A new method to compute the probability of collision for short-term space encounters

R. Serra, D. Arzelier, M. Joldes, J-B. Lasserre, A. Rondepierre and B. Salvy

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Fiction...

Credit Gravity (2013)
On-orbit collision

Figure: Cerise hit by a debris in 1996 (source: CNES/D. Ducros)
On-orbit collision

Figure: Space debris population model (source: ESA)
On-orbit collision

Context

- Two objects: primary $P$ (operational satellite) and secondary $S$ (space debris)
- Information about their geometry, position, velocity at a given time
  - Affected by uncertainty
- Needs:
  - Risk assessment
  - Design of a collision avoidance strategy
  - Compute the probability of collision
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Classical assumptions

- Spherical objects
- Gaussian probability density functions
- Independent probability distribution laws

Figure: Combined spherical object
On-orbit collision

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Probability of collision

- Generally: 12-dimensional, Gaussian integrand, Complex integration domain
- Computation: Monte-Carlo trials and/or simplified models

Figure: Combined spherical object
Short-term encounter model and probability of collision

- **Framework**: High relative velocity
- **Assumptions**:
  - Rectilinear relative motion
  - No velocity uncertainty
  - Infinite encounter time horizon

  \[ P = \frac{1}{2\pi\sigma_x\sigma_y} \int_{B((0,0),R)} \exp\left(-\frac{(x-x_m)^2}{2\sigma_x^2} - \frac{(y-y_m)^2}{2\sigma_y^2}\right) \, dx \, dy, \]

  where
  - \( R \): radius of combined object
  - \( x_m, y_m \): mean relative coordinates
  - \( \sigma_x, \sigma_y \): standard deviations of relative coordinates

  Probability of collision:
  - 2-D integral over a disk.
Short-term encounter model and probability of collision

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  \[ P \approx \text{Probability of collision:} \]
  
  2-D integral over a disk.

**Formula**

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P = \frac{1}{2\pi \sigma_x \sigma_y} \int_{B((0,0),R)} \exp \left( -\frac{(x - x_m)^2}{2\sigma_x^2} - \frac{(y - y_m)^2}{2\sigma_y^2} \right) \, dx \, dy,
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\[ \sim \text{Probability of collision:} \]

2-D integral over a disk.

**Figure:** 2-D Gaussian integral over a disk

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Existing methods

- Methods based on numerical integration schemes: Foster '92, Patera '01, Alfano '05.
- Analytic methods: Chan '97 uses some simplifying assumptions ($\sigma_x = \sigma_y$)

Pro’s and Con’s
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- use truncated power series, but no rigorous proof about convergence rate
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- Use truncated power series, but no rigorous proof about convergence rate
- Truncation orders fixed by trial and error and by comparing with other existing software: Approximately 60,000 test cases were used to evaluate the numerical expression [...] The reference ("truth") probability was computed with MATHCAD 11 [...] - Alfano'05
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- Fast and already used in practice
Existing methods

- Methods based on numerical integration schemes: Foster '92, Patera '01, Alfano '05.
- Analytic methods: Chan '97 → uses some simplifying assumptions ($\sigma_x = \sigma_y$) → provides truncation error bounds

Pro’s and Con’s

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- Fast and already used in practice

Our purpose

Give a "simple", "analytic" formula, suitable for double-precision evaluation and effective error bounds.
Laplace transform:

Our method - Underlying techniques

1. **Laplace transform:**
   \[ \sim \sim \text{Lasserre and Zeron, Solving a Class of Multivariate Integration Problems via Laplace Techniques, Applicationes Mathematicae, 2001.} \]

2. **D-finite functions**
   \[ \sim \sim \text{solution of linear differential equation with polynomial coefficients} \]
   \[ \sim \sim \text{power series coefficients satisfy a linear recurrence relation with polynomial coefficients} \]

Example:

\[ f(x) = \exp(x) \iff \{ f' - f = 0, \ f(0) = 1 \} \iff \{(n + 1)f_{n+1} = f_n, f_0 = 1\} \]
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3. Finite-precision evaluation of power series prone to cancellation
   \( \leadsto \) Chevillard, Mezzarobba, *Multiple-Precision Evaluation of the Airy Ai Function with Reduced Cancellation*, 21st IEEE SYMPOSIUM on Computer Arithmetic, 2013
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∀z ∈ ℝ⁺:

\[ g(z) := \mathcal{P}(\sqrt{z}) = \frac{1}{2\pi \sigma_x \sigma_y} \int_{B((0,0),\sqrt{z})} \exp \left( -\frac{(x - x_m)^2}{2\sigma_x^2} - \frac{(y - y_m)^2}{2\sigma_y^2} \right) \, dx \, dy, \quad (1) \]
Sketch of the proof - Laplace Transform

\[ \forall z \in \mathbb{R}^+ : \]
\[ g(z) := \mathcal{P}(\sqrt{z}) = \frac{1}{2\pi \sigma_x \sigma_y} \int_{B((0,0),\sqrt{z})} \exp \left( -\frac{(x-x_m)^2}{2\sigma_x^2} - \frac{(y-y_m)^2}{2\sigma_y^2} \right) \, dx \, dy, \quad (1) \]

\[ \mathcal{L}(g)(t) = \int_0^{+\infty} g(z) \exp(-tz) \, dz \quad (2) \]

\[ (3) \]
Sketch of the proof - Laplace Transform

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\[ \mathcal{L}(g)(t) = \int_0^{+\infty} g(z) \exp(-t z) \, dz \quad (2) \]

\[ \exp \left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t \sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t \sigma_x^2 + 1)} \right) \quad (3) \]

\[ \frac{t \sqrt{(2t \sigma_x^2 + 1)(2t \sigma_y^2 + 1)}}{t \sqrt{(2t \sigma_x^2 + 1)(2t \sigma_y^2 + 1)}} \]
Sketch of the proof - Laplace Transform

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\]

(3)

\(\mathcal{L}(g)\) is D-finite !
Sketch of the proof - Borel-Laplace

\[ g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1} \]

\[ \mathcal{L}(g)(t) = \exp\left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t\sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t\sigma_x^2 + 1)} \right) \frac{t}{t\sqrt{(2t\sigma_x^2 + 1)(2t\sigma_y^2 + 1)}} \]

\[ \hat{\mathcal{L}}(g)(t) := t^2 \mathcal{L}(g) \left( \frac{1}{t} \right) = \sum_{i=0}^{\infty} l_i \left( \frac{1}{t} \right)^i \]

\( g(z) \) is:
- D-finite
- Entire function of exponential type
- Type \( \sigma = \frac{1}{2\sigma_y^2} \)

\( \hat{\mathcal{L}}(g)(t) \) is:
- D-finite
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Sketch of the proof - Borel-Laplace

\[ \mathcal{L}(g)(t) = \exp \left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t\sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t\sigma_x^2 + 1)} \right) \frac{t}{\sqrt{(2t\sigma_x^2 + 1)(2t\sigma_y^2 + 1)}} \]

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\( l_i \) satisfy a linear recurrence with polynomial coefficients.

\( \leadsto \) Compute everything with gfun
Sketch of the proof - Borel-Laplace

\[ g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1} \]

\[ \hat{L}(g)(t) := t^2 L(g) \left( \frac{1}{t} \right) = \sum_{i=0}^{\infty} l_i \left( \frac{1}{t} \right)^i \]

\[ L(g)(t) = \exp \left( \frac{-\sigma_x^2 y m^2 + \sigma_y^2 x m^2}{2 \sigma_x^2 \sigma_y^2} + \frac{y m^2}{2 \sigma_y^2 (2t \sigma_y^2 + 1)} + \frac{x m^2}{2 \sigma_x^2 (2t \sigma_x^2 + 1)} \right) \frac{t}{t \sqrt{(2t \sigma_x^2 + 1)(2t \sigma_y^2 + 1)}} \]

\( g(z) \) is:
- D-finite
- entire function of exponential type
- type \( \sigma = \frac{1}{2 \sigma_y^2} \)
- sum prone to cancellation

\( \hat{L}(g)(t) \) is:
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- Finite radius of convergence \( 2\sigma_y^2 \)

\( l_i \) satisfy a linear recurrence with polynomial coefficients.
\[ \leadsto \text{Compute everything with gfun} \]
Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1}$$
Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots$$
Cancellation in finite precision power series evaluation

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Values of $\left| \frac{l_i 225^{i+1}}{(i+1)!} \right|$, compared to $g(225) \simeq 0.1004$:

Lost Digits: $d_g(z) \simeq \log \max_i \left| \frac{g_i z^i}{|g(z)|} \right|$
Example: \( \exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \)

\[
\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots
\]
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Values of \( \left| \frac{(-1)^i 20^i}{i!} \right| \), compared to \( \exp(-20) \simeq 2.06 \cdot 10^{-9} \):

Lost Digits: \( d_g(z) \simeq \log \frac{\max_i \left| g_i z^i \right|}{\left| g(z) \right|} \)
Cancellation in finite precision power series evaluation

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**Lost Digits:**
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d_g(z) \simeq \log \max_i \frac{|g_i z^i|}{|g(z)|}
\]

**BUT...**
\[
\exp(-x) = \frac{1}{\exp(x)}
\]

No cancellation!
Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

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Values of $\left| \frac{l_i \cdot 225^{i+1}}{(i+1)!} \right|$, compared to $g(225) \approx 0.1004$:  

![Graph showing values of $l_i \cdot 225^{i+1}/(i+1)!$]
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Values of \( \left| \frac{l_i \cdot 225^{i+1}}{(i+1)!} \right| \), compared to \( g(225) \approx 0.1004 \):

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No cancellation!
Cancellation in finite precision power series evaluation

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No cancellation!

Gawronski, Müller, Reinhard (2007) Method:

Find \( F(z) \) and \( G(z) \) using "complex analysis tricks" \( \sim \) indicator function.
Sketch of the proof - Borel-Laplace + GMR Method

\[ g(z) = \frac{F(z)}{G(z)} \]

Laplace

\[ L(g)(t) = \exp\left( -\frac{\sigma_x^2 y m^2 + \sigma_y^2 x m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y m^2}{2\sigma_y^2} + \frac{x m^2}{2\sigma_x^2 (2t\sigma_y^2 + 1)} \right) \]

expansion at \( \infty \)

\[ \hat{L}(g)(t) := t^2 L(g)\left(\frac{1}{t}\right) = \sum_{i=0}^{\infty} l_i \left(\frac{1}{t}\right)^i \]

\[ g(z) = \sum_{i=0}^{\infty} \frac{l_i}{(i+1)!} z^{i+1} \]

Borel Transform

\[ g(z) \text{ is:} \]
- D-finite
- entire function of exponential type
- type \( \sigma = \frac{1}{2\sigma_y^2} \)

\[ \hat{L}(g)(t) \text{ is:} \]
- D-finite
- Finite radius of convergence \( 2\sigma_y^2 \)
- \( l_i \) - P-recursive
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\[ \mathcal{L}(g)(t) = \exp \left( -\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2} + \frac{y_m^2}{2\sigma_y^2 (2t\sigma_y^2 + 1)} + \frac{x_m^2}{2\sigma_x^2 (2t\sigma_x^2 + 1)} \right) \]

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\( g(z) \) is:
- D-finite
- Entire function of exponential type
- Type \( \sigma = \frac{1}{2\sigma_y^2} \)
- Indicator function:
  \[ |g(re^{i\theta})| \sim \exp(h(\theta)r) \text{ for large } r \]

\[ h(\theta) = \begin{cases} 
-\frac{\cos \theta}{2\sigma_y^2} & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\
0 & \text{if } \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right].
\end{cases} \]

\( \hat{\mathcal{L}}(g)(t) \) is:
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Zoom on Indicator functions

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\[ \text{lost precision } \sim |\sigma - h(0)| \]
Zoom on Indicator functions

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0 & \text{otherwise.} 
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- \( G(z) = \exp(\sigma z) \)
- indicator of \( G \):

\[ \text{lost precision } \sim |\sigma - h(0)| \]
Zoom on Indicator functions

\[ G(z)g(z) = F(z) \]

- \(|g(re^{i\theta})| \sim \exp(h(\theta)r)\) for large \(r\)
- indicator of \(g\):

\[ h(\theta) = \begin{cases} \frac{-\cos \theta}{2\sigma y^2} & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ 0 & \text{otherwise.} \end{cases} \]

- \(G(z) = \exp(\sigma z)\)
- indicator of \(G\):

lost precision \(\sim |\sigma - h(0)|\)
Sketch of the proof - Borel-Laplace + GMR Method

\[ G(z)g(z) = F(z) \]

Laplace

\[ \mathcal{L}(g)(t - \sigma) \]

expansion at \( \infty \)

\[ G(z)g(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \]

Borel Transform

\[ \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k \]

- \( G(z) = \exp(\sigma z) \)
- \( F \) is D-finite
- reduced cancellation for evaluating \( F, G \) on positive real line
- recurrence for \( \alpha_k \):

\[
G(z)g(z) = F(z) \quad \rightarrow \quad \mathcal{L}(g)(t - \sigma) \quad \rightarrow \quad \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k
\]
Sketch of the proof - Borel-Laplace + GMR Method

\[ G(z)g(z) = F(z) \]

\[ \mathcal{L}(g)(t - \sigma) \]

\[ G(z)g(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{(k+1)!} z^{k+1} \]

\[ \sum_{k=0}^{\infty} \alpha_k \left( \frac{1}{t} \right)^k \]

- \( G(z) = \exp(\sigma z) \)
- \( F \) is D-finite
- reduced cancellation for evaluating \( F, G \) on positive real line
- recurrence for \( \alpha_k \):

\[ -(32k \sigma_x^4 \sigma_y^{10} + 128 \sigma_x^4 \sigma_y^{10}) \alpha_k + 4 = \sigma_x^4 \sigma_y^2 - 2 \sigma_x^2 \sigma_y^2 \sigma_y^{10} + \sigma_y^{10} \]

\[ + ((-4 \sigma_x^4 \sigma_y^2 + 8 \sigma_x^2 \sigma_y^6 - 4 \sigma_y^8)k - 10 \sigma_x^4 \sigma_y^2 - 6 \sigma_x^4 \sigma_y^2 \sigma_y^{10} + 2 \sigma_x^2 \sigma_y^6 + 8 \sigma_x^2 \sigma_y^4 \sigma_y^{10} - 10 \sigma_y^8 - 2 \sigma_y^6 \sigma_y^{10}) \alpha_k + 1 \]

\[ + ((24 \sigma_x^4 \sigma_y^6 - 32 \sigma_x^2 \sigma_y^8 + 8 \sigma_y^{10})k + 72 \sigma_x^4 \sigma_y^6 + 12 \sigma_x^4 \sigma_y^6 \sigma_y^{10} - 92 \sigma_x^2 \sigma_y^8 - 8 \sigma_x^2 \sigma_y^6 \sigma_y^{10} + 20 \sigma_y^{10} + 4 \sigma_y^8 \sigma_y^{10}) \alpha_k + 2 \]

\[ + ((-48 \sigma_x^4 \sigma_y^8 + 32 \sigma_x^2 \sigma_y^{10})k - 168 \sigma_x^4 \sigma_y^8 - 8 \sigma_x^4 \sigma_y^6 \sigma_y^{10} + 104 \sigma_x^2 \sigma_y^{10} - 8 \sigma_y^{10} \sigma_y^{10}) \alpha_k + 3, \]
Suppose $\sigma_x > \sigma_y$.

\[
\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ satisfies } \hat{L}'(x) = F(x)\hat{L}(x), \quad \hat{L}(0) = \exp\left(-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2}\right) / 2\sigma_x \sigma_y,
\]

where

\[
F(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2\sigma_y^2 + x} + \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.
\]
Suppose $\sigma_x > \sigma_y$.

\[
\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ satisfies } \hat{L}'(x) = F(x)\hat{L}(x), \quad \hat{L}(0) = \exp \left( \frac{-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x \sigma_y^2}}{2\sigma_x \sigma y} \right), \quad \text{where}
\]

\[
F(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2 \sigma_y^2 + x} + \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.
\]

\[
f_k = \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} \right)^k \left( k + 1 \right) \left( \frac{x_m \sigma_y^2}{\sigma_x^4} \right) + 1 - \frac{\sigma_y^2}{\sigma_x^2} \right) \right) \
1 + \frac{2}{(2\sigma_y^2)^{k+1}} \quad \begin{cases} 
0, & k > 0 \\
\frac{y_m^2}{4\sigma_y^4}, & k = 0,
\end{cases}
\]
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$.

$$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$ satisfies $$\hat{L}'(x) = F(x)\hat{L}(x), \quad \hat{L}(0) = \exp \left( \frac{-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2}}{2\sigma_x \sigma_y} \right),$$ where

$$F(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2\sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2\sigma_y^2 + x}$$

$$+ \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4\sigma_x^2 \sigma_y^2}.$$

$$0 \leq f_k = \frac{2}{(2\sigma_y^2)^{k+1}} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} \right)^k (k + 1) \left( \frac{x_m \sigma_y^2}{\sigma_x^4} \right) + 1 - \frac{\sigma_y^2}{\sigma_x^2} \right) + \begin{cases} 0, & k > 0 \\ \frac{y_m}{4\sigma_y^4}, & k = 0, \end{cases}$$
Prop. Coefficients $\alpha_k$ are positive

Suppose $\sigma_x > \sigma_y$.

$$\hat{L}(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$ satisfies $\hat{L}'(x) = F(x)\hat{L}(x)$, $\hat{L}(0) = \exp\left(-\frac{\sigma_x^2 y_m^2 + \sigma_y^2 x_m^2}{2\sigma_x^2 \sigma_y^2}\right)$, where

$$F(x) = \frac{y_m^2}{4\sigma_y^4} + \frac{\sigma_y^4 x_m^2}{(x (\sigma_x^2 - \sigma_y^2) - 2 \sigma_x^2 \sigma_y^2)^2} - \frac{1}{-2 \sigma_y^2 + x}$$

$$+ \frac{-\sigma_x^2 + \sigma_y^2}{2x (\sigma_x^2 - \sigma_y^2) - 4 \sigma_x^2 \sigma_y^2}.$$

$$0 \leq f_k = 1 + \frac{2}{(2\sigma_y^2)^{k+1}} \left(1 - \frac{\sigma_y^2}{\sigma_x^2}\right)^k \left((k+1) \left(\frac{x_m \sigma_y^2}{\sigma_x^4}\right) + 1 - \frac{\sigma_y^2}{\sigma_x^2}\right)$$

$$\left\{ \begin{array}{ll}
0, & k > 0 \\
\frac{ym^2}{4\sigma_y^4}, & k = 0,
\end{array} \right.$$

$$0 \leq (n + 1)\alpha_{n+1} = \sum_{i=0}^{n} f_i \cdot \alpha_{n-i}, \rightsquigarrow \text{by induction.}$$
Bounds using majorant series

\[ 1 + \frac{1}{2} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} + \frac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{y_m^2}{\sigma_y^2} \right) \]

Let \( \gamma := \frac{1}{2} \left( 1 - \frac{\sigma_y^2}{\sigma_x^2} + \frac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{y_m^2}{\sigma_y^2} \right) \), \( \overline{\alpha}_k := \alpha_0 \gamma^k \) and \( \underline{\alpha}_k := \alpha_0 \left( \frac{1}{2\sigma_y^2} \right)^k \).

Then \( \underline{\alpha}_k \leq \alpha_k \leq \overline{\alpha}_k \), \( \forall k \in \mathbb{N} \).
Bounds using majorant series

Let \( \gamma := \frac{1 + \frac{1}{2} \left( 1 - \frac{\sigma_x^2}{\sigma_y^2} + \frac{x_m^2 \sigma_y^2}{\sigma_x^4} + \frac{y_m^2}{\sigma_y^2} \right)}{2 \sigma_y^2} \), \( \bar{\alpha}_k := \alpha_0 \gamma^k \) and \( \alpha_k := \alpha_0 \left( \frac{1}{2 \sigma_y^2} \right)^k \).

Then \( \bar{\alpha}_k \leq \alpha_k \leq \alpha_k, \forall k \in \mathbb{N} \).

Let \( \tilde{P}_N(z) := \sum_{k=0}^{N-1} \frac{\alpha_k z^k}{(k+1)!} \). Then we have the following error bounds:

\[
\varepsilon_N(z) \leq g(z) - \tilde{P}_N(z) \leq \bar{\varepsilon}_N(z), \quad \text{where}
\]

\[
\varepsilon_N(z) := 2 \alpha_0 \sigma_y^2 e^{-\frac{z}{2 \sigma_y^2}} \left( \frac{z}{2 \sigma_y^2} \right)^{N+1},
\]

\[
\varepsilon_N(z) := \frac{\alpha_0}{\gamma} e^{-\frac{z}{2 \sigma_y^2}} \left( z \gamma \right)^{N+1},
\]

\[
\bar{\varepsilon}_N(z) := \frac{\alpha_0}{\gamma} e^{-\frac{z}{2 \sigma_y^2}} \left( z \gamma \right)^{N+1}.
\]
### Examples

#### Sample 1

| Case # | Input parameters (km) |  |
| --- | --- | --- | --- | --- | --- |
| | $\sigma_x$ | $\sigma_y$ | $R$ | $x_m$ | $y_m$ |
| 1 | 0.05 | 0.025 | 0.005 | 0.01 | 0 |
| 2 | 0.05 | 0.025 | 0.005 | 0 | 0.01 |
| 3 | 0.075 | 0.025 | 0.005 | 0.01 | 0 |
| 4 | 0.075 | 0.025 | 0.005 | 0 | 0.01 |
| 5 | 3 | 1 | 0.01 | 1 | 0 |
| 6 | 3 | 1 | 0.01 | 0 | 1 |
| 7 | 3 | 1 | 0.01 | 10 | 0 |
| 8 | 3 | 1 | 0.01 | 0 | 10 |
| 9 | 10 | 1 | 0.01 | 10 | 0 |
| 10 | 10 | 1 | 0.01 | 0 | 10 |
| 11 | 3 | 1 | 0.05 | 5 | 0 |
| 12 | 3 | 1 | 0.05 | 0 | 5 |
Examples: quality
\[ \eta = \frac{\bar{e}_{10}(z) - \bar{e}_{10}(\tilde{z})}{\bar{e}_{10}(z) + \tilde{P}_{10}(z)} \]
and plot \( \log(\tilde{p}_i z^i) \)

Case 1: \( \eta = 23 \)

Case 2: \( \eta = 22 \)

Case 4: \( \eta = 22 \)

Case 6: \( \eta = 47 \)

Case 8: \( \eta = 33 \)

Case 11: \( \eta = 34 \)
## Numerical study

**Sample 1**

<table>
<thead>
<tr>
<th>Case #</th>
<th>Probability of Collision (-)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alfano</td>
<td>Patera</td>
<td>Chan</td>
<td>New method</td>
</tr>
<tr>
<td>1</td>
<td>$9.742 \times 10^{-3}$</td>
<td>$9.741 \times 10^{-3}$</td>
<td>$9.754 \times 10^{-3}$</td>
<td>$9.742 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$9.181 \times 10^{-3}$</td>
<td>$9.181 \times 10^{-3}$</td>
<td>$9.189 \times 10^{-3}$</td>
<td>$9.181 \times 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>$6.571 \times 10^{-3}$</td>
<td>$6.571 \times 10^{-3}$</td>
<td>$6.586 \times 10^{-3}$</td>
<td>$6.571 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>$6.125 \times 10^{-3}$</td>
<td>$6.125 \times 10^{-3}$</td>
<td>$6.135 \times 10^{-3}$</td>
<td>$6.125 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
<td>$1.577 \times 10^{-5}$</td>
</tr>
<tr>
<td>6</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
<td>$1.011 \times 10^{-5}$</td>
</tr>
<tr>
<td>7</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
<td>$6.443 \times 10^{-8}$</td>
</tr>
<tr>
<td>8</td>
<td>$0$</td>
<td>$3.219 \times 10^{-27}$</td>
<td>$3.216 \times 10^{-27}$</td>
<td>$3.219 \times 10^{-27}$</td>
</tr>
<tr>
<td>9</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
<td>$3.033 \times 10^{-6}$</td>
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<tr>
<td>10</td>
<td>$0$</td>
<td>$9.656 \times 10^{-28}$</td>
<td>$9.645 \times 10^{-28}$</td>
<td>$9.656 \times 10^{-28}$</td>
</tr>
<tr>
<td>11</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
<td>$1.039 \times 10^{-4}$</td>
</tr>
<tr>
<td>12</td>
<td>$1.564 \times 10^{-9}$</td>
<td>$1.564 \times 10^{-9}$</td>
<td>$1.556 \times 10^{-9}$</td>
<td>$1.564 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

— equal to reference value (from [Chan 2008])
Examples: quality $\eta_N = \frac{\overline{\epsilon}_N(z) - \underline{\epsilon}_N(z)}{\overline{\epsilon}_N(z) + \tilde{P}_N(z)}$ and plot $\log(\tilde{p}_i z^i)$

Case 3 Alfano: $\eta_{800} = 30$

Case 5 Alfano: $\eta_{121000} = 20$

Sample 2 (from [Alfano 2009])

<table>
<thead>
<tr>
<th>Case #</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>$R$</th>
<th>$x_m$</th>
<th>$y_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>114.25</td>
<td>1.41</td>
<td>15</td>
<td>0.15</td>
<td>3.88</td>
</tr>
<tr>
<td>5</td>
<td>177.8</td>
<td>0.038</td>
<td>10</td>
<td>2.12</td>
<td>-1.22</td>
</tr>
</tbody>
</table>
## Examples

Sample 2 (from [Alfano 2009])

<table>
<thead>
<tr>
<th>Case #</th>
<th>Input parameters (m)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>$\sigma_y$</td>
<td>$R$</td>
<td>$x_m$</td>
</tr>
<tr>
<td>3</td>
<td>114.25</td>
<td>1.41</td>
<td>15</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>177.8</td>
<td>0.038</td>
<td>10</td>
<td>2.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Alfano's test case number</th>
<th>Probability of collision (-)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alfano</td>
<td>New method</td>
</tr>
<tr>
<td>3</td>
<td>0.10038</td>
<td>0.10038</td>
</tr>
<tr>
<td>5</td>
<td>0.044712</td>
<td>0.045509</td>
</tr>
</tbody>
</table>
Conclusion

New method

- Analytical formula
- Reduced cancellation evaluation
- Error bounds
- No simplifying assumption

Current and future work

- Saddle-point method for "degenerate" cases
- Long-term 3D encounter model
- Extension to polygonal cross-sections
- Extensive testings and comparisons with existing methods