)$

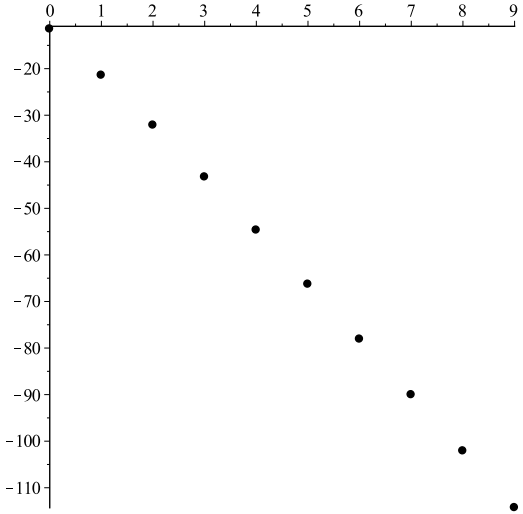
Case 1: $\eta = 23$



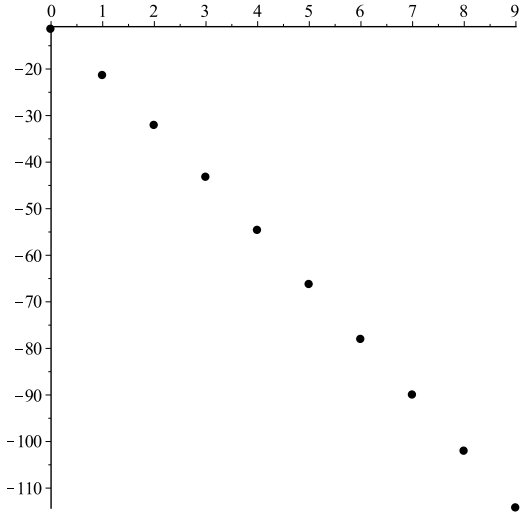
Case 2: $\eta = 22$



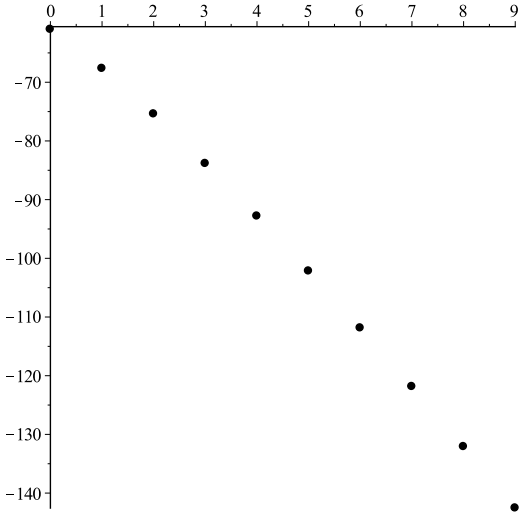
Case 4: $\eta = 22$



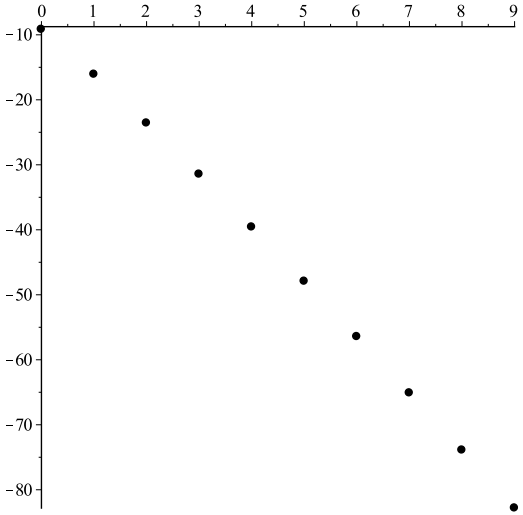
Case 6: $\eta = 47$



Case 8: $\eta = 33$



Case 11: $\eta = 34$



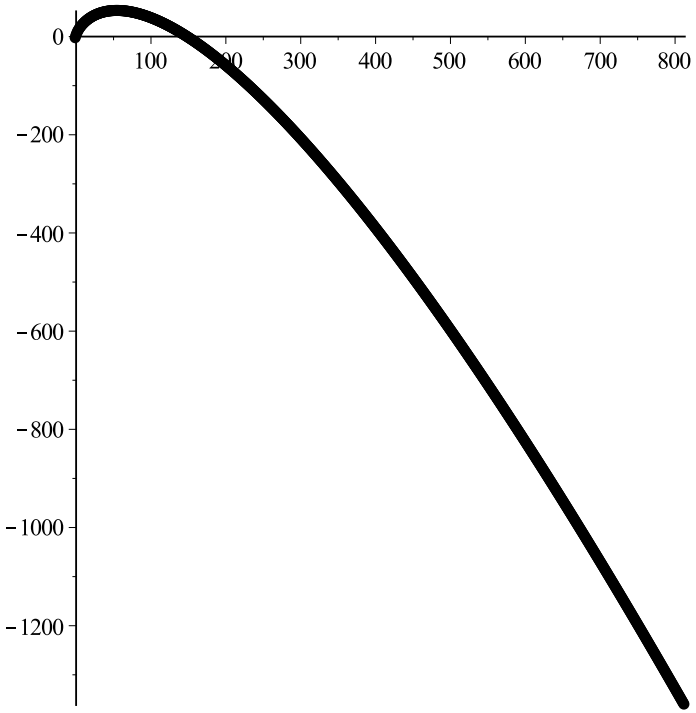
Numerical study

Sample 1

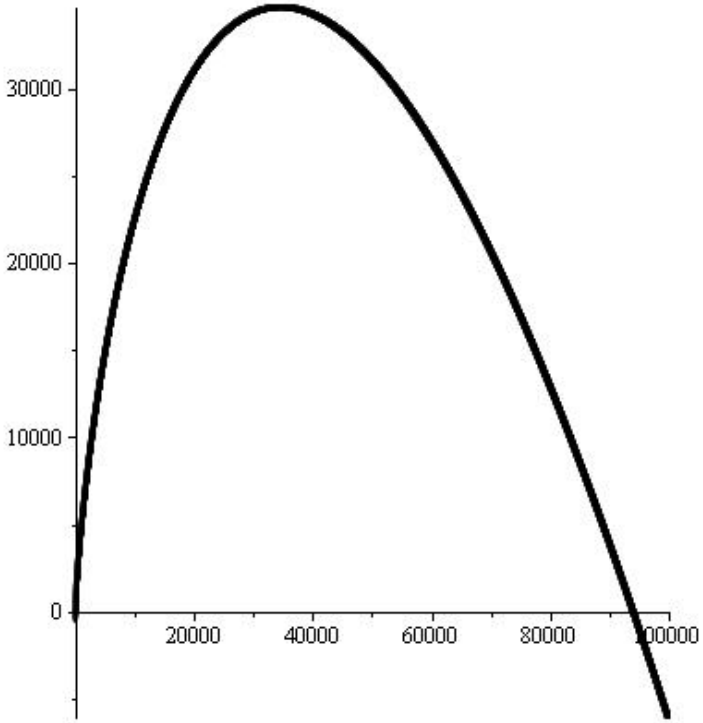
Case #	Probability of Collision (-)			
	Alfano	Patera	Chan	New method
1	9.742×10^{-3}	9.741×10^{-3}	9.754×10^{-3}	9.742×10^{-3}
2	9.181×10^{-3}	9.181×10^{-3}	9.189×10^{-3}	9.181×10^{-3}
3	6.571×10^{-3}	6.571×10^{-3}	6.586×10^{-3}	6.571×10^{-3}
4	6.125×10^{-3}	6.125×10^{-3}	6.135×10^{-3}	6.125×10^{-3}
5	1.577×10^{-5}	1.577×10^{-5}	1.577×10^{-5}	1.577×10^{-5}
6	1.011×10^{-5}	1.011×10^{-5}	1.011×10^{-5}	1.011×10^{-5}
7	6.443×10^{-8}	6.443×10^{-8}	6.443×10^{-8}	6.443×10^{-8}
8	0	3.219×10^{-27}	3.216×10^{-27}	3.219×10^{-27}
9	3.033×10^{-6}	3.033×10^{-6}	3.033×10^{-6}	3.033×10^{-6}
10	0	9.656×10^{-28}	9.645×10^{-28}	9.656×10^{-28}
11	1.039×10^{-4}	1.039×10^{-4}	1.039×10^{-4}	1.039×10^{-4}
12	1.564×10^{-9}	1.564×10^{-9}	1.556×10^{-9}	1.564×10^{-9}

Examples: quality $\eta_N = -\log \dfrac{\overline{\varepsilon}_N(z) - \underline{\varepsilon}_N(z)}{\underline{\varepsilon}_N(z) + \tilde{P}_N(z)}$ and plot $\log\left(\tilde{p}_i z^i\right)$

Case 3 Alfano: $\eta_{800} = 30$



Case 5 Alfano: $\eta_{121000} = 20$



Sample 2 (from [Alfano 2009])

Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

Examples

Sample 2 (from [Alfano 2009])

Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

Alfano's test case number	Probability of collision (-)		
	Alfano	New method	Reference (MC)
3	0.10038	0.10038	0.10085
5	0.044712	0.045509	0.044499

Collision Probability: Sum up

New method

- Analytical formula
- Reduced cancellation evaluation
- Error bounds
- No simplifying assumption
- CNES implemented/tested and now uses it in practice

Current and future work

- Saddle-point method for "degenerate" cases
- Long-term/multiple encounter (new collaboration with CNES)

Safety-critical space applications (2)

A goal of space missions

Achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption
e.g., rendezvous planning tools for formation flight (PRISMA), on-orbit satellite servicing, supply missions (ISS)...

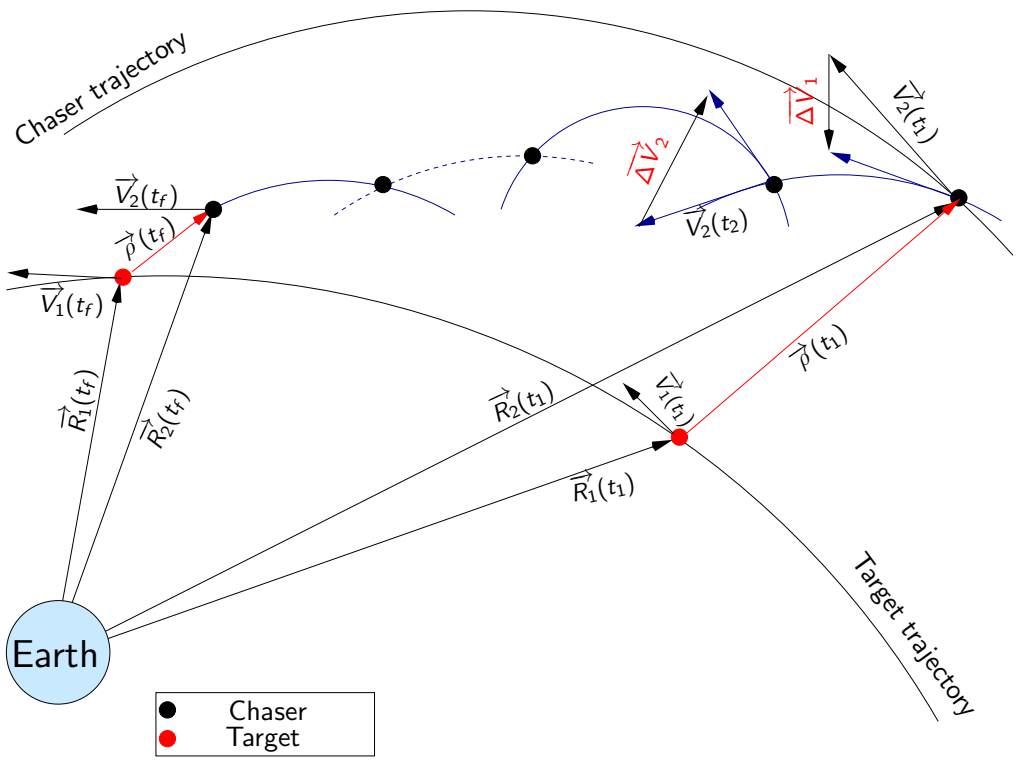
Figure: Copyright Gravity (2013):Ryan's RdV with the Chinese Space Station using a fire extinguisher

Second goal of this talk:
efficient and reliable numerical optimal control for space rendezvous.

Definition of space RdV

RdV Problem

Compute the fuel-optimal guidance law necessary for the chaser to reach the target in fixed-time $t_f - t_0$, given assigned initial and final relative positions and velocities.



Assumption:

- Target is passive
- Time-fixed
- $0 \leq e < 1$
- Relative navigation
- Impulsive thrust
- Keplerian dynamics

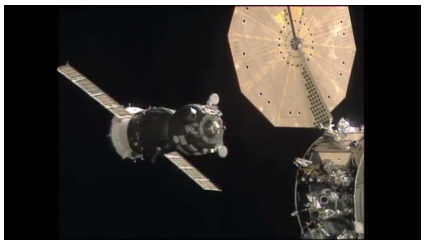
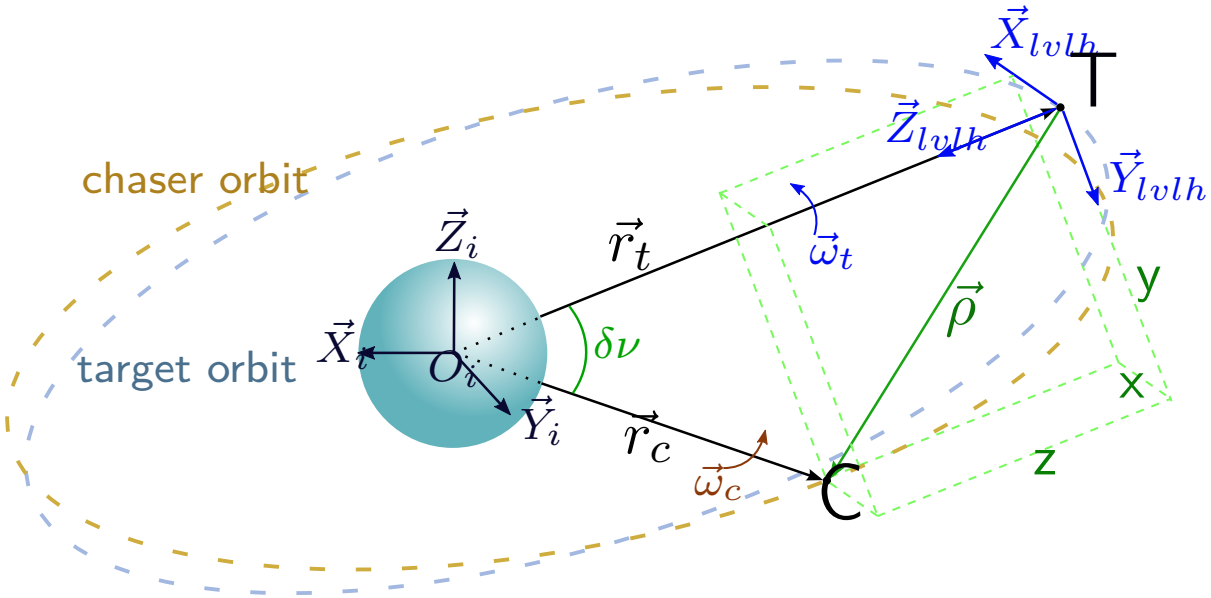


Figure: Soyuz TMA-19M docking to ISS (December 2015) - ISS resupply mission; **NASA YouTube© Chanel**

Relative motion & dynamics



- $\vec{r}_t,\ \vec{r}_c$ target and chaser position in Earth Centered Inertial Frame (ECI) $(\vec{X}_i,\ \vec{Y}_i,\ \vec{Z}_i)$;
- Target centered rotating frame, LVLH convention $(\vec{X}_{lvlh},\ \vec{Y}_{lvlh},\ \vec{Z}_{lvlh})$:
- Target–Chaser distance small wrt. Target–Earth radius: $\|\vec{\rho}\| \ll \|\vec{r}_t\|$

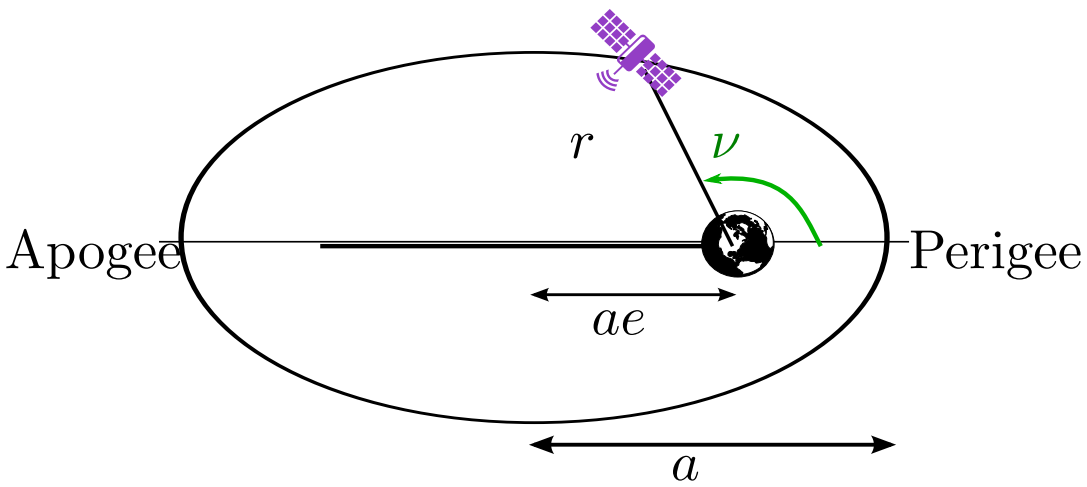
Linearized relative motion:
Tschauner-Hempel eqs.

$$\begin{cases} x'' = 2z \\ y'' = -y \\ z'' = -2x' + \frac{3}{1+e\cos\nu} z \end{cases}$$

out-of-plane

in-plane

The target orbit is elliptical: eccentricity $0 \leq e < 1$, semi-major axis a, true anomaly ν:

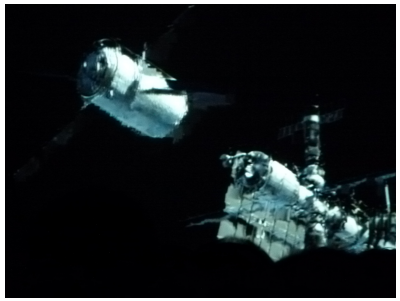


- $r_t = \frac{a(1-e^2)}{(1+e\cos(\nu))}$;
- Kinetic moment is constant : $r^2\dot{\nu} = \sqrt{\mu a(1-e^2)} \rightsquigarrow \dot{\nu} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1+e\cos(\nu))^2$.

Example for the in-plane case

Automated Transfer Vehicle (ATV) mission

- European unmanned vehicle 2008-2014 (ESA, ADS, CNES);
- Logistic servicing of the ISS (dry cargo, water and gas delivery, ISS refueling, ISS reboost, contribution to ISS attitude control, and waste disposal)

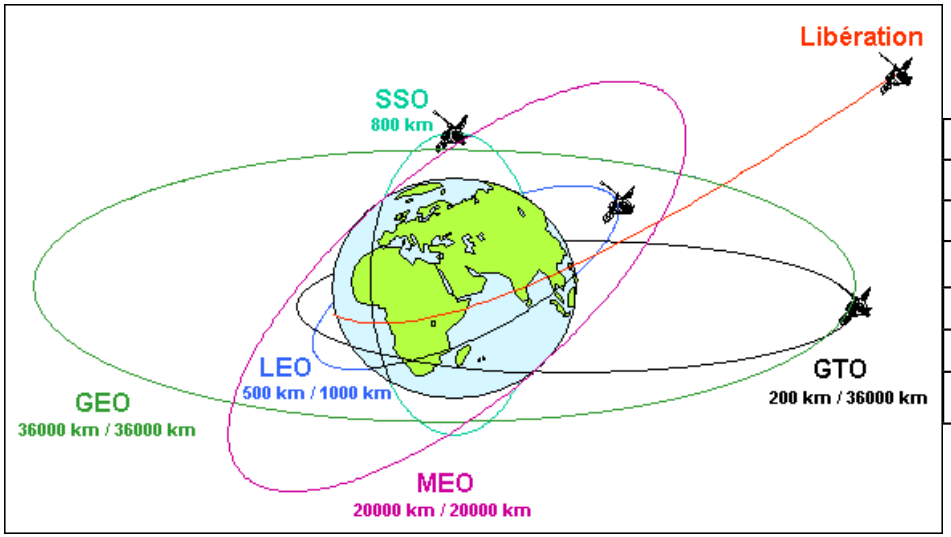


Semi-major axis	a = 6763 km.
Inclination	i=52 deg.
Argument of perigee	ω=0 deg.
Longitude of the ascending node	Ω= 0 deg.
Eccentricity	e = 0.0052
Initial time	ν_0 = 0 rad.
Initial state vector X_0^T	[-30 0.5 8.514 0] km. - m/s.
Final anomaly	ν_f = 8.1832 rad.
Duration	$t_f - t_0$ = 7200 s.
Final state vector X_f^T	[-100 0 0 0] m. - m/s.

Parameters of the ATV example [Labourdette08]

Example for out-of-plane GTO Mission [Zhou 2011]

- Target satellite on Geostationary Transfer Orbit (GTO)
- Temporary elliptic orbit to inject a satellite into the Geostationary Earth Orbit (GEO)
- Perigee on LEO (altitude $\simeq$ 200 km) and apogee on GEO (altitude = 35786 km)
- Requires a change of orbital plane and out-of-plane maneuvers



Semi-major axis	a = 24616 km
Eccentricity	e = 0.73074
Initial anomaly	$\theta_0 = 0.1\pi$ rad
Initial state vector	X_0^T = [10000 -3] m - m/s
Final anomaly	$\theta_f = 5.2$ rad
Duration	$t_f - t_0 = 29888$ s
Final state vector	X_f^T = [0 0] m - m/s

Goals of this study

- Many works since '60s $\rightsquigarrow$ Revisit mathematical formulation of this problem
- New convergent iterative numerical algorithm; lower/upper bounds on optimal cost

Mathematical formulation of RdV problem

- Simplified Tschauner-Hempel equations with ν as indpt var;

$$\left.\begin{array}{ll} x'' = & 2z \\ y'' = & -y \\ z'' = & -2x' + \frac{3}{1+e\cos\nu}\tilde{z} \end{array}\right\} \Rightarrow X' = A(\nu)X$$

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Optimal control Problem (after variable change $t \to \nu$)

Find $\bar{u} \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

$$\begin{array}{ll} \inf\limits_u & \|u\|_{1,p} = \inf\limits_u \displaystyle\int_{\nu_0}^{\nu_f} \|u(\nu)\|_p \mathrm{d}\nu \\ \text{s.t.} & X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu),\ \forall\ \nu \in\ [\nu_0, \nu_f] \\ & X(\nu_0) = X_0,\ X(\nu_f) = X_f \in \mathbb{R}^n,\ \nu_0,\ \nu_f \text{ fixed}, \end{array} \tag{1}$$

Thruster configuration determine p-norm:

– a gimbaled single thruster: $p = 2$
– ungimbaled identical thrusters: $p = 1$

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Previous Works

- Indirect approach, Pontryagin max principle, primer vector theory [Lawden63], [Carter&Brient95]
- Iterative algorithm, calculus of variations, [Lion and Handelsman1968], [Prussing2010]
- Mixed iterative algorithm [Arzelier et al.(2013)]
- Relaxation scheme, duality theory, minimum-norm problem [Neustadt64],[Claeys et al.(2013)]

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Our approach:

- [Neustadt64] $\rightsquigarrow$ Moment Problem $\rightsquigarrow$ Duality [Luenberger1969] $\rightsquigarrow$ Semi-Infinite Convex Programming (SICP)[Reemtsen&Ruckman1998] $\rightsquigarrow$ New convergent iterative numerical algorithm

Mathematical formulation of RdV problem

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Solution of the LODE system:

Mathematical formulation of RdV problem

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- $\varphi(\nu)$ a fundamental matrix for the homogeneous equation [Yamanaka&Ankersen02]

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- $\displaystyle\int_{\nu_0}^{\nu_f} \underbrace{\varphi^{-1}(\sigma)B(\sigma)}_{Y(\sigma)} u(\sigma)\mathrm{d}\sigma = \underbrace{\varphi^{-1}(\nu_f)X(\nu_f) - \varphi^{-1}(\nu_0)X_0}_{c}$

Mathematical formulation of RdV problem

Minimum Norm Moment Problem

$$\begin{aligned} \inf_u \quad & \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p \mathrm{d}\nu \\ \text{s.t.} \quad & \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)\mathrm{d}\sigma = c, \ \nu_0, \ \nu_f \text{ fixed.} \end{aligned} \tag{4}$$

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- Relax problem on the set of functions of bounded variation g:
  $\rightsquigarrow$ Duality with linear functional ℓ: $\ell(y_i) = \langle y_i, \ell \rangle = \int_{\nu_0}^{\nu_f} y_i(\nu)^T \mathrm{d}g(\nu)$.

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- [Neustadt64] proved existence of optimal solution g as a step function with at most n points of discontinuity: 2 for out-of-plane and 4 for in-plane.

Semi-Infinite Convex Problem Formulation

[Neustadt64] proves the equivalence with the following problem:

SICP problem

Find $\bar{\lambda} \in \mathbb{R}^n$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^n} \quad -c^T \lambda$$
$$\underbrace{\|Y^T(\nu)\lambda\|_q}_{\text{primer vector}} \leq 1, \ \nu \in [\nu_0, \nu_f]. \tag{5}$$

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– ungimbaled identical thrusters: $p = 1, \ q = \infty$

Linear programing (LP) problem:

$$\begin{aligned} &\inf_{\lambda \in \mathbb{R}^n} \quad -c^T\lambda \\ &\text{s.t.} \ \left| \sum_{i=1}^{n} \lambda_i y_{i,s}(\nu) \right| \leq 1, \\ &\nu \in [\nu_0, \nu_f], \ s = 1, \ldots, r. \end{aligned} \tag{7}$$

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– gimbaled single thruster: $p = 2, \ q = 2$

Positive semi-definite (SDP) problem:

$$\begin{aligned} \inf_{\lambda \in \mathbb{R}^n} \quad & -c^T\lambda \\ \text{s.t.} \quad & \begin{bmatrix} -1 & \lambda^T Y(\nu) \\ Y^T(\nu)\lambda & -\mathbf{1} \end{bmatrix} \preceq \mathbf{0}, \\ & \nu \in [\nu_0, \nu_f]. \end{aligned} \tag{6}$$

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$\rightsquigarrow$ Convergent iterative numerical approach[Reemsten90s]

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

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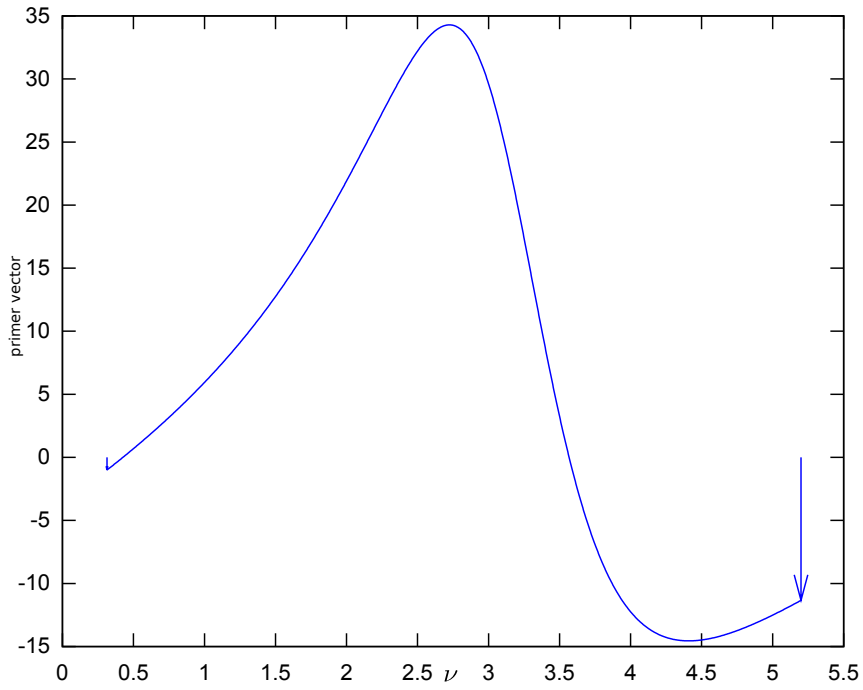
Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;



iteration 0

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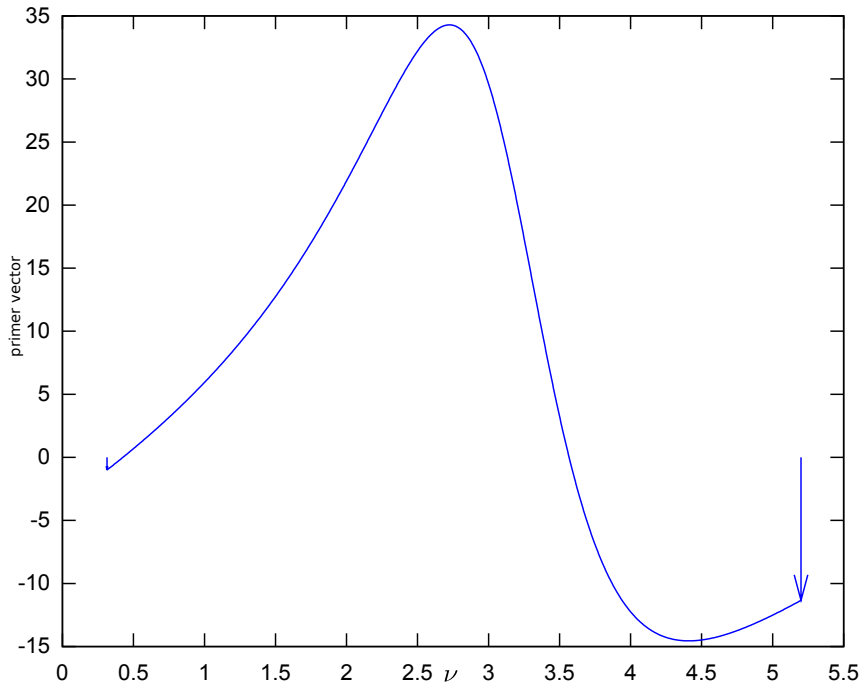
While $\max_{\nu} \mathrm{pv} > 1$

Step 1 Find $\nu_1 = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_1, \nu_f\}$



iteration 0

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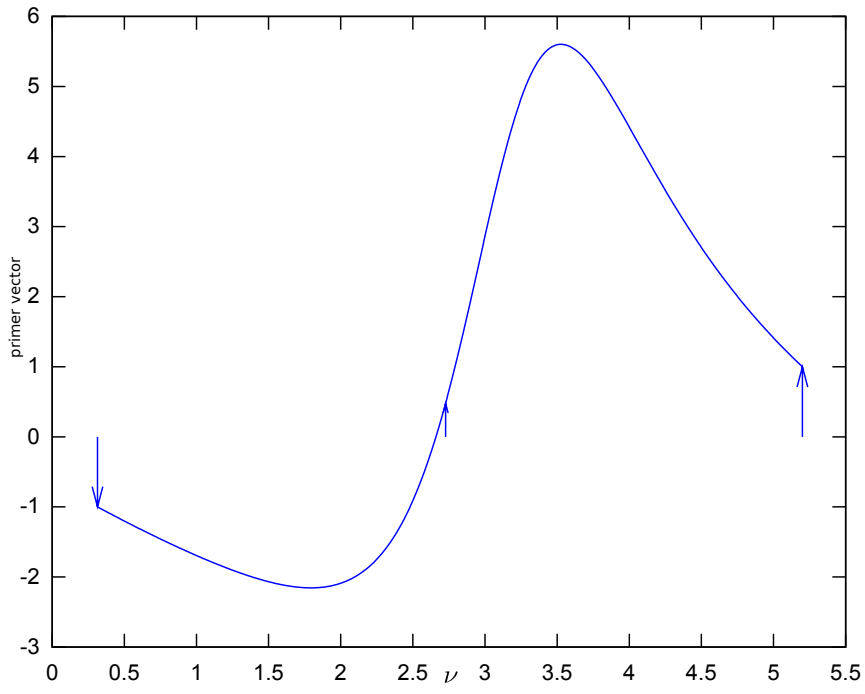
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector } \mathrm{pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \mathrm{pv} > 1$

Step 1 Find $\nu_1 = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_1, \nu_f\}$
Cost $-\mu_1 = 19.4198$



iteration 1

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

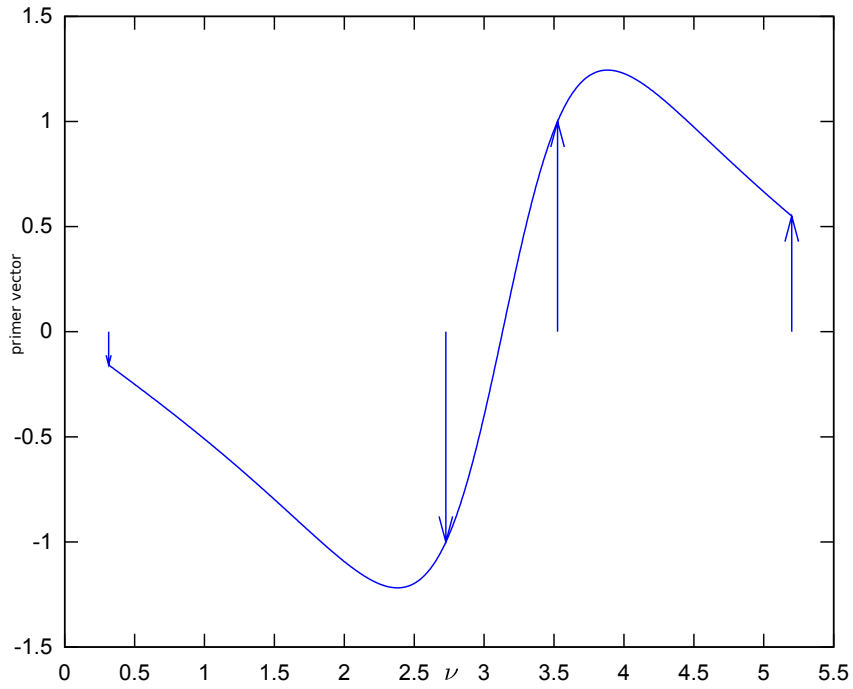
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \text{pv} > 1$

Step 2 Find $\nu_2 = \arg\left[\max_{\nu} \text{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_2, \nu_f\}$
Cost $-\mu_2 = 7.7249$



iteration 2

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

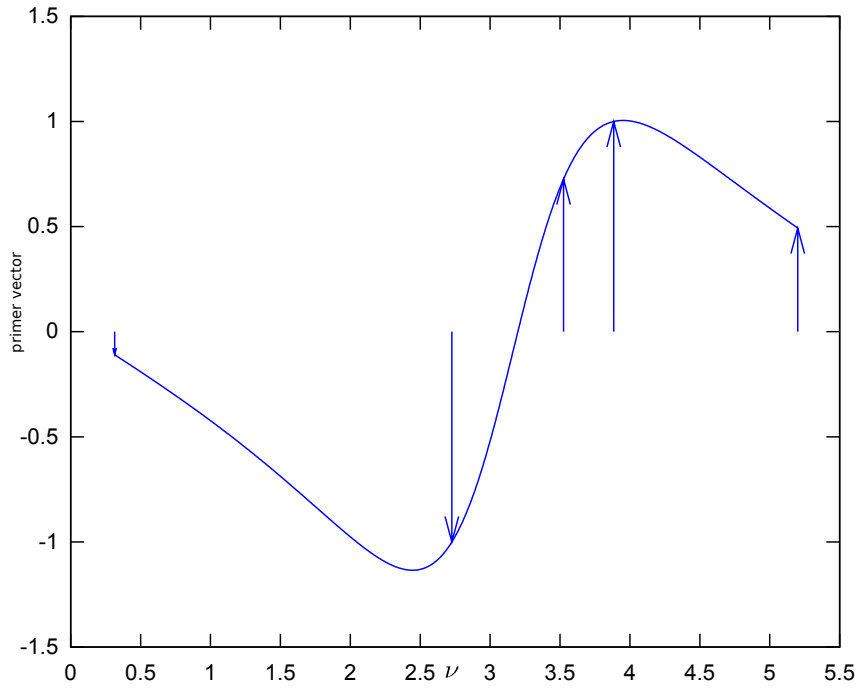
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \text{pv} > 1$

Step 3 Find $\nu_3 = \arg\left[\max_{\nu} \text{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_3, \nu_f\}$
Cost $-\mu_3 = 6.6837$



iteration 3

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

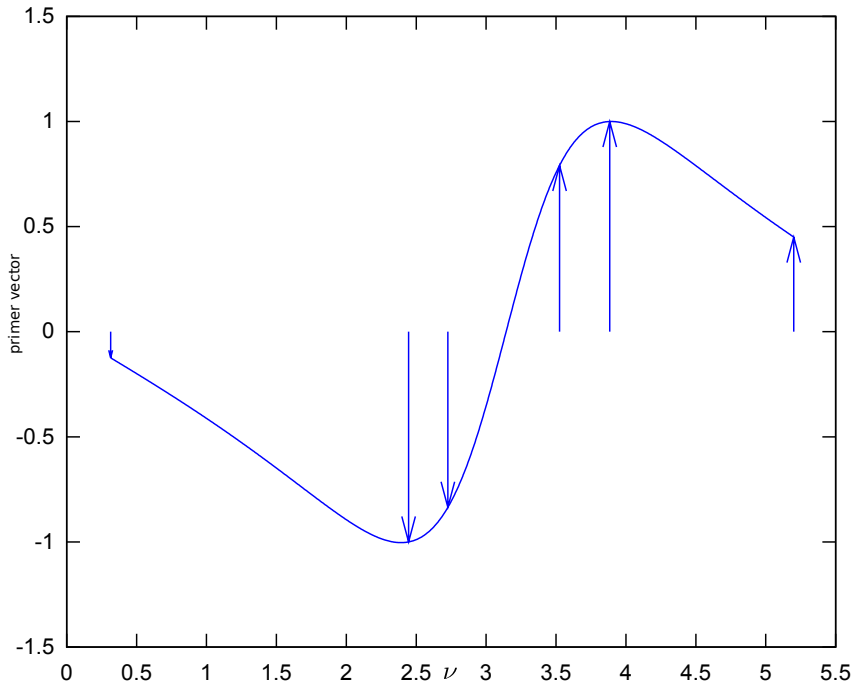
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \text{pv} > 1$

Step 4 Find $\nu_4 = \arg\left[\max_{\nu} \text{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_4, \nu_f\}$
Cost $-\mu_4 = 6.2832$



iteration 4

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

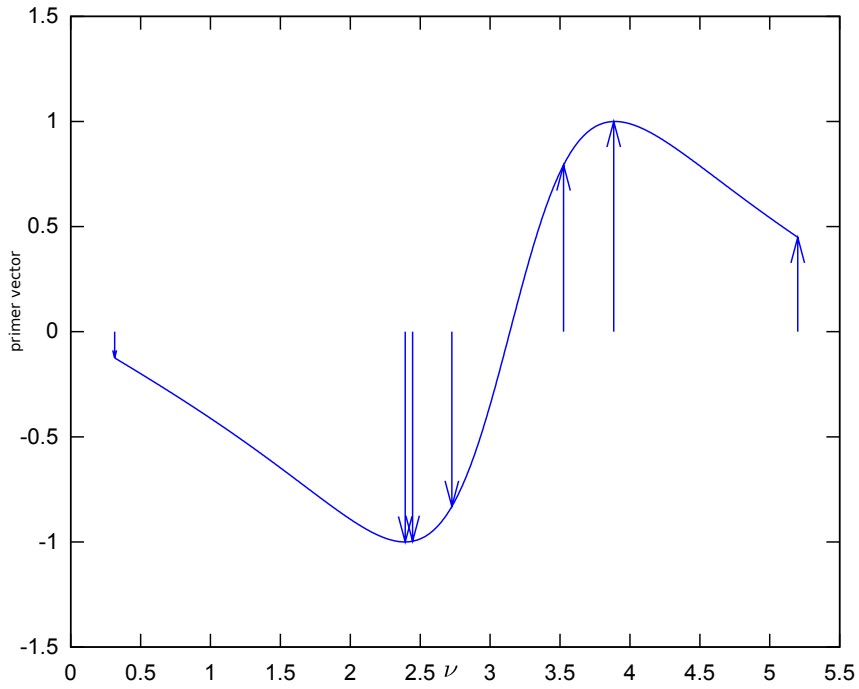
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \text{pv} > 1$

Step 5 Find $\nu_5 = \arg\left[\max_{\nu} \text{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_5, \nu_f\}$
Cost $-\mu_5 = 6.273242$



iteration 5

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

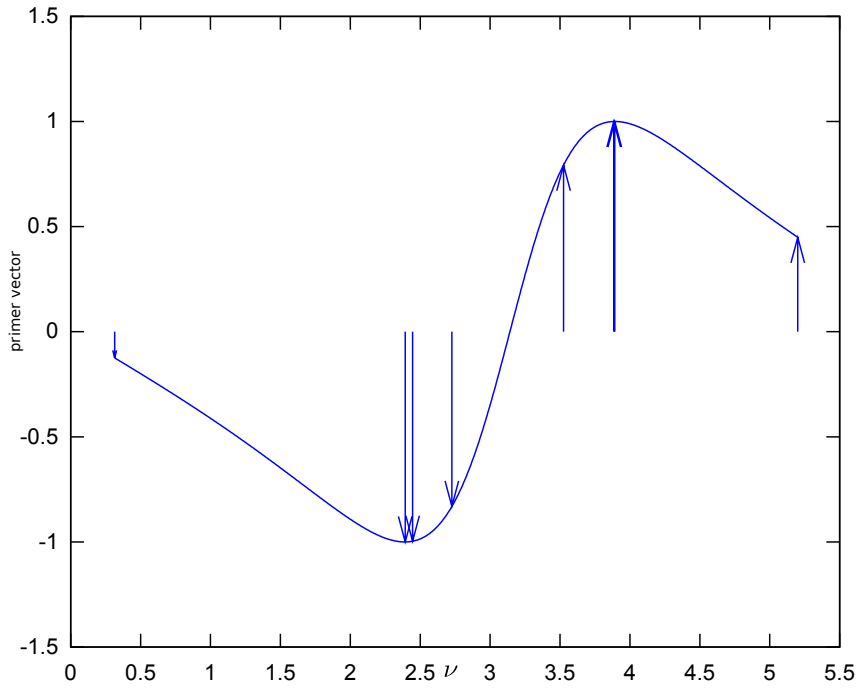
For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector } \mathrm{pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \mathrm{pv} > 1$

Step 6 Find $\nu_6 = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_6, \nu_f\}$
Cost $-\mu_6 = 6.273240$



iteration 6

Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

For GTO Mission[Zhou et al.2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \quad -0.0954]$.
Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

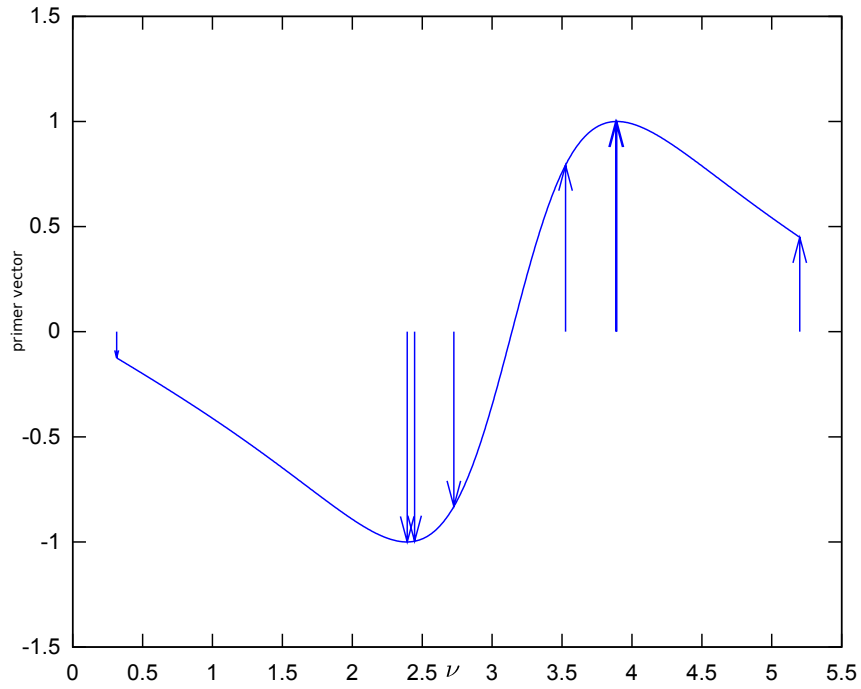
$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \quad -c^T \lambda$$
$$\underbrace{\left| \frac{\sin \nu\lambda_1 + \cos \nu\lambda_2}{1 + e \cos \nu} \right|}_{\text{primer vector } \mathrm{pv}(\nu)} \leq 1. \tag{8}$$

Step 0 Initialize (λ_1, λ_2), e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

While $\max_{\nu} \mathrm{pv} > 1$

Step 7 Find $\nu_7 = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_7, \nu_f\}$
Cost $-\mu_7 = 6.272461$

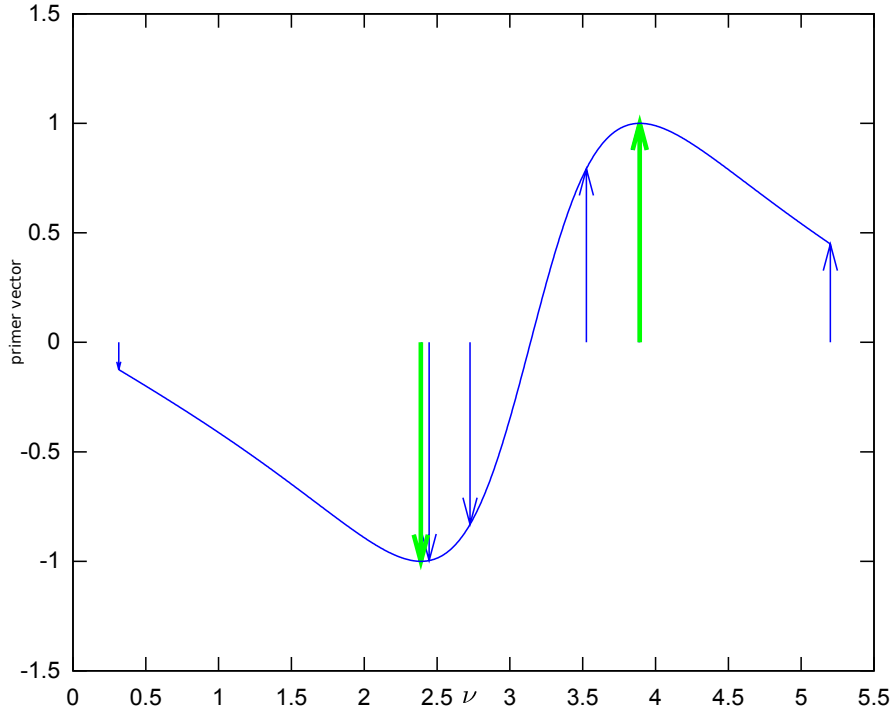
Final cost: 6.272461,
$\bar{\lambda} = [0.68270 \quad 0.000001246]$



iteration 7

Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.

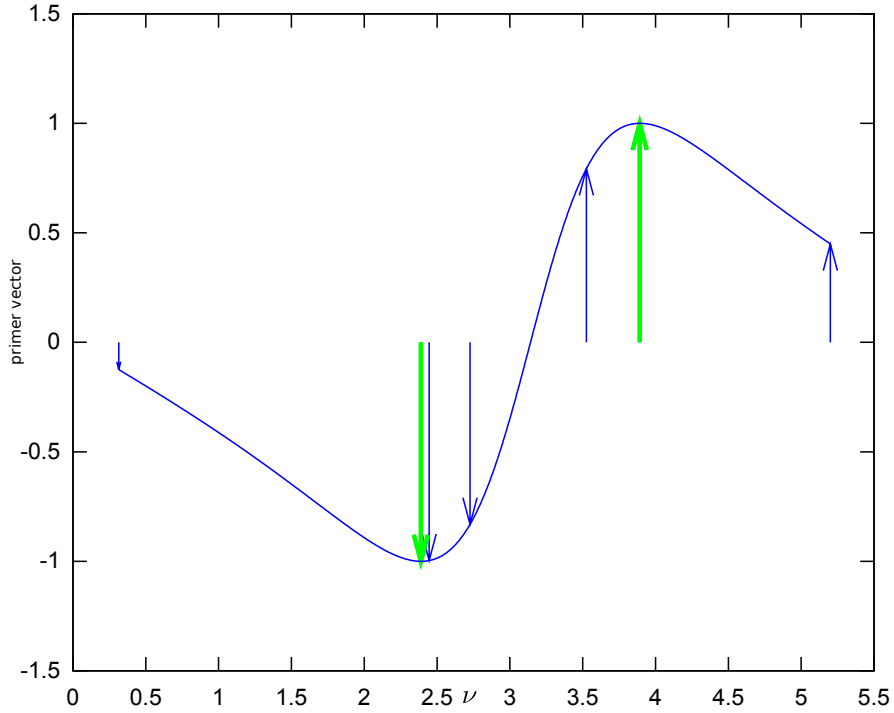


Example GTO mission (out-of-plane)

- primer vector $\simeq 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)
- impulse locations: $2.3883;$ 3.8919

Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.



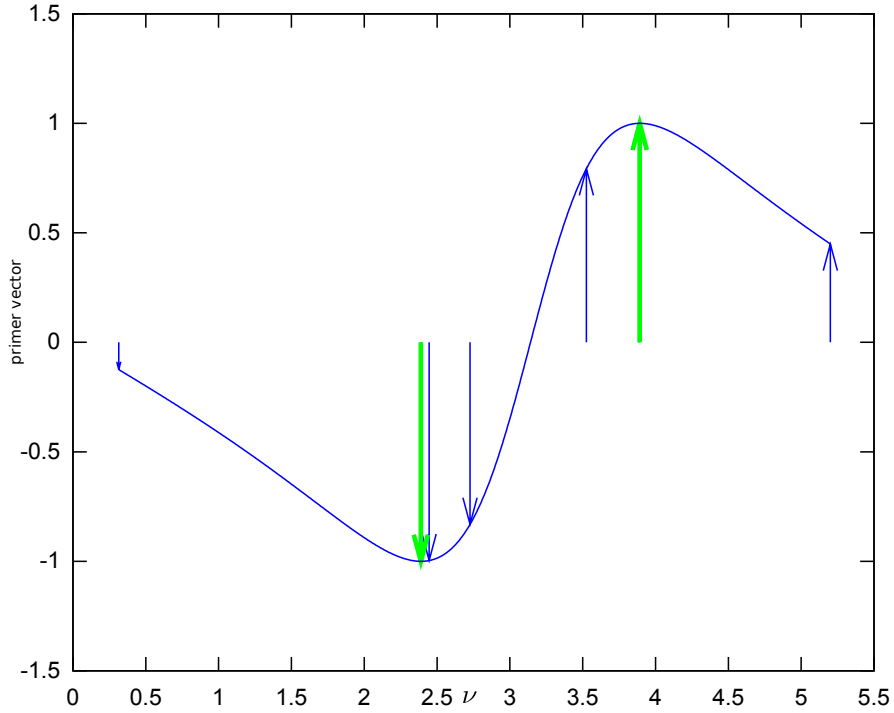
Example GTO mission (out-of-plane)

- primer vector $\simeq 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)
- impulse locations: 2.3883; 3.8919

- There exists an optimal solution with at most n (2 for out-of-plane and 4 for in-plane) impulses $\hat{\nu}_i, i = 1, \ldots, N \leq n$.

Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.



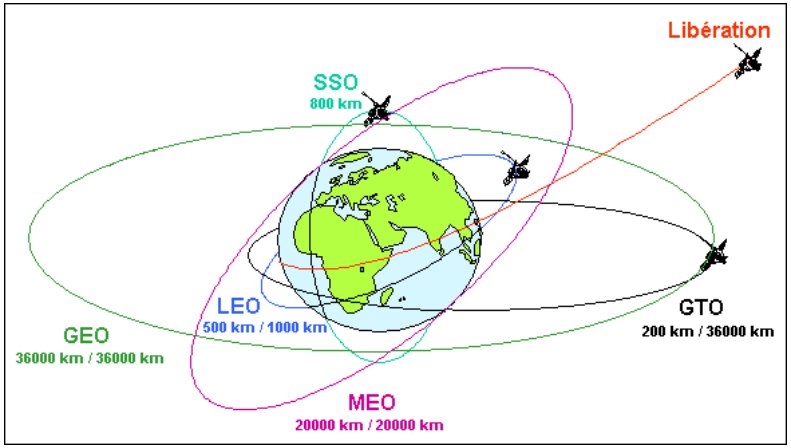
Example GTO mission (out-of-plane)

- primer vector $\simeq 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)
- impulse locations: 2.3883; 3.8919
- velocities increments: $\Delta V_1 = 3.115020$ m/s, $\Delta V_2 = 3.157441$ m/s.

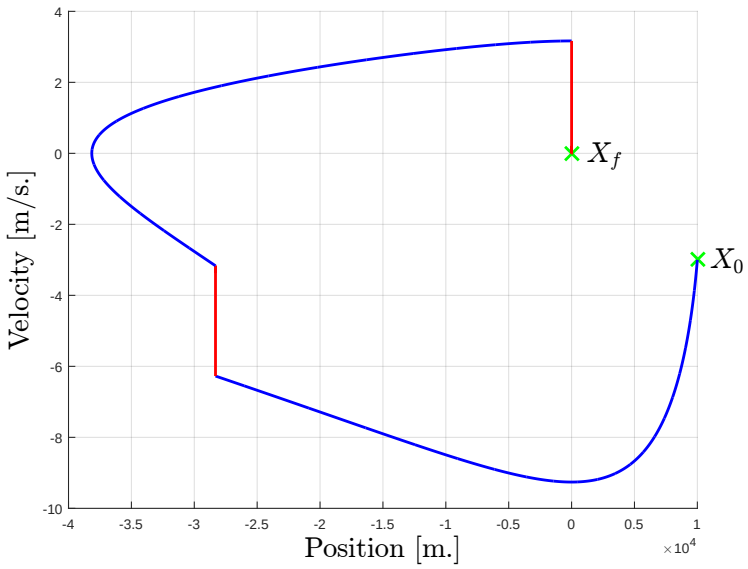
- There exists an optimal solution with at most n (2 for out-of-plane and 4 for in-plane) impulses $\hat{\nu}_i, i = 1, \ldots, N \leq n$.
- **Solve** for $\Delta V_i, i = 1, \ldots, N$, the linear system $c = \sum_i Y(\hat{\nu}_i)\Delta V_i$.

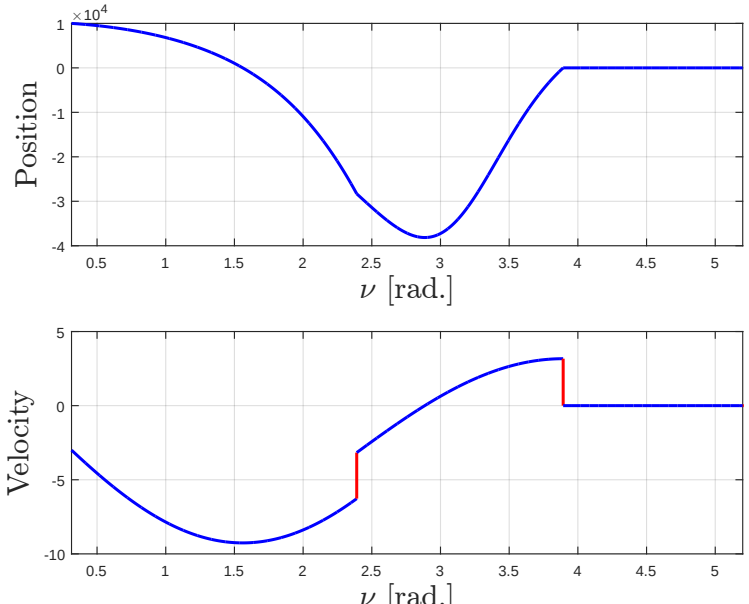
Results for GTO Mission [Zhou 2011]

- Target satellite on Geostationary Transfer Orbit (GTO)
- Temporary elliptic orbit to inject a satellite into the Geostationary Earth Orbit (GEO)
- Perigee on LEO (altitude $\simeq$ 200 km) and apogee on GEO (altitude = 35786 km)
- Requires a change of orbital plane and out-of-plane maneuvers



Semi-major axis	a = 24616 km
Eccentricity	e = 0.73074
Initial anomaly	$\theta_0 = 0.1\pi$ rad
Initial state vector	X_0^T = [10000 -3] m - m/s
Final anomaly	$\theta_f = 5.2$ rad
Duration	$t_f - t_0 = 29888$ s
Final state vector	X_f^T = [0 0] m - m/s





Example for the in-plane case

SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Example for the in-plane case

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Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

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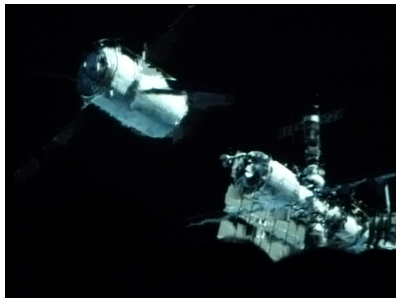
Automated Transfer Vehicle (ATV) mission - European unmanned vehicle 2008-2014 (ESA, ADS, CNES); Logistic servicing of the ISS [Labourdette08]:



Semi-major axis	a = 6763 km.
Inclination	i=52 deg.
Argument of perigee	ω=0 deg.
Longitude of the ascending node	Ω= 0 deg.
Eccentricity	e = 0.0052
Initial time	ν_0 = 0 rad.
Initial state vector X_0^T	[-30 0.5 8.514 0] km. - m/s.
Initial state vector $\tilde{X}_0^T$	[-51.9222 0.0865 0.95734 0].10^4
Final anomaly	ν_f = 8.1832 rad.
Duration	$t_f - t_0$ = 7200 s.
Final state vector X_f^T	[-100 0 0 0] m. - m/s.
Final state vector $\tilde{X}_f^T$	[-76.3818 0 69.1519 0]

Example for the in-plane case

SICP problem for in-plane, $q = 2$

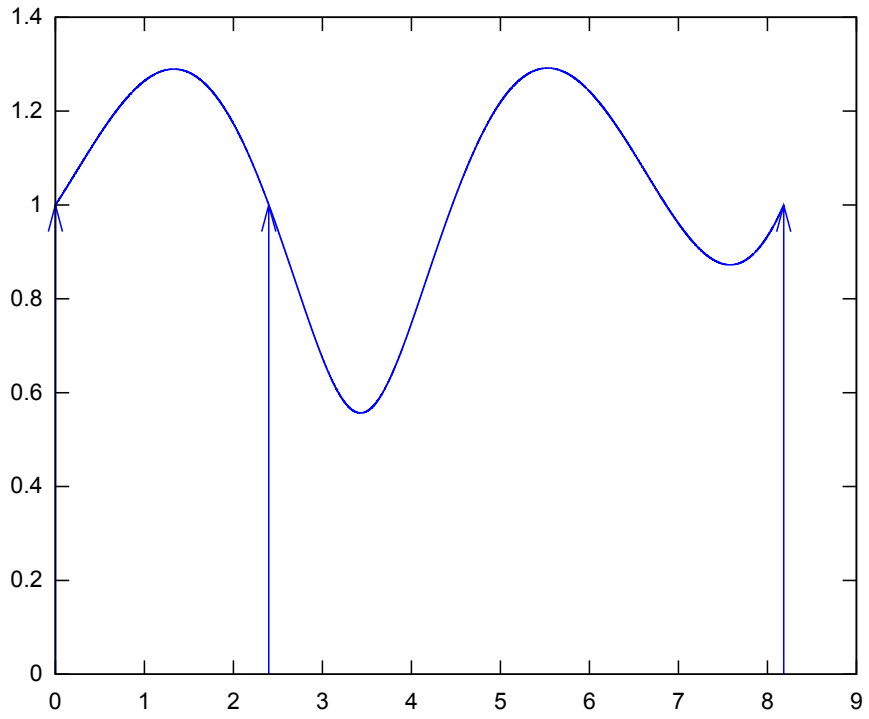
Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$



iteration 1

Example for the in-plane case

SICP problem for in-plane, $q = 2$

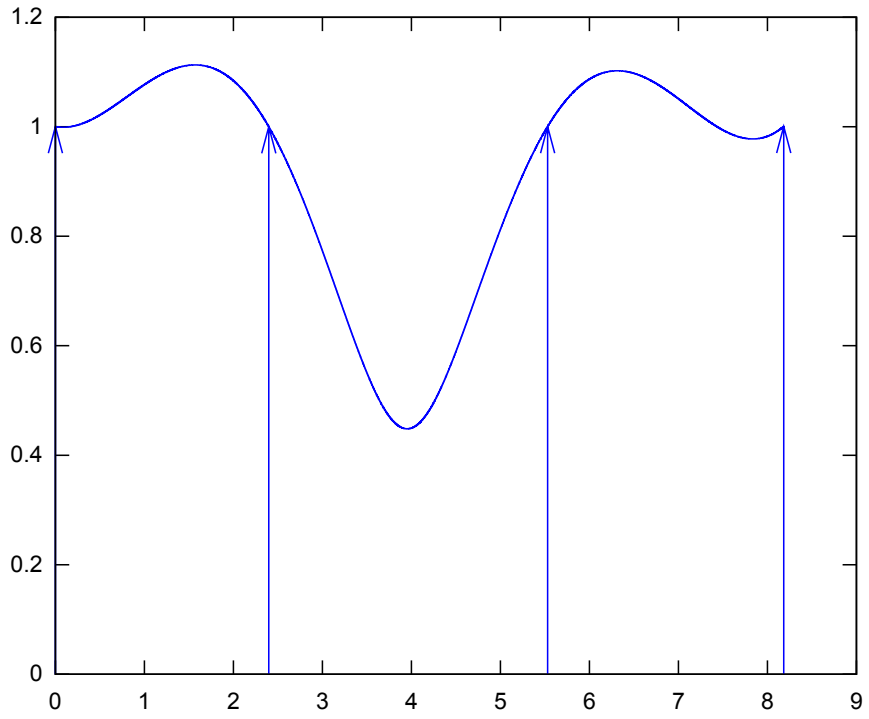
Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$



iteration 2

Example for the in-plane case

SICP problem for in-plane, $q = 2$

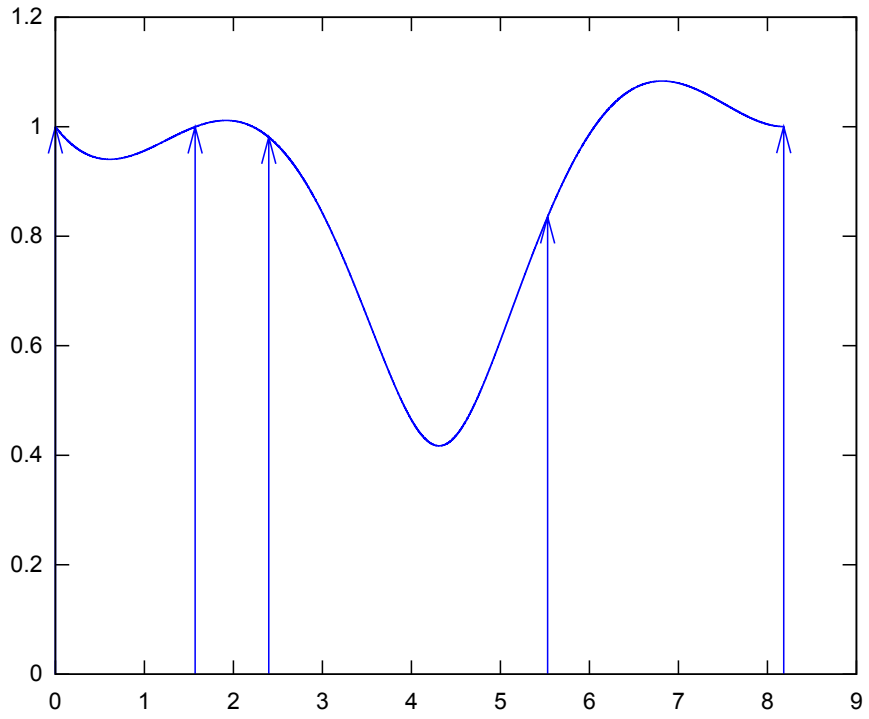
Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$



iteration 3

Example for the in-plane case

SICP problem for in-plane, $q = 2$

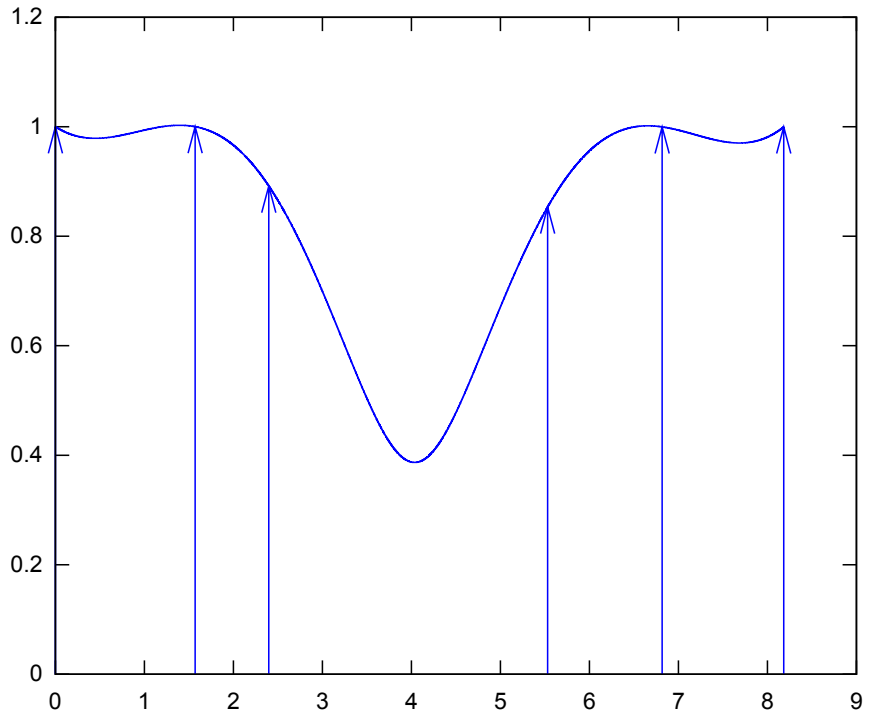
Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$



iteration 4

Example for the in-plane case

SICP problem for in-plane, $q = 2$

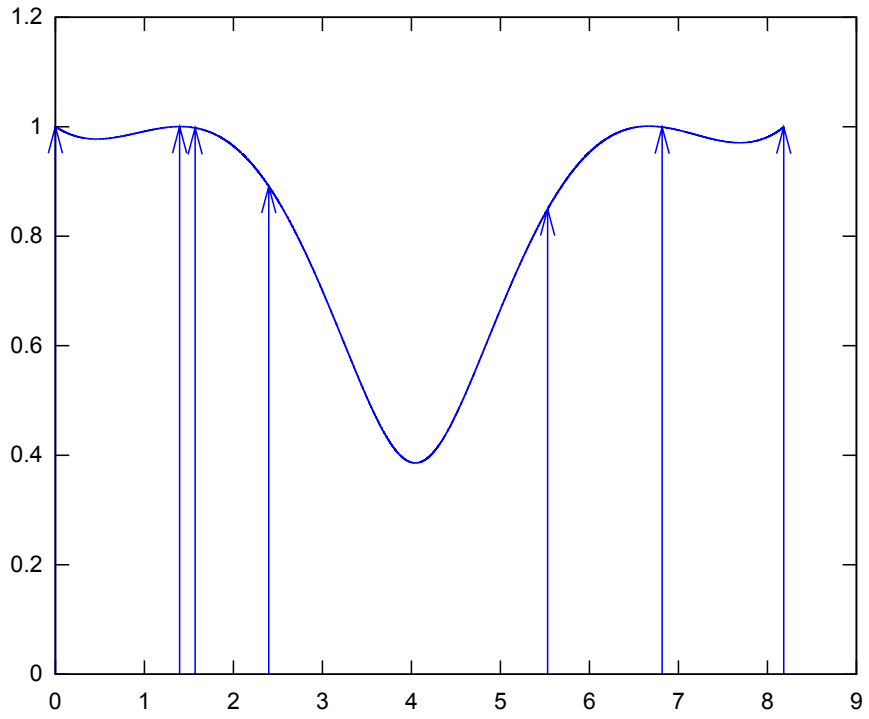
Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$



iteration 5

Example for the in-plane case

SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda \qquad \left\| Y^T(\nu)\lambda \right\|_2 \leq 1. \tag{9}$$

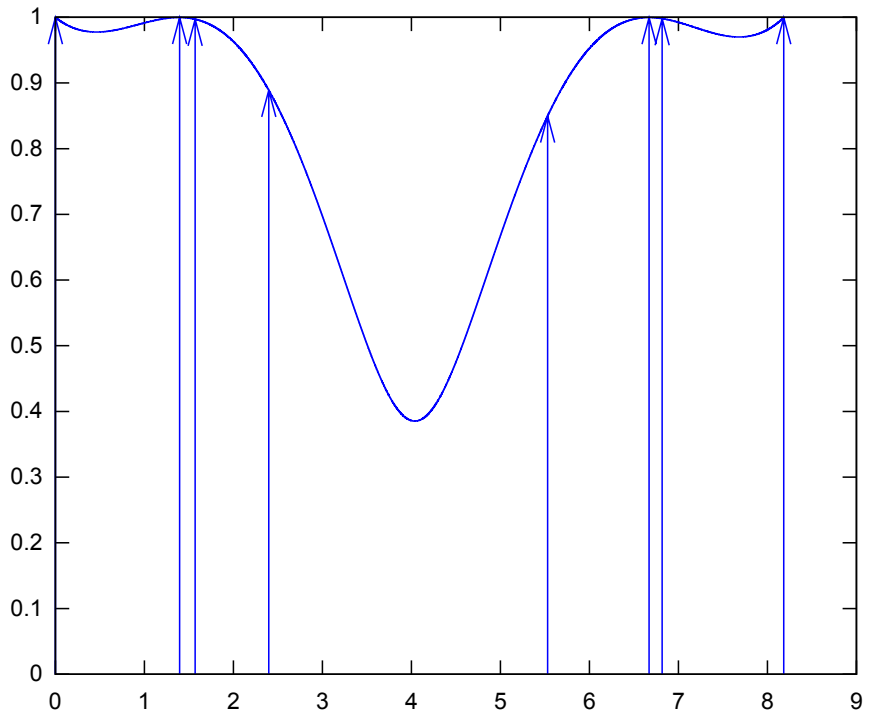
Step 0 Initialize λ, e.g. from a solution with initial and final impulses; $\mathrm{Cost}_0 = 18.064877$;

While $\max_{\nu} \mathrm{pv} > 1$

Step i Find $\nu_i = \arg\left[\max_{\nu} \mathrm{pv}\right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$

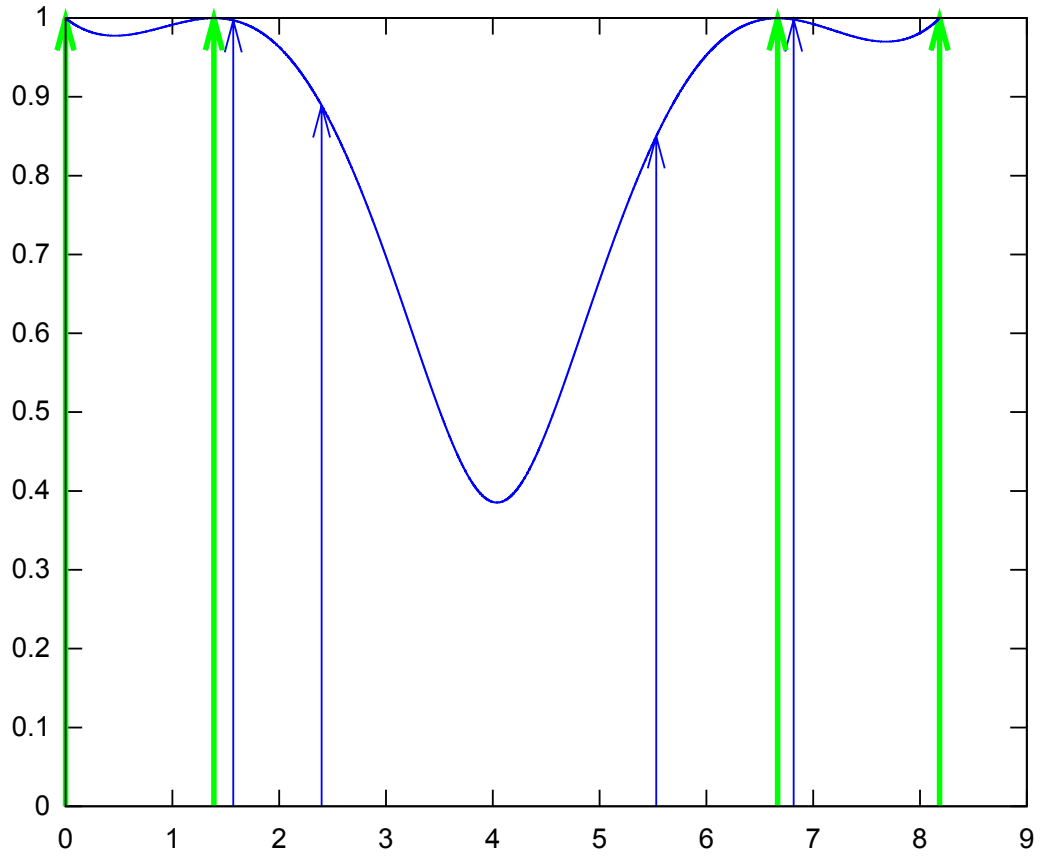
Final cost: 6.272461,

$$\bar{\lambda}^T = \begin{bmatrix} -0.0001177 \\ 0.0001231 \\ -0.0001571 \\ -0.001436 \end{bmatrix}$$



iteration 6

Retrieval of the impulse locations and velocities: example ATV mission (in-plane)



- primer vector module $\simeq 1 \pm 10^{-4}$ (after 6 iterations)
- impulse locations: 0; 1.388128; 6.666595; 8.183058 [rad]
- velocities increments:
  $\Delta V_1 = [7.50230589; -0.742372034]$ m/s,
  $\Delta V_2 = [1.55579123; -0.088346857]$ m/s
  $\Delta V_3 = [-0.62565013; -0.03325936]$ m/s
  $\Delta V_4 = [-1.06509710; -0.11440204]$ m/s.

Results for ATV mission (in-plane) (I)

Semi-major axis	a = 6763 km.
Inclination	i=52 deg.
Argument of perigee	ω=0 deg.
Longitude of the ascending node	Ω= 0 deg.
Eccentricity	e = 0.0052
Initial time	ν_0 = 0 rad.
Initial state vector X_0^T	[-30 0.5 8.514 0] km. - m/s.
Initial state vector $\tilde{X}_0^T$	[-51.9222 0.0865 0.95734 0].10^4
Final anomaly	ν_f = 8.1832 rad.
Duration	$t_f - t_0$ = 7200 s.
Final state vector X_f^T	[-100 0 0 0] m. - m/s.
Final state vector $\tilde{X}_f^T$	[-76.3818 0 69.1519 0]

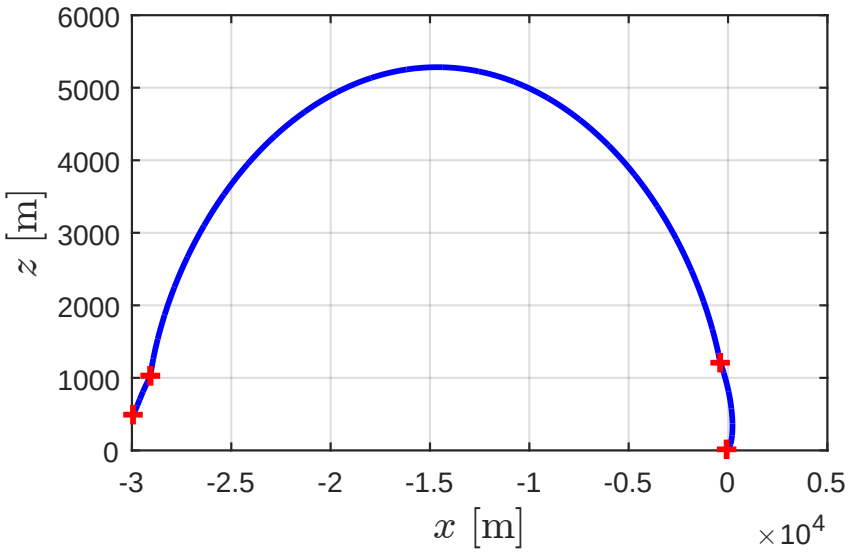


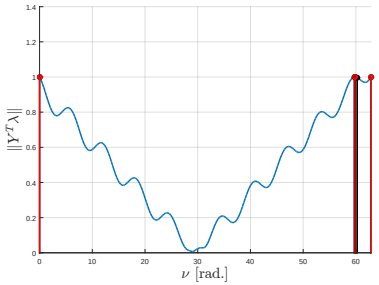
Figure: Optimal trajectory in (x, z) plane.

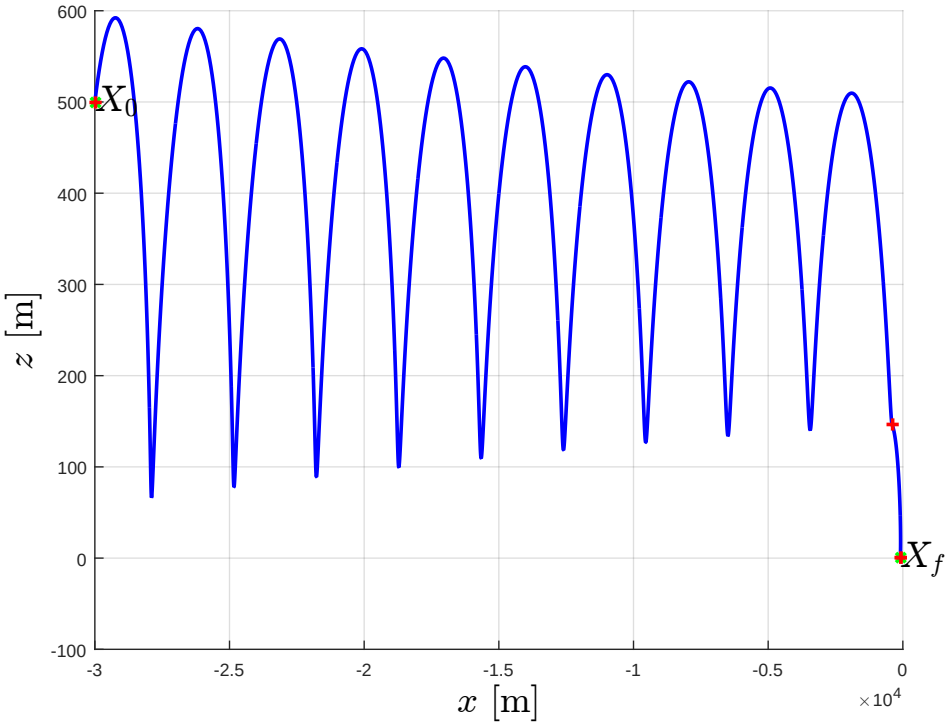
Results for ATV-Long mission (in-plane) (II)

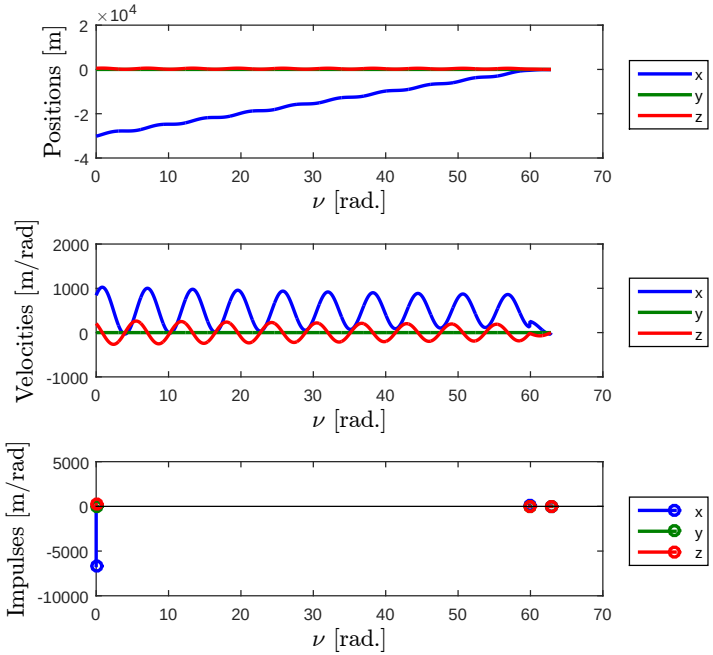
Semi-major axis	a = 6763 km.
Inclination	i=52 deg.
Argument of perigee	ω=0 deg.
Longitude of the ascending node	Ω= 0 deg.
Eccentricity	e = 0.0052
Initial time	ν_0 = 0 rad.
Initial state vector X_0^T	[-30 0.5 8.514 0] km. - m/s.
Initial state vector $\tilde{X}_0^T$	[-51.9222 0.0865 0.95734 0].10^4
Final anomaly	ν_f = 8.1832 rad.
Duration	$t_f - t_0$ =55350 s.
Final state vector X_f^T	[-100 0 0 0] m. - m/s.
Final state vector $\tilde{X}_f^T$	[-76.3818 0 69.1519 0]

Control:

imp.[rad]	velocity[m/s]
0	-7.55418436 -0.23360120
59.88676846	0.14408063 0.001028313
62.83136216	0.0416620 0.0012880





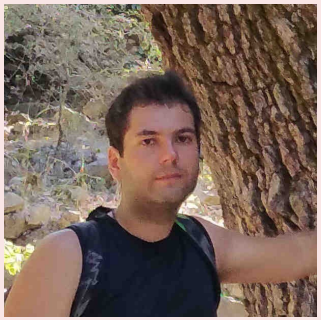


Sum up: RdV Problem

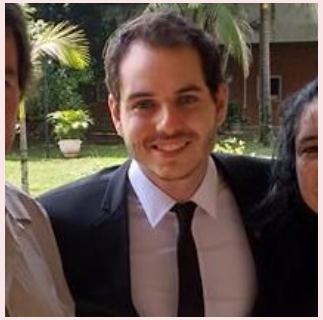
- Convergent SICP iterative algorithm for solving the optimal control problem in the linearized elliptic case
- Use state of the art linear/SDP solvers
- Gives simple geometrical insight to obtain analytical solution in the out-of-plane case.

More on this...

16h45-17h15 Florent's talk:
Approximations, Fixed-Point Methods and Algorithms for Function Space Problems



17h15-17h45 Paulo's talk:
Model predictive control for spacecraft rendezvous hovering phases based on validated Chebyshev series approximations of the transition matrices



Roubíček's example [Roubíček 2006], [Kružík 1998]

Let $T > 1$. Solve for $x \in W^{1,1}([0,T],\mathbb{R})$ and $u \in \mathcal{L}^1([0,T],\mathbb{R})$:

$$\min_{u} \quad J(x,u) = \int_0^T \underbrace{(2-2t+t^2)}_{Y(t)} |u(t)| \mathrm{d}t + (x(T)-1)^2$$

$$\text{s.t.} \quad \frac{dx}{dt} = u, \quad x(0) = 0.$$

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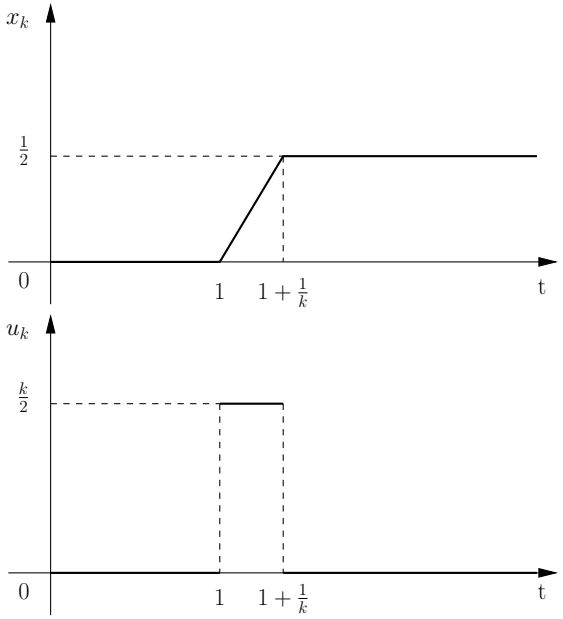
Solution via a minimizing sequence, e.g. $\{(x_k, u_k)\}_{k\in\mathbb{N}}$:

$$x_k(t) = \begin{cases} 0, & t \in [0,1] \\ \frac{k}{2}(t-1), & t \in \left(1, 1+\frac{1}{k}\right] \\ \frac{1}{2}, & t \in \left(1+\frac{1}{k}, T\right] \end{cases}$$

$$u_k(t) = \begin{cases} 0, & t \in [0,1] \\ \frac{k}{2}, & t \in \left(1, 1+\frac{1}{k}\right] \\ 0, & t \in \left(1+\frac{1}{k}, T\right] \end{cases}$$



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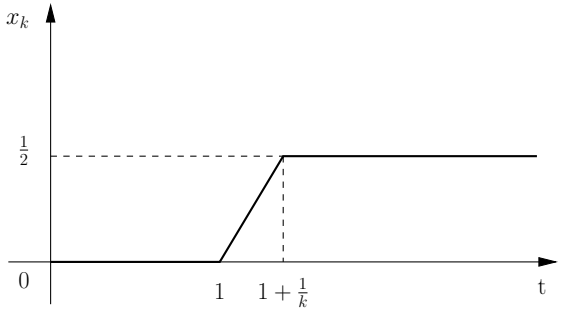

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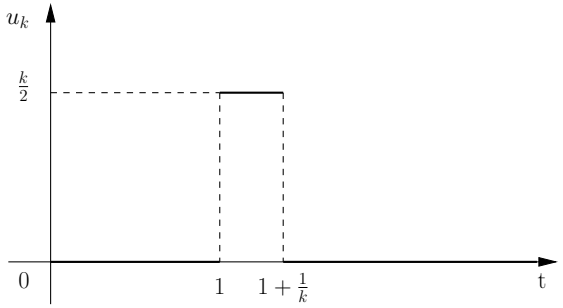

$$J(x_k, u_k) = \int_0^T Y(t)|u_k(t)|\mathrm{d}t + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \mathcal{O}\left(\frac{1}{k}\right) \to \frac{3}{4}$$

Roubíček's example [Roubíček 2006], [Kružík 1998]

$\rightsquigarrow$ $\inf J$ cannot be achieved and the optimal control does not exist in $\mathcal{L}^1([0,T],\mathbb{R})$
$\rightsquigarrow$ Solution via step functions or discrete measures:

$$\min_{x(T),t_0} \quad J = \int_0^T Y(t)|\mathrm{d}x(t)| + (x(T)-1)^2$$
$$x(t) = \begin{cases} 0 & t \leq t_0 \\ x(T) & t > t_0 \end{cases}$$

$$\min_{\mu,t_0} \quad J = \int_0^T Y(t)|\mathrm{d}\mu(t)| + (x(T)-1)^2$$
$$\mu([0,T]) = \delta_{t_0}([0,T]) = \begin{cases} 0 & t_0 \notin [0,T] \\ x(T) & t_0 \in [0,T] \end{cases}$$

Back

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$$\mu([0,T]) = \delta_{t_0}([0,T]) = \begin{cases} 0 & t_0 \notin [0,T] \\ x(T) & t_0 \in [0,T] \end{cases}$$

$$\min_{x(T),t_0} \quad x(T)Y(t_0) + (x(T)-1)^2$$ with solution: $$\begin{aligned} t_0^* &= 1 \\ x^*(T) &= 1/2 \\ J^* &= 3/4 \end{aligned}$$

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Zoom on Indicator functions

$$g(z) = \frac{F(z)}{G(z)}$$

- $\left|g(re^{i\theta})\right| \sim \exp\left(h(\theta)r\right)$ for large r
- indicator of g:

$$h(\theta) = \begin{cases} \frac{-\cos\theta}{2{\sigma_y}^2} & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

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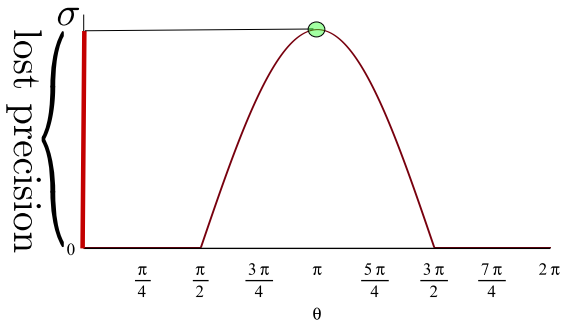

lost precision $\sim |\sigma - h(0)|$

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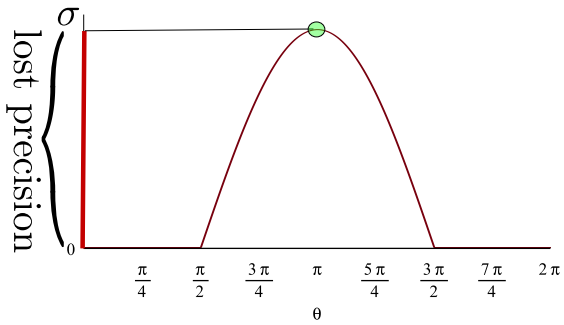

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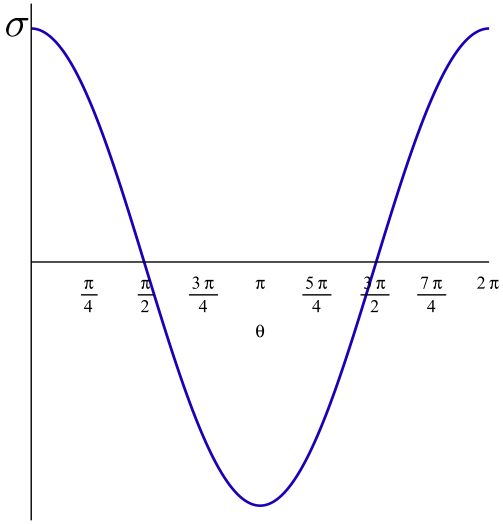

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$G(z)g(z) = F(z)$

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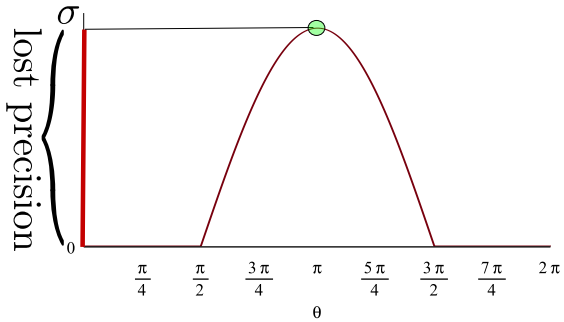

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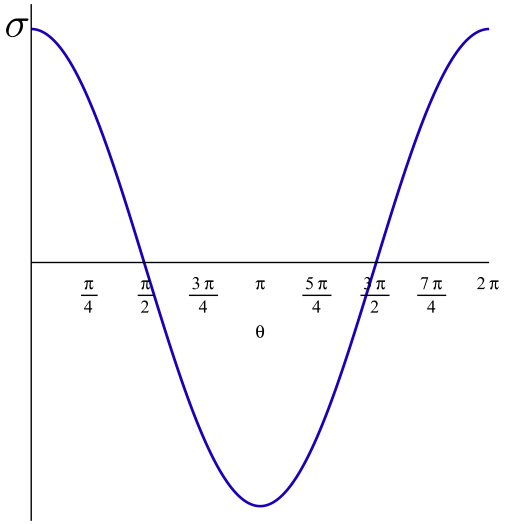
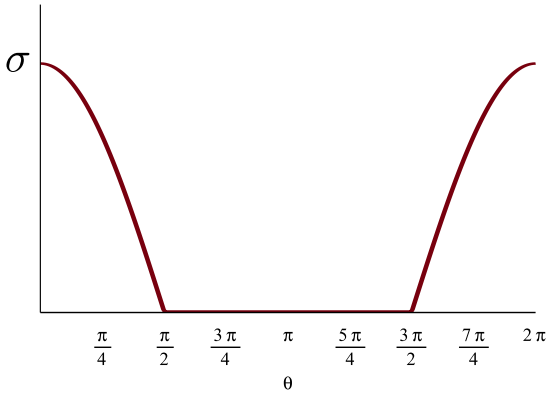


- indicator of F:



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Prop. Coefficients α_k are positive

Suppose $\sigma_x > \sigma_y$.

$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ satisfies $\hat{L}'(x) = \varphi(x)\hat{L}(x), \quad \hat{L}(0) = \dfrac{\exp\left(-\dfrac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2\sigma_y^2}\right)}{2\sigma_x\sigma_y}$, where

$$\varphi(x) = \frac{{y_m}^2}{4{\sigma_y}^4} + \frac{{\sigma_y}^4 {x_m}^2}{\left(x\left({\sigma_x}^2 - {\sigma_y}^2\right) - 2\,{\sigma_x}^2{\sigma_y}^2\right)^2} - \frac{1}{-2\,{\sigma_y}^2 + x} + \frac{-{\sigma_x}^2 + {\sigma_y}^2}{2x\left({\sigma_x}^2 - {\sigma_y}^2\right) - 4\,{\sigma_x}^2{\sigma_y}^2}.$$

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$0 \le (n+1)\alpha_{n+1} = \sum_{i=0}^{n} \varphi_i \cdot \alpha_{n-i}$, $\rightsquigarrow$ by induction.

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