Validated numerics for robust space mission design

M. Joldes
D. Arzelier, F. Bréhard, N. Brisebarre, N. Deak, J.-B. Lasserre, C. Louembet, A. Rondepierre, B. Salvy

LAAS-CNRS
Joint Projects with CNES and Airbus Defence and Space

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Context

- Numerical Computing: floating-point arithmetic
  - High Performance Computing (MultiCores, GPUs, FPGAs):
    - Fast numerical solutions: global optimization, systems of differential equations, integration
    - Usually, solutions lack certification of the output accuracy

Catastrophic cancellation example:

Evaluate

\[(333.75 - a^2)^{b6} + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8\]

\[\rightarrow\text{eval to 0 by cancellation}\]

for \(a = 77617.0, b = 33096.0\) (Rump '88)

Results of C program, gcc, Linux:

- 1.1726039400531787 in binary64;
- 1.1726039400531786318588349045201838 in binary128.

Exact result is \(-0.827396\...\).
Numerical Computing: floating-point arithmetic

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→ Computer Algebra Systems (eg. Maple):
  - **Exact solution**, e.g. \(-\frac{54767}{66192}\)
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5.5b^8 - 2 - 5.5b^8 \rightarrow \text{eval to 0 by cancellation}
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Computer Algebra Systems (eg. Maple):
- **Exact solution**, e.g. \(-\frac{54767}{66192}\)
### Another example: Cancellation in finite precision power series evaluation

**Example:**

\[
\exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}
\]

\[
\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots
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Lost Digits: \( \simeq \log \frac{\max_{i} \frac{20^i}{i!}}{\exp(-20)} \)

\( \simeq 54 \) bits lost, hence binary64 result: 0.01583705682...
Taylor series: \( \exp = \sum \frac{1}{n!} x^n \)

Recurrence for coefficients:
\[
    u(n + 1) = \frac{u(n)}{n + 1}
\]

- \(u(0) = 1\) \quad 1/0! = 1
- \(u(1) = 1\) \quad 1/1! = 1
- \(u(2) = 0.5\) \quad 1/2! = 0.5

\[\vdots\]

- \(u(50) \approx 3.28 \cdot 10^{-65}\) \quad 1/50! \approx 3.28 \cdot 10^{-65}
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Chebyshev series: \( \exp = \sum I_n(1) T_n(x) \)

Recurrence for coefficients:
\[
u(n + 1) = -2nu(n) + u(n - 1)
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\[
\begin{align*}
    u(0) &= 1.266 & I_0(1) &\approx 1.266 \\
    u(1) &= 0.565 & I_1(1) &\approx 0.565 \\
    u(2) &\approx 0.136 & I_2(1) &\approx 0.136 \\
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Let's consider the Chebyshev Series of D-finite Functions.

### Taylor Series

The Taylor series of the exponential function is given by:

$$ \exp = \sum \frac{1}{n!} x^n $$

**Recurrence for coefficients:**

$$ u(n+1) = \frac{u(n)}{n+1} $$

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More subtle cause:

Convergent and Divergent Solutions of the Recurrence \( u(n + 1) = -2nu(n) + u(n - 1) \):

If \( u(n) \) is solution, then there exists another solution \( v(n) \sim \frac{1}{u(n)} \)
2009, Feb. 10: collision between Iridium 33 and Cosmos 2251, although predicted minimum distance of close approach was of 584m.

Figure: Animation of Iridium 33 and Kosmos 2251’s collision; GNU Free Documentation, Wikipedia
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Can/Should we trust numerics?

- they are fast...
- double precision floating-point numbers: "Sufficient for expressing the distance from the Earth to the Moon with an error less than the thickness of a bacterium";
- Collision probabilities boil down to integral computations...
- Mathematica Book: when Mathematica does a numerical integral, the only information it has about your integrand is a sequence of numerical values for it. If you give a sufficiently pathological integrand, Mathematica may simply give you the wrong answer
Safety-critical space applications

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First focus of this talk: efficient and reliable numerical computation of on-orbit collision probability
A goal of space missions

Achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption e.g., rendezvous planning tools for formation flight (PRISMA), on-orbit satellite servicing, supply missions (ISS)...

Figure: Soyuz TMA-19M docking to ISS (December 2015) - ISS resupply mission; NASA YouTube © Chanel
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Second goal of this talk:
efficient and reliable numerical optimal control for space rendezvous.
On-orbit collision

Figure: Space debris population model (source: ESA)

Thomas Pesquet, when a debris whizzes past the ISS:

"Climb into an escape shuttle, wait and hope. This happened four times"

On-orbit collision

Context

- Two objects: primary $P$ (operational satellite) and secondary $S$ (space debris)
- Information about their geometry, position, velocity at a given time
  \[\leadsto\text{Affected by uncertainty}\]
- Needs:
  - Risk assessment
  - Design of a collision avoidance strategy
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- Spherical objects
- Gaussian probability density functions
- Independent probability distribution laws

Figure: Combined spherical object
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Probability of collision

Generally: 12-dimensional, Gaussian integrand, Complex integration domain
Computation: Monte-Carlo trials and/or simplified models
Short-term encounter model and probability of collision

- Framework: High relative velocity

- Assumptions:
  - Rectilinear relative motion
  - No velocity uncertainty
  - Infinite encounter time horizon

  \[ P = \frac{1}{2\pi\sigma_x\sigma_y} \int_{B((0,0),R)} \exp \left( -\frac{(x-x_m)^2}{2\sigma_x^2} - \frac{(y-y_m)^2}{2\sigma_y^2} \right) \, dx \, dy \]

  where 
  - \( R \): radius of combined object
  - \( x_m, y_m \): mean relative coordinates
  - \( \sigma_x, \sigma_y \): standard deviations of relative coordinates

\[ \text{Probability of collision:} \]
\[ 2\text{-D integral over a disk.} \]
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**Figure:** 2-D Gaussian integral over a disk

**Formula**

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Existing methods

- Methods based on numerical integration schemes: Foster '92, Patera '01, Alfano '05.
- Analytic methods: Chan '97 uses some simplifying assumptions ($\sigma_x = \sigma_y$).

Pro's and Con's
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- Fast and already used in practice
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Our purpose

Give a "simple", "analytic" formula, suitable for double-precision evaluation and effective error bounds.
Our method - Underlying techniques

- **Laplace transform:**
  \[ \Rightarrow \text{Lasserre and Zeron, Solving a Class of Multivariate Integration Problems via Laplace Techniques, Applicationes Mathematicae, 2001.} \]

**D-finite functions**
- Solution of linear differential equation with polynomial coefficients
- Power series coefficients satisfy a linear recurrence relation with polynomial coefficients

Example:
\[ f(x) = e^x = \sum_{n=0}^{\infty} f_n x^n \leftrightarrow \{ f' - f = 0, f(0) = 1 \} \leftrightarrow \{ (n+1)f_{n+1} = f_n, f_0 = 1 \} \]

- \( \cos, \arccos, \text{Airy functions, Bessel functions} \ldots \)

- **Finite-precision evaluation of power series prone to cancellation**
  \[ \Rightarrow \text{Gawronski, Müller, Reinhard, Reduced Cancellation in the Evaluation of Entire Functions and Applications to the Error Function, SIAM Journal on Numerical Analysis, 2007.} \]
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∀z ∈ ℝ⁺:

\[
g(z) := \mathcal{P}(\sqrt{z}) = \frac{1}{2\pi \sigma_x \sigma_y} \int_{B((0,0), \sqrt{z})} \exp \left( - \frac{(x - x_m)^2}{2\sigma_x^2} - \frac{(y - y_m)^2}{2\sigma_y^2} \right) \, dx \, dy,
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... = \frac{\exp \left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t\sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t\sigma_x^2 + 1)} \right)}{t \sqrt{(2t\sigma_x^2 + 1)(2t\sigma_y^2 + 1)}}
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\[ \frac{t\sqrt{(2t\sigma_x^2 + 1)(2t\sigma_y^2 + 1)}} \]

\[ \mathcal{L}(g) \text{ is D-finite!} \]
Sketch of the proof - Borel-Laplace

\[ \hat{L}(g)(t) := t^2 \mathcal{L}(g) \left( \frac{1}{t} \right) = \sum_{i=0}^{\infty} \ell_i \left( \frac{1}{t} \right)^i \]

\[ \mathcal{L}(g)(t) = \exp \left( \frac{-\sigma x^2 y m^2 + \sigma y^2 x m^2}{2\sigma^2 \sigma y^2} + \frac{y m^2}{2\sigma y^2 (2t \sigma y^2 + 1)} + \frac{x m^2}{2\sigma^2 (2t \sigma x^2 + 1)} \right) t^2 \sqrt{(2t \sigma x^2 + 1)(2t \sigma y^2 + 1)} \]

\[ g(z) := \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1} \]

\[ g(z) \text{ is:} \]
- D-finite
- entire function of exponential type
- type \( \sigma = \frac{1}{2\sigma^2 y} \)

\[ \hat{L}(g)(t) \text{ is:} \]
- D-finite
- Finite radius of convergence \( 2\sigma^2 y \)

*B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.*
Sketch of the proof - Borel-Laplace

\[ g(z) \]

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- entire function of exponential type
- type \( \sigma = \frac{1}{2\sigma_y^2} \)

\[ \hat{L}(g)(t) \text{ is:} \]
- D-finite
- Finite radius of convergence \( 2\sigma_y^2 \)

\[ \ell_i \text{ satisfy a linear recurrence with polynomial coefficients.} \]

\[ \sim \text{ Compute everything with gfun*} \]

---

*B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.*
Sketch of the proof - Borel-Laplace

\[ \mathcal{L}(g)(t) = \exp \left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t\sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t\sigma_x^2 + 1)} \right) \frac{t}{\sqrt{(2t\sigma_x^2 + 1)(2t\sigma_y^2 + 1)}} \]

\[ \hat{g}(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1} \]

\[ \hat{L}(g)(t) := t^2 \mathcal{L}(g) \left( \frac{1}{t} \right) = \sum_{i=0}^{\infty} \ell_i \left( \frac{1}{t} \right)^i \]

\( g(z) \) is:
- D-finite
- entire function of exponential type
- type \( \sigma = \frac{1}{2\sigma_y^2} \)
- sum prone to cancellation

\( \hat{L}(g)(t) \) is:
- D-finite
- Finite radius of convergence \( 2\sigma_y^2 \)

\( \ell_i \) satisfy a linear recurrence with polynomial coefficients.

\( \leadsto \) Compute everything with gfun*

---

*B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.*
Cancellation in finite precision power series evaluation

Example: \( \sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15 \)

\[ g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1} \]
Cancellation in finite precision power series evaluation

Example: \( \sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15 \)

\[
g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}
\]

\[
g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots
\]
Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} \ldots + 4.3 - 0.14 - 0.60 \ldots$$

Values of $|\frac{\ell_i 225^{i+1}}{(i+1)!}|$, compared to $g(225) \simeq 0.1004$:

Lost Digits: $d_g(z) \simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$
Cancellation in finite precision power series evaluation

Example: \( \exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \)

\[
\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots
\]
Example: \( \exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \)

\[
\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots
\]

Values of \( \left| \frac{(-1)^i 20^i}{i!} \right| \), compared to \( \exp(-20) \approx 2.06 \cdot 10^{-9} \):

Lost Digits: \( d_g(z) \approx \log \frac{\max_i |g_i z^i|}{|g(z)|} \)
Cancellation in finite precision power series evaluation

Example: \( \exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \)

\[
\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots
\]

Values of \( \left| \frac{(-1)^i 20^i}{i!} \right| \), compared to \( \exp(-20) \approx 2.06 \cdot 10^{-9} \):

Lost Digits: \( d_g(z) \approx \log \max_i |g_i z^i| / |g(z)| \)

BUT...

\[
\exp(-x) = \frac{1}{\exp(x)}
\]

No cancellation!
Cancellation in finite precision power series evaluation

Example: \( \sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15 \)

\[ g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1} \]

\[ g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots \]

Values of \( \left| \frac{\ell_i}{225^{i+1}} \right| \), compared to \( g(225) \approx 0.1004 \):
Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots$$

Values of $\left| \frac{\ell_i 225^{i+1}}{(i+1)!} \right|$, compared to $g(225) \simeq 0.1004$:

$$g(\hat{z}) = \frac{F(\hat{z})}{G(\hat{z})}$$

No cancellation!
Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 \cdot 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots$$

Values of $\left| \frac{\ell_i z^{225(i+1)}}{(i+1)!} \right|$, compared to $g(225) \approx 0.1004$:

Gawronski, Müller, Reinhard (2007) Method:

Find $F(z)$ and $G(z)$ using "complex analysis tricks" $\leadsto$ indicator function more.

$\leadsto G(z) = \exp(\sigma z)$. 
Sketch of the proof - Borel-Laplace + GMR Method

\[ G(z)g(z) = F(z) \]
\[ \mathcal{L}(g)(t - \sigma) \]
\[ \sum_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \]
\[ \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k \]

- \( G(z) = \exp(\sigma z) \)
- \( F \) is D-finite
- reduced cancellation for evaluating \( F, G \) on positive real line
Sketch of the proof - Borel-Laplace + GMR Method

\[ G(z)g(z) = F(z) \]

Laplace

\[ \mathcal{L}(g)(t - \sigma) \]

expansion at \( \infty \)

Borel Transform

\[ \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k \]

- \( G(z) = \exp(\sigma z) \)
- \( F \) is D-finite
- reduced cancellation for evaluating \( F, G \) on positive real line
- recurrence for \( \alpha_k \):

\[
\begin{align*}
- (32k\sigma_x^4\sigma_y^{10} + 128\sigma_x^4\sigma_y^{10})\alpha_{k+4} &= (\sigma_x^4\sigma_y^{2} - 2\sigma_x^2\sigma_y^{2} \sigma_y^{2} + \sigma_y^{4} \sigma_y^{2})\alpha_k \\
+((-4\sigma_x^4\sigma_y^{4} + 8\sigma_x^2\sigma_y^{6} - 4\sigma_y^{8})k - 10\sigma_x^4\sigma_y^{4} - 6\sigma_x^2\sigma_y^{2} \sigma_y^{2} + 20\sigma_x^2\sigma_y^{6} + 8\sigma_x^4\sigma_y^{4} \sigma_y^{2} - 10\sigma_y^{8} - 2\sigma_y^{6} \sigma_y^{2})\alpha_{k+1} \\
+((24\sigma_x^4\sigma_y^{6} - 32\sigma_x^2\sigma_y^{8} + 8\sigma_y^{10})k + 72\sigma_x^4\sigma_y^{6} + 12\sigma_x^2\sigma_y^{6} \sigma_y^{2} - 92\sigma_x^2\sigma_y^{8} - 8\sigma_x^4\sigma_y^{6} \sigma_y^{2} + 20\sigma_y^{10} + 4\sigma_y^{8} \sigma_y^{2})\alpha_{k+2} \\
+((-48\sigma_x^4\sigma_y^{8} + 32\sigma_x^2\sigma_y^{10})k - 168\sigma_x^4\sigma_y^{8} - 8\sigma_x^2\sigma_y^{6} \sigma_y^{2} + 104\sigma_x^2\sigma_y^{10} - 8\sigma_y^{10} \sigma_y^{2})\alpha_{k+3},
\end{align*}
\]
Sketch of the proof - Borel-Laplace + GMR Method

\[ G(z)g(z) = F(z) \]
\[ \mathcal{L}(g)(t - \sigma) \]
\[ G(z)g(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \]
\[ \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k \]

- \( G(z) = \exp(\sigma z) \)
- \( F \) is D-finite
- reduced cancellation for evaluating \( F, G \) on positive real line
- recurrence for \( \alpha_k \):

\[
\begin{align*}
- (32k\sigma_x^4\sigma_y^{10} + 128\sigma_x^4\sigma_y^{10})\alpha_{k+4} &= (\sigma_x^4 y_m^2 - 2\sigma_x^2 \sigma_y^2 y_m^2 + \sigma_y^4 y_m^2)\alpha_k \\
+(( -4 \sigma_x^4 \sigma_y^4 + 8 \sigma_x^2 \sigma_y^6 - 4 \sigma_y^8 ) k - 10 \sigma_x^4 \sigma_y^4 - 6 \sigma_x^2 \sigma_y^2 y_m + 20 \sigma_x^2 \sigma_y^4 y_m + 8 \sigma_x^4 \sigma_y^4 y_m - 10 \sigma_y^8 - 2 \sigma_y^6 y_m)\alpha_{k+1}

+((24 \sigma_x^4 \sigma_y^6 - 32 \sigma_x^2 \sigma_y^8 + 8 \sigma_y^{10}) k + 4 \sigma_x^4 \sigma_y^6 + 6 \sigma_x^2 \sigma_y^8 - 8 \sigma_x^2 \sigma_y^6 y_m + 20 \sigma_y^{10} + 4 \sigma_y^8 y_m)\alpha_{k+2}

+(( -48 \sigma_x^4 \sigma_y^8 + 32 \sigma_x^2 \sigma_y^{10}) k - 168 \sigma_x^4 \sigma_y^8 - 8 \sigma_x^4 \sigma_y^6 y_m + 104 \sigma_x^2 \sigma_y^{10} - 8 \sigma_y^{10} y_m)\alpha_{k+3},
\end{align*}
\]

- We can prove that \( \alpha_k \) are positive
- Lower/upper bounds for \( \alpha_k \) with majorant series
Bounds using majorant series

Let $\gamma := 1 + \frac{1}{2} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} + \frac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{y_m^2}{\sigma_y^2} \right)$, $\bar{\alpha}_k := \alpha_0 \gamma^k$ and $\alpha_k := \alpha_0 \left( \frac{1}{2\sigma_y^2} \right)^k$.

Then $\alpha_k \leq \alpha_k \leq \bar{\alpha}_k$, $\forall k \in \mathbb{N}$. 

Bounds using majorant series

Let \( \gamma := \frac{1}{2} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} + \frac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{y_m^2}{\sigma_y^2} \right) \), \( \bar{\alpha}_k := \alpha_0 \gamma^k \) and \( \alpha_k := \alpha_0 \left( \frac{1}{2\sigma_y^2} \right)^k \).

Then \( \bar{\alpha}_k \leq \alpha_k \leq \alpha_k, \forall k \in \mathbb{N} \).

Let \( \tilde{P}_N(z) := \sum_{k=0}^{N-1} \frac{\alpha_k z^k}{(k+1)!} \). Then we have the following error bounds:

\[
\varepsilon_N(z) \leq g(z) - \tilde{P}_N(z) \leq \bar{\varepsilon}_N(z), \text{ where }
\]

\[
\varepsilon_N(z) := 2\alpha_0 \sigma_y^2 e^{-\frac{z}{2\sigma_y^2}} \left( \frac{z}{2\sigma_y^2} \right)^{N+1} \left( N + 1 \right)! ,
\]

\[
\varepsilon_N(z) := \frac{\alpha_0}{\gamma} e^{-\frac{z}{2\sigma_y^2}} \left( z\gamma \right)^{N+1} \left( N + 1 \right)! .
\]
### Examples

#### Sample 1

<table>
<thead>
<tr>
<th>Case #</th>
<th>Input parameters (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\sigma_x)</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
</tr>
<tr>
<td>4</td>
<td>0.075</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
</tr>
</tbody>
</table>
Examples: quality $\eta = - \log \frac{\bar{\varepsilon}_{10}(z) - \varepsilon_{10}(z)}{\varepsilon_{10}(z) + \tilde{P}_{10}(z)}$ and plot $\log (\tilde{p}_i z^i)$

Case 1: $\eta = 23$

Case 2: $\eta = 22$

Case 4: $\eta = 22$

Case 6: $\eta = 47$

Case 8: $\eta = 33$

Case 11: $\eta = 34$
## Numerical study

### Sample 1

<table>
<thead>
<tr>
<th>Case #</th>
<th>Probability of Collision (−)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alfano</td>
<td>Patera</td>
<td>Chan</td>
<td>New method</td>
</tr>
<tr>
<td>1</td>
<td>$9.742 \times 10^{-3}$</td>
<td>$9.741 \times 10^{-3}$</td>
<td>$9.754 \times 10^{-3}$</td>
<td>$9.742 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$9.181 \times 10^{-3}$</td>
<td>$9.181 \times 10^{-3}$</td>
<td>$9.189 \times 10^{-3}$</td>
<td>$9.181 \times 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>$6.571 \times 10^{-3}$</td>
<td>$6.571 \times 10^{-3}$</td>
<td>$6.586 \times 10^{-3}$</td>
<td>$6.571 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>$6.125 \times 10^{-3}$</td>
<td>$6.125 \times 10^{-3}$</td>
<td>$6.135 \times 10^{-3}$</td>
<td>$6.125 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
</tr>
<tr>
<td>6</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
</tr>
<tr>
<td>7</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
</tr>
<tr>
<td>8</td>
<td>$0$</td>
<td>$3.219 \times 10^{-27}$</td>
<td>$3.216 \times 10^{-27}$</td>
<td>$3.219 \times 10^{-27}$</td>
</tr>
<tr>
<td>9</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
</tr>
<tr>
<td>10</td>
<td>$0$</td>
<td>$9.656 \times 10^{-28}$</td>
<td>$9.645 \times 10^{-28}$</td>
<td>$9.656 \times 10^{-28}$</td>
</tr>
<tr>
<td>11</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
</tr>
<tr>
<td>12</td>
<td>$1.564 \times 10^{-9}$</td>
<td>$1.564 \times 10^{-9}$</td>
<td>$1.556 \times 10^{-9}$</td>
<td>$1.564 \times 10^{-9}$</td>
</tr>
</tbody>
</table>
Examples: quality $\eta_N = -\log \frac{\bar{\epsilon}_N(z) - \epsilon_N(z)}{\bar{\epsilon}_N(z) + \bar{\tilde{P}}_N(z)}$ and plot $\log(\tilde{p}_i z^i)$

Case 3 Alfano: $\eta_{800} = 30$

Case 5 Alfano: $\eta_{121000} = 20$

Sample 2 (from [Alfano 2009])

<table>
<thead>
<tr>
<th>Case #</th>
<th>Input parameters (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_x$</td>
</tr>
<tr>
<td>3</td>
<td>114.25</td>
</tr>
<tr>
<td>5</td>
<td>177.8</td>
</tr>
</tbody>
</table>
### Examples

Sample 2 (from [Alfano 2009])

The table below shows the input parameters and the probability of collision for two cases. The input parameters include

<table>
<thead>
<tr>
<th>Case #</th>
<th>Input parameters (m)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>$\sigma_y$</td>
</tr>
<tr>
<td>3</td>
<td>114.25</td>
<td>1.41</td>
</tr>
<tr>
<td>5</td>
<td>177.8</td>
<td>0.038</td>
</tr>
</tbody>
</table>

The last table shows the probability of collision for the two cases. The probabilities are given for Alfano, New method, and Reference (MC).

<table>
<thead>
<tr>
<th>Alfano’s test case number</th>
<th>Probability of collision (-)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alfano</td>
</tr>
<tr>
<td>3</td>
<td>0.10038</td>
</tr>
<tr>
<td>5</td>
<td>0.044712</td>
</tr>
</tbody>
</table>
Collision Probability: Sum up

New method
- Analytical formula
- Reduced cancellation evaluation
- Error bounds
- No simplifying assumption
- CNES implemented/tested and now uses it in practice

Current and future work
- Saddle-point method for "degenerate" cases
- Long-term/multiple encounter (new collaboration with CNES)
Safety-critical space applications (2)

A goal of space missions

Achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption e.g., rendezvous planning tools for formation flight (PRISMA), on-orbit satellite servicing, supply missions (ISS)...

Figure: Copyright Gravity (2013): Ryan’s RdV with the Chinese Space Station using a fire extinguisher

Second goal of this talk:

efficient and reliable numerical optimal control for space rendezvous.
Definition of space RdV

RdV Problem

Compute the fuel-optimal guidance law necessary for the chaser to reach the target in fixed-time $t_f - t_0$, given assigned initial and final relative positions and velocities.

Assumption:
- Target is passive
- Time-fixed
- $0 \leq e < 1$
- Relative navigation
- Impulsive thrust
- Keplerian dynamics

Figure: Soyuz TMA-19M docking to ISS (December 2015) - ISS resupply mission; NASA YouTube © Chanel
Relative motion & dynamics

- $\vec{r}_t$, $\vec{r}_c$ target and chaser position in Earth Centered Inertial Frame (ECI) $(\vec{X}_i, \vec{Y}_i, \vec{Z}_i)$;
- Target centered rotating frame, LVLH convention $(\vec{X}_{lvlh}, \vec{Y}_{lvlh}, \vec{Z}_{lvlh})$;
- Target–Chaser distance small wrt. Target–Earth radius: $\|\vec{\rho}\| \ll \|\vec{r}_t\|$

Linearized relative motion: Tschauner-Hempel eqs.

\[
\begin{align*}
x'' &= 2z \\
y'' &= -y \\
z'' &= -2x' + \frac{3}{1+e \cos \nu} z
\end{align*}
\]
The target orbit is elliptical: eccentricity \(0 \leq e < 1\), semi-major axis \(a\), true anomaly \(\nu\):

\[ r_t = \frac{a(1-e^2)}{(1+e \cos(\nu))}; \]

- Kinetic moment is constant: \(r^2 \dot{\nu} = \sqrt{\mu a (1-e^2)} \Rightarrow \dot{\nu} = \sqrt{\frac{\mu}{a^3 (1-e^2)^3}} (1 + e \cos(\nu))^2.\)
Automated Transfer Vehicle (ATV) mission

- European unmanned vehicle 2008-2014 (ESA, ADS, CNES);
- Logistic servicing of the ISS (dry cargo, water and gas delivery, ISS refueling, ISS reboost, contribution to ISS attitude control, and waste disposal)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
<td>(a = 6763) km.</td>
</tr>
<tr>
<td>Inclination</td>
<td>(i = 52) deg.</td>
</tr>
<tr>
<td>Argument of perigee</td>
<td>(\omega = 0) deg.</td>
</tr>
<tr>
<td>Longitude of the ascending node</td>
<td>(\Omega = 0) deg.</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>(e = 0.0052)</td>
</tr>
<tr>
<td>Initial time</td>
<td>(\nu_0 = 0) rad.</td>
</tr>
<tr>
<td>Initial state vector (X^T_0)</td>
<td>([-30 0.5 8.514 0]) km. - m/s.</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>(\nu_f = 8.1832) rad.</td>
</tr>
<tr>
<td>Duration</td>
<td>(t_f - t_0 = 7200) s.</td>
</tr>
<tr>
<td>Final state vector (X^T_f)</td>
<td>([-100 0 0 0]) m. - m/s.</td>
</tr>
</tbody>
</table>

Parameters of the ATV example [Labourdette08]
● Target satellite on Geostationary Transfer Orbit (GTO)
● Temporary elliptic orbit to inject a satellite into the Geostationary Earth Orbit (GEO)
● Perigee on LEO (altitude $\approx 200$ km) and apogee on GEO (altitude = 35786 km)
● Requires a change of orbital plane and out-of-plane maneuvers

<table>
<thead>
<tr>
<th>Semi-major axis</th>
<th>$a = 24616$ km</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eccentricity</td>
<td>$e = 0.73074$</td>
</tr>
<tr>
<td>Initial anomaly</td>
<td>$\theta_0 = 0.1\pi$ rad</td>
</tr>
<tr>
<td>Initial state vector</td>
<td>$X_0^T = [10000 -3]$ m - m/s</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>$\theta_f = 5.2$ rad</td>
</tr>
<tr>
<td>Duration</td>
<td>$t_f - t_0 = 29888$ s</td>
</tr>
<tr>
<td>Final state vector</td>
<td>$X_f^T = [0 0]$ m - m/s</td>
</tr>
</tbody>
</table>
RdV Problem

Goals of this study

- Many works since ’60s ⟷ Revisit mathematical formulation of this problem
- New convergent iterative numerical algorithm; lower/upper bounds on optimal cost
Mathematical formulation of RdV problem

- Simplified Tschauner-Hempel equations with $\nu$ as indpt var;

\[
\begin{align*}
    x'' &= 2z \\
    y'' &= -y \\
    z'' &= -2x' + \frac{3}{1+e\cos \nu} \tilde{z}
\end{align*}
\]

\[
\Rightarrow X' = A(\nu)X
\]
Mathematical formulation of RdV problem

- Simplified Tschauner-Hempel equations with $\nu$ as indpt var; $\tilde{u} \approx \frac{F_{\text{prop}}}{m_c}$.

\[
\begin{align*}
x'' &= 2z + \tilde{u}_R \\
y'' &= -y + \tilde{u}_S \\
z'' &= -2x' + \frac{3}{1 + e \cos \nu} \tilde{z} + \tilde{u}_W
\end{align*}
\]
\[\Rightarrow X' = A(\nu)X + B(\nu)U\]
Mathematical formulation of RdV problem

- Simplified Tschauner-Hempel equations with \( \nu \) as indpt var; \( \tilde{u} \approx \frac{F_{\text{prop}}}{m_c} \).

\[
\begin{align*}
x'' &= 2z + \tilde{u}_R \\
y'' &= -y + \tilde{u}_S \\
z'' &= -2x' + \frac{3}{1 + e \cos \nu} \tilde{z} + \tilde{u}_W
\end{align*}
\]
\[
\Rightarrow \quad X' = A(\nu)X + B(\nu)U
\]

Optimal control Problem (after variable change \( t \rightarrow \nu \))

Find \( \bar{u} \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r) \) solution of the optimal control problem:

\[
\begin{align*}
\inf_{u} \quad & ||u||_{1,p} = \inf_{u} \int_{\nu_0}^{\nu_f} ||u(\nu)||_p d\nu \\
\text{s.t.} \quad & X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \quad \forall \ \nu \in [\nu_0, \nu_f] \\
& X(\nu_0) = X_0, \quad X(\nu_f) = X_f \in \mathbb{R}^n, \quad \nu_0, \ \nu_f \ \text{fixed},
\end{align*}
\]

(1)

Thruster configuration determine \( p \)-norm:

- a gimbaled single thruster: \( p = 2 \)
- ungimbaled identical thrusters: \( p = 1 \)
Mathematical formulation of RdV problem

Optimal control Problem (after variable change $t \rightarrow \nu$)

Find $\bar{u} \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

\[
\inf_{u} \quad \|u\|_{1,p} = \inf_{u} \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t.} \quad X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \quad \forall \ \nu \in [\nu_0, \nu_f] \\
X(\nu_0) = X_0, \quad X(\nu_f) = X_f \in \mathbb{R}^n, \quad \nu_0, \ \nu_f \text{ fixed.}
\] (2)

Previous Works

- Indirect approach, Pontryagin max principle, primer vector theory [Lawden63], [Carter&Brient95]
- Iterative algorithm, calculus of variations, [Lion and Handelsman1968], [Prussing2010]
- Mixed iterative algorithm [Arzelier et al.(2013)]
- Relaxation scheme, duality theory, minimum-norm problem [Neustadt64], [Claeys et al.(2013)]
Mathematical formulation of RdV problem

Optimal control Problem (after variable change $t \to \nu$)

Find $\bar{u} \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

$$\inf_{u} \|u\|_{1,p} = \inf_{u} \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu$$

subject to:

$$X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \quad \forall \nu \in [\nu_0, \nu_f]$$

$$X(\nu_0) = X_0, \quad X(\nu_f) = X_f \in \mathbb{R}^n, \quad \nu_0, \nu_f$$ fixed.

Previous Works

- Indirect approach, Pontryagin max principle, primer vector theory [Lawden63], [Carter&Brient95]
- Iterative algorithm, calculus of variations, [Lion and Handelsman1968], [Prussing2010]
- Mixed iterative algorithm [Arzelier et al.(2013)]
- Relaxation scheme, duality theory, minimum-norm problem [Neustadt64],[Claeys et al.(2013)]

Our approach:

- [Neustadt64] $\leadsto$ Moment Problem $\leadsto$ Duality [Luenberger1969] $\leadsto$ Semi-Infinite Convex Programming (SICP)[Reemtsen&Ruckman1998] $\leadsto$ New convergent iterative numerical algorithm
Mathematical formulation of RdV problem

Optimal control Problem (after variable change $t \rightarrow \nu$)

Find $\bar{u} \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

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subject to:

$$X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \ \forall \nu \in [\nu_0, \nu_f]$$

$$X(\nu_0) = X_0, \ X(\nu_f) = X_f \in \mathbb{R}^n, \ \nu_0, \ \nu_f \text{ fixed.}$$

Solution of the LODE system:
Optimal control Problem (after variable change $t \to \nu$)

Find $\bar{u} \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

$$
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X(\nu_0) = X_0, \ X(\nu_f) = X_f \in \mathbb{R}^n, \ \nu_0, \ \nu_f \text{ fixed.}
$$

Solution of the LODE system:

- $\varphi(\nu)$ a fundamental matrix for the homogeneous equation [Yamanaka&Ankersen02]
Mathematical formulation of RdV problem

Optimal control Problem (after variable change $t \rightarrow \nu$)

Find $\bar{u} \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the optimal control problem:

$$\inf_{u} \|u\|_{1,p} = \inf_{u} \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu$$

s.t. $X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \forall \nu \in [\nu_0, \nu_f]$

$X(\nu_0) = X_0, X(\nu_f) = X_f \in \mathbb{R}^n, \nu_0, \nu_f$ fixed. \hspace{1cm} (3)

Solution of the LODE system:

- $\varphi(\nu)$ a fundamental matrix for the homogeneous equation \cite{Yamanaka&Ankersen02}

- $X(\nu) = \varphi(\nu)\varphi^{-1}(\nu_0)X_0 + \varphi(\nu) \int_{\nu_0}^{\nu} \varphi^{-1}(\sigma)B(\sigma)u(\sigma)d\sigma$
Mathematical formulation of RdV problem

**Optimal control Problem (after variable change \( t \rightarrow \nu \))**

Find \( \bar{u} \in L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r) \) solution of the optimal control problem:

\[
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\text{s.t.} \quad X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \ \forall \ \nu \in [\nu_0, \nu_f] \\
X(\nu_0) = X_0, \ X(\nu_f) = X_f \in \mathbb{R}^n, \ \nu_0, \ \nu_f \text{ fixed.} \tag{3}
\]

**Solution of the LODE system:**

- \( \varphi(\nu) \) a fundamental matrix for the homogeneous equation [Yamanaka&Ankersen02]
- \( X(\nu) = \varphi(\nu)\varphi^{-1}(\nu_0)X_0 + \varphi(\nu) \int_{\nu_0}^{\nu} \varphi^{-1}(\sigma)B(\sigma)u(\sigma)d\sigma \)
- \( \int_{\nu_0}^{\nu_f} \varphi^{-1}(\sigma)B(\sigma)u(\sigma)d\sigma \underbrace{Y(\sigma)}_{c} = \varphi^{-1}(\nu_f)X(\nu_f) - \varphi^{-1}(\nu_0)X_0 \)
Mathematical formulation of RdV problem

Minimum Norm Moment Problem

\[
\inf_u \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t.} \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)d\sigma = c, \ \nu_0, \ \nu_f \text{ fixed.}
\]

(4)
Mathematical formulation of RdV problem

Minimum Norm Moment Problem

\[
\inf_u \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t.} \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)d\sigma = c, \ \nu_0, \ \nu_f \ \text{fixed.} 
\]

- optimal solution may not be reached due to concentration effects [Roubicek 2006]
Mathematical formulation of RdV problem

Minimum Norm Moment Problem

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\inf_u \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
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\] (4)

- optimal solution may not be reached due to concentration effects [Roubicek 2006]

- \( \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r) \) is not the topological dual of any other functional space
Mathematical formulation of RdV problem

Minimum Norm Moment Problem

\[
\begin{align*}
\inf_u \|u\|_{1,p} &= \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t.} \quad \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)d\sigma &= c, \ \nu_0, \ \nu_f \text{ fixed.}
\end{align*}
\]

- optimal solution may not be reached due to concentration effects [Roubicek 2006]
- \(\mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^T)\) is not the topological dual of any other functional space
- Relax problem on the set of functions of bounded variation \(g\):
  \[\sim \text{ Duality with linear functional } \ell: \ell(y_i) = \langle y_i, \ell \rangle = \int_{\nu_0}^{\nu_f} y_i(\nu)^T d\mathcal{g}(\nu).\]
Mathematical formulation of RdV problem

**Minimum Norm Moment Problem**

\[
\inf_u \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p d\nu \\
\text{s.t. } \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)d\sigma = c, \ \nu_0, \ \nu_f \ \text{fixed.}
\]

Optimal solution may not be reached due to concentration effects [Roubicek 2006]

\[L_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)\] is not the topological dual of any other functional space

Relax problem on the set of functions of bounded variation \(g\):

\[\mapsto \text{Duality with linear functional } \ell: \ell(y_i) = \langle y_i, \ell \rangle = \int_{\nu_0}^{\nu_f} y_i(\nu)^T d\sigma(\nu).\]

[Neustadt64] proved existence of optimal solution \(g\) as a step function with at most \(n\) points of discontinuity: 2 for out-of-plane and 4 for in-plane.
[Neustadt64] proves the equivalence with the following problem:

**SICP problem**

Find $\bar{\lambda} \in \mathbb{R}^n$ solution of

$$-ar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^n} -c^T \lambda$$

$$\|Y^T(\nu)\lambda\|_q \leq 1, \ \nu \in [\nu_0, \nu_f].$$

(5)
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\|Y^T(\nu)\lambda\|_q \leq 1, \ \nu \in [\nu_0, \nu_f].
\]

(5)

- ungimbaled identical thrusters: $p = 1, \ q = \infty$

Linear programing (LP) problem:

\[
\inf_{\lambda \in \mathbb{R}^n} -c^T \lambda \\
\text{s.t. } \left| \sum_{i=1}^{n} \lambda_i y_{i,s}(\nu) \right| \leq 1, \ \nu \in [\nu_0, \nu_f], \ s = 1, \ldots, r.
\]

(7)
[Neustadt64] proves the equivalence with the following problem:

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Find \( \bar{\lambda} \in \mathbb{R}^n \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^n} -c^T \lambda \\
\|Y^T(\nu)\lambda\|_q \leq 1, \ \nu \in [\nu_0, \nu_f].
\]

(5)

- gimbaled single thruster: \( p = 2, \ q = 2 \)
- ungimbaled identical thrusters: \( p = 1, \ q = \infty \)

Positive semi-definite (SDP) problem:

\[
\inf_{\lambda \in \mathbb{R}^n} -c^T \lambda \\
\text{s.t.} \quad \begin{bmatrix} -1 & \lambda^T Y(\nu) \\ Y^T(\nu) \lambda & -1 \end{bmatrix} \preceq 0, \quad \nu \in [\nu_0, \nu_f].
\]

(6)

Linear programing (LP) problem:

\[
\inf_{\lambda \in \mathbb{R}^n} -c^T \lambda \\
\text{s.t.} \quad \left| \sum_{i=1}^{n} \lambda_i y_{i,s}(\nu) \right| \leq 1, \quad \nu \in [\nu_0, \nu_f], \ s = 1, \ldots, r.
\]

(7)

\( \leadsto \) Convergent iterative numerical approach [Reemsten90s]
Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

For GTO Mission [Zhou et al. 2011], \( \nu_0 = 0.1\pi, \nu_f = 5.2, \ c^T = [9.1889 \ - 0.0954] \).
Find \( \lambda \in \mathbb{R}^2 \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \ -c^T \lambda \leq 1.
\]

\[
\left[ \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right]_{\text{primer vector pv(\nu)}} 
\leq 1.
\]

Step 0 Initialize \((\lambda_1, \lambda_2)\), e.g. from a solution with initial and final impulses; The cost is: \(-\mu_0 = 19.422948\). While \(\max \nu_{\text{pv}} > 1\)
Step Find \(\nu = \arg \left[ \max \nu_{\text{pv}} \right]\). Solve discretized version of (8) on \(\{\nu_0, ..., \nu, \nu_f\}\).

Final cost: 6.272461, \(\bar{\lambda} = [0.68270, 0.000001246]\).
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], \( \nu_0 = 0.1\pi, \nu_f = 5.2, c^T = [9.1889 \ - 0.0954] \).

Find \( \bar{\lambda} \in \mathbb{R}^2 \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda \quad \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \leq 1.
\]

**(8)**

**Step 0** Initialize \((\lambda_1, \lambda_2)\), e.g. from a solution with initial and final impulses; The cost is: \(-\mu_0 = 19.422948\);
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \ -0.0954]$. Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda$$

$$\left| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right| \leq 1.$$  \hspace{1cm} (8)

**Step 0** Initialize $(\lambda_1, \lambda_2)$, e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

**While** $\max_{\nu} pv > 1$

**Step 1** Find $\nu_1 = \arg \left[ \max_{\nu} pv \right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_1, \nu_f\}$
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], \( \nu_0 = 0.1\pi, \nu_f = 5.2, \ c^T = [9.1889\ -\ 0.0954] \).

Find \( \bar{\lambda} \in \mathbb{R}^2 \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda
\]

\[
\frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \leq 1.
\]

(8)

**Step 0** Initialize \((\lambda_1, \lambda_2)\), e.g. from a solution with initial and final impulses; The cost is:

\( -\mu_0 = 19.422948 \);

While \( \max_{\nu} \text{pv} > 1 \)

**Step 1** Find \( \nu_1 = \arg \left[ \max_{\nu} \text{pv} \right] \). Solve discretized version of (8) on \( \{\nu_0, \ldots, \nu_1, \nu_f\} \)

Cost \( -\mu_1 = 19.4198 \)
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \ - 0.0954]$.

Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda$$

$$\left| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right| \leq 1.$$

(8)

**Step 0** Initialize $(\lambda_1, \lambda_2)$, e.g. from a solution with initial and final impulses; The cost is:

$$-\mu_0 = 19.422948;$$

While $\max_{\nu} pv > 1$

**Step 2** Find $\nu_2 = \arg \left[ \max_{\nu} pv \right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_2, \nu_f\}$

Cost $-\mu_2 = 7.7249$
Example of the iterative algorithm for solving SICP

SICP problem for out-of-plane

For GTO Mission [Zhou et al. 2011], $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \ - 0.0954]$. Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda
\leq \left| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right| \leq 1.
$$

(8)

**Step 0** Initialize $(\lambda_1, \lambda_2)$, e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

**Step 3** Find $\nu_3 = \arg \left[ \max_{\nu} pv(\nu) \right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_3, \nu_f\}$

Cost $-\mu_3 = 6.6837$
Example of the iterative algorithm for solving SICP

### SICP problem for out-of-plane

For GTO Mission [Zhou et al. 2011], \( \nu_0 = 0.1\pi, \nu_f = 5.2, c^T = [9.1889 - 0.0954] \).
Find \( \bar{\lambda} \in \mathbb{R}^2 \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda \\
\left\| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right\| \leq 1.
\]

(8)

**Step 0** Initialize \((\lambda_1, \lambda_2)\), e.g. from a solution with initial and final impulses; The cost is: \( -\mu_0 = 19.422948 \);

**While** \( \max_\nu \text{pv} > 1 \)

**Step 4** Find \( \nu_4 = \arg \left[ \max_\nu \text{pv} \right] \). Solve discretized version of (8) on \( \{\nu_0, \ldots, \nu_4, \nu_f\} \)
Cost \( -\mu_4 = 6.2832 \)
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], \( \nu_0 = 0.1\pi, \nu_f = 5.2, \ c^T = [9.1889, -0.0954] \).

Find \( \bar{\lambda} \in \mathbb{R}^2 \) solution of

\[
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda \quad \left| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right| \leq 1.
\]

(8)

**Step 0** Initialize \((\lambda_1, \lambda_2)\), e.g. from a solution with initial and final impulses; The cost is:

\(-\mu_0 = 19.422948\);

**While** \( \max_{\nu} \nu \text{pv} > 1 \)

**Step 5** Find \( \nu_5 = \arg \left[ \max_{\nu} \nu \text{pv} \right] \). Solve discretized version of (8) on \( \{\nu_0, \ldots, \nu_5, \nu_f\} \)

Cost \(-\mu_5 = 6.273242\)
Example of the iterative algorithm for solving SICP

### SICP problem for out-of-plane

For GTO Mission\cite{Zhou et al.2011}, $\nu_0 = 0.1\pi$, $\nu_f = 5.2$, $c^T = [9.1889 \ -0.0954]$.

Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$-ar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} -c^T \lambda$$

$$\left| \frac{\sin \nu \lambda_1 + \cos \nu \lambda_2}{1 + e \cos \nu} \right| \leq 1.$$  \hspace{1cm} (8)

**Step 0** Initialize $(\lambda_1, \lambda_2)$, e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

**While** $\max_{\nu} \text{pv} > 1$

**Step 6** Find $\nu_6 = \arg \left[ \max_{\nu} \text{pv} \right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_6, \nu_f\}$

Cost $-\mu_6 = 6.273240$
Example of the iterative algorithm for solving SICP

**SICP problem for out-of-plane**

For GTO Mission [Zhou et al. 2011], $\nu_0 = 0.1 \pi$, $\nu_f = 5.2$, $c^T = [9.1889 \; -0.0954]$.

Find $\bar{\lambda} \in \mathbb{R}^2$ solution of

$$
-\bar{\eta} = \bar{\mu} = \min_{\lambda \in \mathbb{R}^2} \left(-c^T \lambda \right) \quad \sin \nu \lambda_1 + \cos \nu \lambda_2 \left| \frac{1 + e \cos \nu}{1 + e \cos \nu} \right| \leq 1. \tag{8}
$$

**Step 0** Initialize $(\lambda_1, \lambda_2)$, e.g. from a solution with initial and final impulses; The cost is: $-\mu_0 = 19.422948$;

**While** $\max_{\nu} \text{pv} > 1$

**Step 7** Find $\nu_7 = \arg \left[ \max_{\nu} \text{pv} \right]$. Solve discretized version of (8) on $\{\nu_0, \ldots, \nu_7, \nu_f\}$

Cost $-\mu_7 = 6.272461$

**Final cost:** 6.272461

$\bar{\lambda} = [0.68270 \; 0.000001246]$
Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.

**Example GTO mission (out-of-plane)**

- Primer vector $\approx 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)

- Impulse locations: 2.3883; 3.8919
Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.

Example GTO mission (out-of-plane)

- Primer vector $\approx 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)
- Impulse locations: 2.3883; 3.8919

- There exists an optimal solution with at most $n$ (2 for out-of-plane and 4 for in-plane) impulses $\hat{\nu}_i, i = 1, \ldots, N \leq n$. 
Retrieval of the impulse locations and velocities

- Impulses are located where primer vector is 1.

- There exists an optimal solution with at most $n$ (2 for out-of-plane and 4 for in-plane) impulses $\hat{\nu}_i, i = 1, \ldots, N \leq n$.

- Solve for $\Delta V_i, i = 1, \ldots, N$, the linear system $c = \sum_i Y(\hat{\nu}_i)\Delta V_i$.

Example GTO mission (out-of-plane)

- Primer vector $\simeq 1 \pm 10^{-4}$ (after 6 iterations, with rigorous computations)

- Impulse locations: $2.3883, 3.8919$

- Velocities increments:
  
  $\Delta V_1 = 3.115020$ m/s, $\Delta V_2 = 3.157441$ m/s.
Results for GTO Mission [Zhou 2011]

- Target satellite on Geostationary Transfer Orbit (GTO)
- Temporary elliptic orbit to inject a satellite into the Geostationary Earth Orbit (GEO)
- Perigee on LEO (altitude $\approx 200$ km) and apogee on GEO (altitude = 35786 km)
- Requires a change of orbital plane and out-of-plane maneuvers

<table>
<thead>
<tr>
<th>Semi-major axis</th>
<th>$a = 24616$ km</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eccentricity</td>
<td>$e = 0.73074$</td>
</tr>
<tr>
<td>Initial anomaly</td>
<td>$\theta_0 = 0.1\pi$ rad</td>
</tr>
<tr>
<td>Initial state vector</td>
<td>$X_0^T = [10000 -3]$ m - m/s</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>$\theta_f = 5.2$ rad</td>
</tr>
<tr>
<td>Duration</td>
<td>$t_f - t_0 = 29888$ s</td>
</tr>
<tr>
<td>Final state vector</td>
<td>$X_f^T = [0 0]$ m - m/s</td>
</tr>
</tbody>
</table>

![Graphs showing orbital dynamics](image-url)
Example for the in-plane case

SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda$$
$$\left\| Y^T(\nu)\lambda \right\|_2 \leq 1.$$  \hfill (9)
Example for the in-plane case

SICP problem for in-plane, \( q = 2 \)

Find \( \bar{\lambda} \in \mathbb{R}^4 \) solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \\
\left\| Y^T(\nu) \lambda \right\|_2 \leq 1.
\] (9)
Example for the in-plane case

**SICP problem for in-plane, \( q = 2 \)**

Find \( \bar{\lambda} \in \mathbb{R}^4 \) solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \\
\left\| Y^T(\nu)\lambda \right\|_2 \leq 1. 
\]  

Automated Transfer Vehicle (ATV) mission - European unmanned vehicle 2008-2014 (ESA, ADS, CNES); Logistic servicing of the ISS [Labourdette08]:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
<td>( a = 6763 \text{ km.} )</td>
</tr>
<tr>
<td>Inclination</td>
<td>( i = 52 \text{ deg.} )</td>
</tr>
<tr>
<td>Argument of perigee</td>
<td>( \omega = 0 \text{ deg.} )</td>
</tr>
<tr>
<td>Longitude of the ascending node</td>
<td>( \Omega = 0 \text{ deg.} )</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>( e = 0.0052 )</td>
</tr>
<tr>
<td>Initial time</td>
<td>( \nu_0 = 0 \text{ rad.} )</td>
</tr>
<tr>
<td>Initial state vector ( X^T_0 )</td>
<td>([-30 0.5 8.514 0] \text{ km. - m/s.})</td>
</tr>
<tr>
<td>Initial state vector ( \bar{X}^T_0 )</td>
<td>([-51.9222 0.0865 0.95734 0].10^4)</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>( \nu_f = 8.1832 \text{ rad.} )</td>
</tr>
<tr>
<td>Duration</td>
<td>( t_f - t_0 = 7200 \text{ s.} )</td>
</tr>
<tr>
<td>Final state vector ( X^T_f )</td>
<td>([-100 0 0 0] \text{ m. - m/s.})</td>
</tr>
<tr>
<td>Final state vector ( \bar{X}^T_f )</td>
<td>([-76.3818 0 69.1519 0])</td>
</tr>
</tbody>
</table>
Example for the in-plane case

SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda$$

$$\left\| Y^T(\nu) \lambda \right\|_2 \leq 1.$$  \hfill (9)

Step 0 Initialize $\lambda$, e.g. from a solution with initial and final impulses;
Cost$_0 = 18.064877$;

While $\max_{\nu} \nu_v > 1$

Step $i$ Find $\nu_i = \arg \left[ \max_{\nu} \nu_v \right]$. Solve discretized version of (9) on \{\nu$_0$, ..., \nu$_i$, \nu$_f$\}
Example for the in-plane case

### SICP problem for in-plane, \( q = 2 \)

Find \( \bar{\lambda} \in \mathbb{R}^4 \) solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \\
\left\| Y^T(\nu)\lambda \right\|_2 \leq 1.
\]  \( (9) \)

---

**Step 0** Initialize \( \lambda \), e.g. from a solution with initial and final impulses;  
Cost \( C_{0} = 18.064877 \);

**While** \( \max_{\nu} \nu p_v > 1 \)

**Step i** Find \( \nu_i = \arg \left[ \max_{\nu} \nu p_v \right] \). Solve discretized version of (9) on \( \{\nu_0, \ldots, \nu_i, \nu_f\} \)

---

![Graph](image-url)
Example for the in-plane case

### SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \\
\|Y^T(\nu)\lambda\|_2 \leq 1.
\]

(9)

**Step 0** Initialize $\lambda$, e.g. from a solution with initial and final impulses; Cost$_0 = 18.064877$;

**While** $\max_{\nu} \nu > 1$

**Step $i$** Find $\nu_i = \arg \left[ \max_{\nu} \nu \right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$
Example for the in-plane case

SICP problem for in-plane, $q = 2$

Find $\bar{\lambda} \in \mathbb{R}^4$ solution of

$$-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} \quad -c^T \lambda$$

$$\left\| Y^T(\nu) \lambda \right\|_2 \leq 1.$$  \hspace{1cm} (9)

Step 0  Initialize $\lambda$, e.g. from a solution with initial and final impulses;
          Cost$_0 = 18.064877$;

While $\max \nu \text{pv} > 1$

Step i  Find $\nu_i = \arg \left[ \max \nu \text{pv} \right]$. Solve discretized version of (9) on $\{\nu_0, \ldots, \nu_i, \nu_f\}$
Example for the in-plane case

**SICP problem for in-plane, \( q = 2 \)**

Find \( \bar{\lambda} \in \mathbb{R}^4 \) solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \\
\|Y^T(\nu)\lambda\|_2 \leq 1.
\]  

(9)

**Step 0** Initialize \( \lambda \), e.g. from a solution with initial and final impulses;
\[ \text{Cost}_0 = 18.064877; \]

**While** \( \max_{\nu} \nu v > 1 \)

**Step i** Find \( \nu_i = \arg \left[ \max_{\nu} \nu v \right] \). Solve discretized version of (9) on \( \{\nu_0, \ldots, \nu_i, \nu_f\} \)
Example for the in-plane case

SICP problem for in-plane, \( q = 2 \)

Find \( \tilde{\lambda} \in \mathbb{R}^4 \) solution of

\[
-\eta = \bar{\mu} = \min_{\lambda \in \mathbb{R}^4} -c^T \lambda \quad -c^T \lambda \\
\|Y^T(\nu)\lambda\|_2 \leq 1.
\] (9)

Step 0 Initialize \( \lambda \), e.g. from a solution with initial and final impulses;
Cost \( C_0 = 18.064877 \);

While \( \max_{\nu} \nu > 1 \)

Step \( i \) Find \( \nu_i = \arg \left[ \max_{\nu} \nu \right] \). Solve discretized version of (9) on \( \{\nu_0, \ldots, \nu_i, \nu_f\} \)

Final cost: 6.272461,

\[
\tilde{\lambda}^T = \begin{bmatrix}
-0.0001177 \\
0.0001231 \\
-0.0001571 \\
-0.001436
\end{bmatrix}
\]

iteration 6
Retrieval of the impulse locations and velocities: example ATV mission (in-plane)

- Primer vector module $\simeq 1 \pm 10^{-4}$ (after 6 iterations)
- Impulse locations: $0, 1.388128, 6.666595, 8.183058$ [rad]
- Velocities increments:
  - $\Delta V_1 = [7.50230589; -0.742372034]$ m/s,
  - $\Delta V_2 = [1.55579123; -0.088346857]$ m/s
  - $\Delta V_3 = [-0.62565013; -0.03325936]$ m/s
  - $\Delta V_4 = [-1.06509710; -0.11440204]$ m/s.
Results for ATV mission (in-plane) (I)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
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<tr>
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<td>$i = 52$ deg.</td>
</tr>
<tr>
<td>Argument of perigee</td>
<td>$\omega = 0$ deg.</td>
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<tr>
<td>Longitude of the ascending node</td>
<td>$\Omega = 0$ deg.</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>$e = 0.0052$</td>
</tr>
<tr>
<td>Initial time</td>
<td>$\nu_0 = 0$ rad.</td>
</tr>
<tr>
<td>Initial state vector $X^T_0$</td>
<td>[-30 0.5 8.514 0] km. - m/s.</td>
</tr>
<tr>
<td>Initial state vector $\tilde{X}^T_0$</td>
<td>[-51.9222 0.0865 0.95734 0].10$^4$</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>$\nu_f = 8.1832$ rad.</td>
</tr>
<tr>
<td>Duration</td>
<td>$t_f - t_0 = 7200$ s.</td>
</tr>
<tr>
<td>Final state vector $X^T_f$</td>
<td>[-100 0 0 0] m. - m/s.</td>
</tr>
<tr>
<td>Final state vector $\tilde{X}^T_f$</td>
<td>[-76.3818 0 69.1519 0]</td>
</tr>
</tbody>
</table>

![Figure: Optimal trajectory in $(x, z)$ plane.](image)

Figure: Optimal trajectory in $(x, z)$ plane.
Results for ATV-Long mission (in-plane) (II)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
<td>$a = 6763$ km.</td>
</tr>
<tr>
<td>Inclination</td>
<td>$i = 52$ deg.</td>
</tr>
<tr>
<td>Argument of perigee</td>
<td>$\omega = 0$ deg.</td>
</tr>
<tr>
<td>Longitude of the ascending node</td>
<td>$\Omega = 0$ deg.</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>$e = 0.0052$</td>
</tr>
<tr>
<td>Initial time</td>
<td>$\nu_0 = 0$ rad.</td>
</tr>
<tr>
<td>Initial state vector $X_0^T$</td>
<td>$[-30 \ 0.5 \ 8.514 \ 0]$ km. - m/s.</td>
</tr>
<tr>
<td>Initial state vector $\tilde{X}_0^T$</td>
<td>$[-51.9222 \ 0.0865 \ 0.95734 \ 0]$</td>
</tr>
<tr>
<td>Final anomaly</td>
<td>$\nu_f = 8.1832$ rad.</td>
</tr>
<tr>
<td>Duration</td>
<td>$t_f - t_0 = 55350$ s.</td>
</tr>
<tr>
<td>Final state vector $X_f^T$</td>
<td>$[-100 \ 0 \ 0 \ 0]$ m. - m/s.</td>
</tr>
<tr>
<td>Final state vector $\tilde{X}_f^T$</td>
<td>$[-76.3818 \ 0 \ 69.1519 \ 0]$</td>
</tr>
</tbody>
</table>

Control:

<table>
<thead>
<tr>
<th>imp.[rad]</th>
<th>velocity[m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-7.55418436$</td>
</tr>
<tr>
<td></td>
<td>$-0.23360120$</td>
</tr>
<tr>
<td>59.88676846</td>
<td>$0.14408063$</td>
</tr>
<tr>
<td></td>
<td>$0.001028313$</td>
</tr>
<tr>
<td>62.83136216</td>
<td>$0.0416620$</td>
</tr>
<tr>
<td></td>
<td>$0.0012880$</td>
</tr>
</tbody>
</table>

**Diagram:**

- Positions [m] vs $\nu$ [rad.]
- Velocities [m/rad] vs $\nu$ [rad.]
- Impulses [m/rad] vs $\nu$ [rad.]
Sum up: RdV Problem

- Convergent SICP iterative algorithm for solving the optimal control problem in the linearized elliptic case
- Use state of the art linear/SDP solvers
- Gives simple geometrical insight to obtain analytical solution in the out-of-plane case.

More on this...

16h45-17h15 Florent’s talk:
*Approximations, Fixed-Point Methods and Algorithms for Function Space Problems*

17h15-17h45 Paulo’s talk:
*Model predictive control for spacecraft rendezvous hovering phases based on validated Chebyshev series approximations of the transition matrices*
Let $T > 1$. Solve for $x \in W^{1,1}([0, T], \mathbb{R})$ and $u \in \mathcal{L}^1([0, T], \mathbb{R})$:

$$
\begin{align*}
\min_{u} & \quad J(x, u) = \int_{0}^{T} \left( 2 - 2t + t^2 \right) |u(t)| dt + (x(T) - 1)^2 \\
\text{s.t.} & \quad \frac{dx}{dt} = u, \quad x(0) = 0.
\end{align*}
$$
Let $T > 1$. Solve for $x \in W^{1,1}([0, T], \mathbb{R})$ and $u \in L^1([0, T], \mathbb{R})$: 

$$\min_u J(x, u) = \int_0^T \left(2 - 2t + t^2\right) |u(t)| dt + (x(T) - 1)^2$$

s.t. $\frac{dx}{dt} = u$, $x(0) = 0$.

Solution via a minimizing sequence, e.g. $\{(x_k, u_k)\}_{k \in \mathbb{N}}$:

$$x_k(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{k}{2} (t - 1), & t \in \left(1, 1 + \frac{1}{k}\right] \\ \frac{1}{2}, & t \in \left(1 + \frac{1}{k}, T\right] \end{cases}$$

$$u_k(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{k}{2}, & t \in \left(1, 1 + \frac{1}{k}\right] \\ 0, & t \in \left(1 + \frac{1}{k}, T\right] \end{cases}$$
Let $T > 1$. Solve for $x \in W^{1,1}([0, T], \mathbb{R})$ and $u \in L^1([0, T], \mathbb{R})$:

$$\min_u \ J(x, u) = \int_0^T \left(2 - 2t + t^2\right) |u(t)| dt + (x(T) - 1)^2$$

subject to

$$\frac{dx}{dt} = u, \quad x(0) = 0.$$

Solution via a minimizing sequence, e.g. $\{(x_k, u_k)\}_{k \in \mathbb{N}}$:

$$x_k(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{k}{2}(t - 1), & t \in \left(1, 1 + \frac{1}{k}\right) \\ \frac{1}{2}, & t \in \left(1 + \frac{1}{k}, T\right) \end{cases}$$

$$u_k(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{k}{2}, & t \in \left(1, 1 + \frac{1}{k}\right) \\ 0, & t \in \left(1 + \frac{1}{k}, T\right) \end{cases}$$

$$J(x_k, u_k) = \int_0^T Y(t) |u_k(t)| dt + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + O \left(\frac{1}{k}\right) \to \frac{3}{4}$$
\( \inf J \) cannot be achieved and the optimal control does not exist in \( \mathcal{L}^1([0, T], \mathbb{R}) \).

Solution via step functions or discrete measures:

\[
\min_{x(T), t_0} J = \int_0^T Y(t)|dx(t)| + (x(T) - 1)^2
\]

\[
x(t) = \begin{cases} 
0 & t \leq t_0 \\
x(T) & t > t_0 
\end{cases}
\]

\[
\min_{\mu, t_0} J = \int_0^T Y(t)|d\mu(t)| + (x(T) - 1)^2
\]

\[
\mu([0, T]) = \delta_{t_0}([0, T]) = \begin{cases} 
0 & t_0 \notin [0, T] \\
x(T) & t_0 \in [0, T] 
\end{cases}
\]
Roubíček’s example [Roubíček 2006], [Kružíck 1998]

\[ \inf J \text{ cannot be achieved and the optimal control does not exist in } L^1([0, T], \mathbb{R}) \]

\[ \text{Solution via step functions or discrete measures:} \]

\[
\begin{align*}
\min_{x(T), t_0} J &= \int_0^T Y(t) |dx(t)| + (x(T) - 1)^2 \\
\quad x(t) &= \begin{cases} 
0 & t \leq t_0 \\
x(T) & t > t_0 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{\mu, t_0} J &= \int_0^T Y(t) |d\mu(t)| + (x(T) - 1)^2 \\
\quad \mu([0, T]) &= \delta_{t_0}([0, T]) = \begin{cases} 
0 & t_0 \not\in [0, T] \\
x(T) & t_0 \in [0, T] 
\end{cases}
\end{align*}
\]

\[
\min_{x(T), t_0} x(T) Y(t_0) + (x(T) - 1)^2 \quad \text{with solution:} \quad x^*(T) = \frac{1}{2}, \quad J^* = \frac{3}{4}
\]
Zoom on Indicator functions

\[ g(z) = \frac{F(z)}{G(z)} \]

- \(|g(re^{i\theta})| \sim \exp(h(\theta)r)\) for large \(r\)

- indicator of \(g\):

\[ h(\theta) = \begin{cases} 
\frac{-\cos \theta}{2\sigma_y^2} & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\
0 & \text{otherwise.}
\end{cases} \]
Zoom on Indicator functions

\[ g(z) = \frac{F(z)}{G(z)} \]

- \(|g(re^{i\theta})| \sim \exp(h(\theta)r)\) for large \(r\)

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0 & \text{otherwise.} 
\end{cases} \]

\[ \text{lost precision } \sim |\sigma - h(0)| \]
Zoom on Indicator functions

\[ g(z) = \frac{F(z)}{G(z)} \]

- \[ |g(re^{i\theta})| \sim \exp(h(\theta)r) \] for large \( r \)
- indicator of \( g \):

\[ h(\theta) = \begin{cases} \frac{-\cos \theta}{2\sigma y^2} & \text{if } \theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \\ 0 & \text{otherwise.} \end{cases} \]

- \( G(z) = \exp(\sigma z) \) 
  indicator of \( G \):

lost precision \( \sim |\sigma - h(0)| \)
**Zoom on Indicator functions**

\[ G(z)g(z) = F(z) \]

- \(|g(re^{i\theta})| \sim \exp(h(\theta)r)\) for large \(r\)
- indicator of \(g\):

\[ h(\theta) = \begin{cases} 
-\frac{\cos \theta}{2\sigma y^2} & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\
0 & \text{otherwise.} 
\end{cases} \]

\[ G(z) = \exp(\sigma z) \] indicator of \(G\):

\[ \text{indicator of } F:\]

lost precision \(\sim |\sigma - h(0)|\)
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$.

$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ satisfies $\hat{L}'(x) = \varphi(x)\hat{L}(x)$, $\hat{L}(0) = \exp\left(-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2}\right)$, where

$$\varphi(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2 \sigma_y^2 + x} + \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.$$
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$.

$$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$ satisfies $\hat{L}'(x) = \varphi(x)\hat{L}(x)$, $\hat{L}(0) = \exp\left(-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2}\right)$, where

$$\varphi(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{x \left(\sigma_x^2 - \sigma_y^2\right) - 2\sigma_x^2 \sigma_y^2} - \frac{1}{-2\sigma_y^2 + x} + \frac{-\sigma_x^2 + \sigma_y^2}{2x \left(\sigma_x^2 - \sigma_y^2\right) - 4\sigma_x^2 \sigma_y^2}.$$

$$\varphi_k = \frac{1 + \left(1 - \frac{\sigma_y^2}{\sigma_x^2}\right)^k \left(k + 1\right) \left(\frac{x_m^2 \sigma_y^2}{\sigma_x^4}\right) + 1 - \frac{\sigma_y^2}{\sigma_x^2}}{\left(2\sigma_y^2\right)^{k+1}} + \begin{cases} 0, & k > 0 \\ \frac{y_m^2}{4\sigma_y^4}, & k = 0, \end{cases}$$
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$. 

$$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$ satisfies $\hat{L}'(x) = \varphi(x)\hat{L}(x)$, $\hat{L}(0) = \exp\left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} \right)$, where

$$\varphi(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2 \sigma_y^2 + x}$$

$$+ \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.$$

$$0 \leq \varphi_k = \frac{1 + \left(1 - \frac{\sigma_y^2}{\sigma_x^2}\right)^k \left(k + 1\right) \left(\frac{x_m^2 \sigma_y^2}{\sigma_x^4}\right) + 1 - \frac{\sigma_y^2}{\sigma_x^2}}{\left(2\sigma_y^2\right)^{k+1}} + \begin{cases} 
0, & k > 0 \\
\frac{y_m^2}{4\sigma_y^4}, & k = 0,
\end{cases}$$
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$.

\[
\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ satisfies } \hat{L}'(x) = \varphi(x)\hat{L}(x), \quad \hat{L}(0) = \frac{\exp \left( - \frac{\sigma^2_y y_m^2 + \sigma^2_x x_m^2}{2\sigma_x^2 \sigma_y^2} \right)}{2\sigma_x \sigma_y}, \quad \text{where}
\]

\[
\varphi(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2 \sigma_y^2 + x} + \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.
\]

\[
0 \leq \varphi_k = 1 + \frac{2}{(2\sigma_y^2)^{k+1}} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} \right)^k \left( k + 1 \right) \left( \frac{x_m \sigma_y^2}{\sigma_x^4} \right) + 1 - \frac{\sigma_y^2}{\sigma_x^2}\right)
\]

\[
0 \leq (n + 1)\alpha_{n+1} = \sum_{i=0}^{n} \varphi_i \cdot \alpha_{n-i}, \quad \sim \text{ by induction.}
\]