DC motors: dynamic model and control techniques

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1 Magnetic considerations on rotating coils

1.1 Magnetic field and conductors

A **magnetic field** is generated by a flowing current: it is in particular experienced in the neighborhood of a moving charge. The effect of a magnetic field may be experimented, for instance, by positioning a magnet close to a wire where a current is flowing; what can be observed is that the magnet experiences a force, which is due to the magnetic field generated by the current flowing in the wire. If the magnet is moved around the wire, the force changes depending on the positions assumed by the magnet. In particular, it can be observed that the magnetic field decreases as the distance from the wire increases and increases as the current increases.

The above experiment aims to verify that a flowing current indeed generates a vector field $\vec{B}$. This field may be thought of as the sum of infinite contributions $d\vec{B}$ due to all the infinitesimal segments of wire $d\vec{l}$, where the current i flows. Each segment induces a magnetic field at any point in the surrounding space. In particular, the **Biot and Savart law** states that, if $\vec{r}$ is the vector connecting the wire segment $d\vec{l}$ to a generic point p in the space, the contribution $d\vec{B}$ of the magnetic field in the point p due to the segment $d\vec{l}$ is given by

$$d\vec{B} = i\,\frac{d\vec{l} \times \vec{r}}{|r|^3}, \tag{1}$$

where the symbol $\vec{\ }$ denotes that the considered quantity is a vector of $\mathbb{R}^3$ and the symbol $\times$ denotes the vector product operation.

Equation (1) provides a tool for computing the magnetic field associated to any conductor where a current is flowing. In particular, by suitably integrating it for the case of a solenoid, it turns out that, if the corner effects are neglected (namely, the solenoid is long enough), the field inside the windings is constant and parallel to the axis of the solenoid, and its magnitude is given by [6, p. 136]

$$|\vec{B}| = \frac{\mu}{l}\, N\, i, \tag{2}$$

where l is the length of the solenoid, μ is the magnetic permeability of the dielectric inside the solenoid, N is the number of turns of the wire (see Figure 1).

Consider now a surface S located in a magnetic field $\vec{B}$; the **magnetic flux**, or **flux** Φ flowing through the surface is defined as the integral along the surface of the normal component of the magnetic field:

$$\Phi := \int_S \vec{B} \cdot \vec{n}\, dS,$$

where $\vec{n}$ is the perpendicular to the surface.

In particular, if a uniform magnetic field $\vec{B}$ approaches a flat surface with an angle of incidence β and the area of the surface is A, then

$$\Phi \;=\; |\vec{B}|\, A \cos\beta. \tag{3}$$

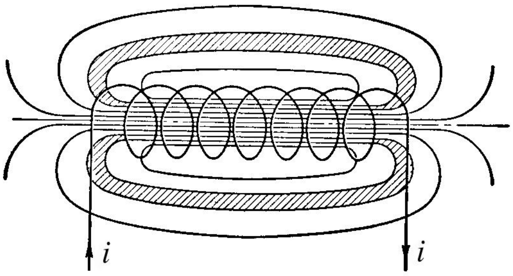

Figure 1: The magnetic field in the neighborhood of a solenoid where a current i flows.

Consider now equation (2). Since the cross section of the solenoid has constant area, the flux flowing through a generic surface normal to the solenoid axis is constant along the whole solenoid. Denoting by A the constant area of the cross section of the solenoid, since the magnetic field is parallel to the solenoid axis (*i.e.*, $\beta = 0$), the flux may be computed substituting equation (2) in equation (3):

$$\Phi = K_0 \, N \, i, \tag{4}$$

where $K_0 := \frac{\mu}{l} \, A$.

1.2 The magneto-motive force in a rotating coil

Once understood that a moving charge generates a magnetic field, what about if a moving charge passes through an exogenous magnetic field? The **Lorentz force equation** is a good starting point to understand the mechanical aspects of this electromagnetic phenomenon.

The Lorentz force is the force experienced by a moving charge in an electromagnetic field. If $\vec{E}$ denotes the electric field, $\vec{B}$ denotes the magnetic field, q is a charge moving with a speed $\vec{v}$ in the space, then the charge experiences a force given by

$$\vec{F} = q \, (\vec{E} + \vec{v} \times \vec{B}). \tag{5}$$

Now, consider a wire where a uniform current i is flowing. Assuming the electric field to be zero (this is the case of a DC motor), with reference to the quantity of charge dq found in an infinitesimal section $d\vec{\ell}$ of the wire, the force in equation (5) may be computed as a function of i:

$$\begin{aligned} d\vec{F} &= dq \, \vec{v} \times \vec{B} \\ &= dq \, \frac{d\vec{\ell}}{dt} \times \vec{B} \\ &= \frac{dq}{dt} \, d\vec{\ell} \times \vec{B} \\ &= i \, d\vec{\ell} \times \vec{B}. \end{aligned}$$

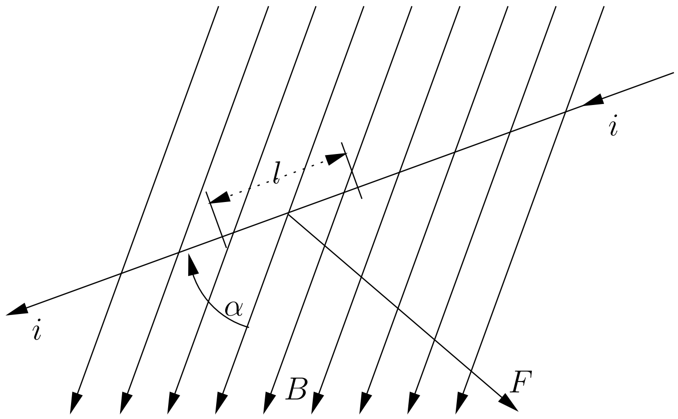

Figure 2: Force experienced by a current-carrying conductor located in a uniform magnetic field.

It turns out that, if α denotes the angle of incidence between the magnetic field and a straight wire of length l, then the magnitude of the force $\vec{F}$ is given by

$$|\vec{F}| = i\,l\,|\vec{B}|\,\sin\alpha, \tag{6}$$

and $\vec{F}$ is oriented on the perpendicular to the plane spanned by the magnetic field and the wire following the right-hand-screw rule (see Figure 2).

Now, on the basis of equation (6), consider a rigid rectangular coil constituted by a single wire where a current i flows, suitably located in an uniform exogenous magnetic field. As it can be seen from Figure 3; [1] if l is the length of the wire perpendicular to the magnetic field, then two forces are applied to the coil. Since the angle α of equation (6) is $\pm\frac{\pi}{2}$ (depending on which side of the coil is considered), then it turns out that the magnitudes of the two forces are the same:

$$|\vec{F}| = |\vec{B}|\,i\,l. \tag{7}$$

Since the coil is square, the current i flows in opposite directions on the two sides of the coil; thus, the two forces F generate a torque T exerted at the center of the coil that is dependent on the angular position θ of the coil with respect to the magnetic field. In particular, if the length of an edge of the square is d, then

$$T = 2\,|\vec{F}|\,\frac{d}{2}\,\sin\theta = |\vec{F}|\,d\,\sin\theta; \tag{8}$$

whence, taking into account equation (7), equation (8) yields

$$T = |\vec{F}|\,d\,\sin\theta = |\vec{B}|\,i\,l\,d\,\sin\theta. \tag{9}$$

[1] As usual, a dot $\cdot$ means that the flowing current is exiting the page, while a cross $\times$ means that the current is entering the page.

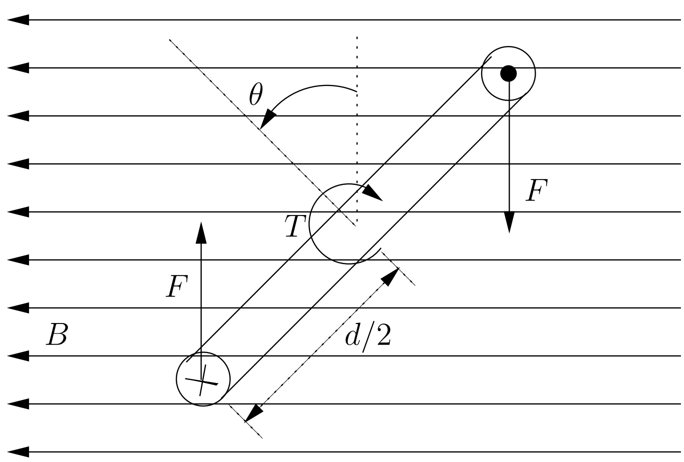

Figure 3: Torque experienced by a coil in a uniform magnetic field.

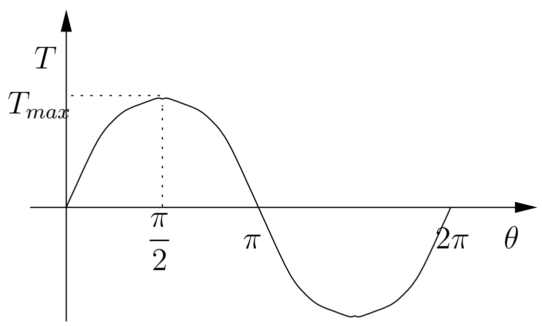

Figure 4: Profile of the torque T experienced by the coil in a magnetic field, as the rotation angle θ varies.

Consider now equation (9) and notice that the torque highly depends on the rotor coil angular position θ. Imagine that the coil is in the rotor of a motor; then the resulting torque is highly dependent on the motor position; moreover, if no load torque is present, the motor keeps turning clockwise and counter-clockwise; as a matter of fact, if the coil turns of an angle π, the torque exerted has the same amplitude but opposite sign (see Figure 4).

Since the goal is to have the motor to exert a constant torque for any position θ, the solution adopted is to insert on the rotor shaft a commutator constituted by two segments connected to the rotor windings and brushes that slide between the segments as the rotor turns(see Figure 5). In such a configuration, the sign of the current flowing in the coil changes at each half revolution of the motor; thus, if the segments are properly positioned with respect to the coil position, the torque profile of Figure 4 will be suitably inverted during half revolution (see Figure 5).

Now, once this solution is adopted, although the torque has always the same sign, still it is highly dependent on the rotor position; the obvious solution to this problem is to increase the number of coils in the rotor and the segments of the commutator, connecting each pair of opposite segments to a coil in such a way that when the brushes activate that coil, the rotor angle is $\theta = \pm\frac{\pi}{2}$ (*i.e.*, the maximum torque position). It turns out that, if N independent

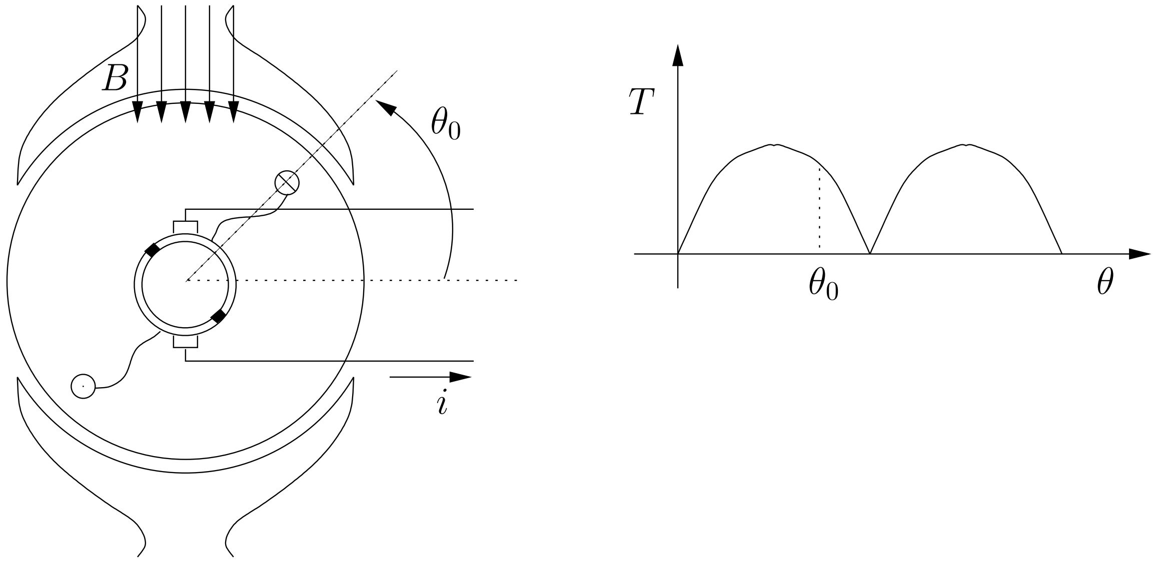

Figure 5: Torque exerted by the motor, when a two segments commutator is used.

windings are located on the rotor, the commutator will have $2\,N$ segments and the torque profile will be constituted by $2\,N$ half sinusoids (related to the torques T_i experienced by each coil) overlapped in one single period (see Figure 6, where θ_c denotes a generic commutation angle). In such a way the torque ripple may be decreased as much as needed by increasing N and the residual ripple may be assumed to be filtered by the mechanical system, so that the resulting torque may be approximated as

$$T = i\,l\,d\,|\vec{B}|. \tag{10}$$

Finally, considering that, by equation (3), the flux Φ flowing through the rotor is proportional to $|\vec{B}|$ (note that the angle β in equation (3) is equal to $\pi/2$; as a matter of fact the flux is assumed to be flowing straight inside the motor), it turns out

$$T = K_\Phi\,\Phi\,i, \tag{11}$$

where $K_\Phi := l\,d/A$.

1.3 The back EMF effect

In Figure 4, the rotor conductors carry current supplied from an external source and are located in an exogenous magnetic field; whence, forces are experienced by the coil and a torque is exerted on the rotor shaft. However, due to the rotation of the coil, the conductors themselves cut the magnetic flux, thus generating an electro-motive force (*i.e.*, an induced voltage in the rotor windings) whose magnitude may be computed by means of the **Faraday's law of induction**. This law states that if there is a variation of the flux Φ_c flowing in the internal surface of a closed wire, then an electro-motive force e is induced in the wire, according to:

$$e = -\frac{d\Phi_c}{dt}. \tag{12}$$

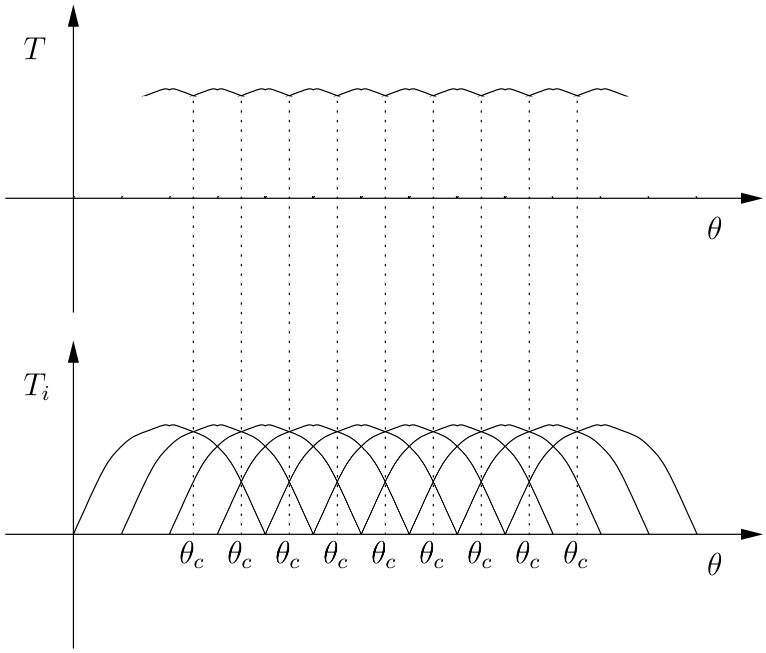

Figure 6: Torque exerted by the motor, when a multiple segments commutator is used.

In this particular case, notice that the flux flowing in the coil is given by equation (3) substituting β with the angle $\theta(t)$ of the motor and A with the area of the internal surface of the coil. Whence, computing the derivative in equation (12), the **back electro-motive force** or, more easily, the back EMF may be computed as

$$\begin{aligned} e &= -\frac{d}{dt}(|\vec{B}|\, A \cos\theta(t)) \\ &= |\vec{B}|\, A\, \dot{\theta}(t) \sin\theta(t) \\ &= |\vec{B}|\, A\, \omega(t) \sin\theta(t), \end{aligned}$$

where $\omega := \dfrac{d\theta(t)}{dt}$ denotes the angular speed of the motor.

Noticing that, due to the presence of the commutator, the coil always operates in a neighborhood the position $\theta = \pi/2$, then $\sin\theta(t) \approx 1$ and the back EMF may be written as $e = |\vec{B}|\, A\, \omega(t)$.

Finally, similarly to the case studied in equations (10) and (11), the back EMF may be expressed as a function of the flux Φ, and it can be verified easily that

$$e = K_\Phi\, \Phi\, \omega, \tag{13}$$

where K_Φ is the same constant as the one in equation (11).

2 The basic equations of the DC motor

The set of equations here reported, constitutes a model of the DC motor, which may be represented as a nonlinear dynamic system. The main restrictions of this model, with respect to a real motor are

1. the assumption that the *magnetic circuit is linear* (such an assumption is approximate, since the metal parts are not perfectly smooth and there is some flux dispersion inside the motor, moreover, due to saturation of the metal, equation (4) does not hold for high values of i);

2. the assumption that the mechanical friction is only linear in the motor speed; namely, *only viscous friction* is assumed to be present in the motor (such an assumption is approximate since Coulomb friction is usually experienced in motors).

2.1 The electric equations

Following the process described in Section 1.1, in a DC motor, the magnetic flux is generated by windings located on the stator.

Although the physical reason why electrical power is transformed in mechanical power is the one explained in Section 1.2, the actual implementations of this result are various, as a matter of fact, since the magnetic field B arises from the stator coils, not only the rotor coils may rotate with respect to the stator, but also the stator supply may rotate (in an electrical sense) by increasing the number of coils and by a more sophisticated supply.

In this handout, a simple model, which applies to the above cases (provided proper transformations are performed on the system variables) will be introduced.

The stator of the motor will be assumed to have a single coil characterized by an inductance L_e due to the windings and a resistance R_e due to dispersions in the conductor (see Figure 7). The equation associated with such an electric circuit is given by

$$v_e(t) = L_e \frac{\mathrm{d}\, i_e}{\mathrm{d}t} + R_e\, i_e. \tag{14}$$

Since relation (14) is linear, by transforming in the Laplace domain the signals, it can be written

$$\frac{i_e(s)}{v_e(s)} = \frac{K_e}{1 + \tau_e\, s}, \tag{15}$$

where $K_e := \dfrac{1}{R_e}$ is the **stator gain** and $\tau_e := \dfrac{L_e}{R_e}$ is the **stator time constant**.

The rotor is assumed to be a single coil characterized by inductance L_a and resistance R_a (see Figure 7), but it has to be taken into account the back EMF of the motor in equation (13). The equation associated with such an electric circuit is given by

$$v_a(t) = L_a \frac{\mathrm{d}\, i_a}{\mathrm{d}t} + R_a\, i_a + e. \tag{16}$$

Again, since relation (16) is linear, by transforming in the Laplace domain the signals, it can be written

$$\frac{i_a(s)}{v_a(s) - e(s)} = \frac{K_a}{1 + \tau_a\, s}, \tag{17}$$

where $K_a := \dfrac{1}{R_a}$ is the **rotor gain** and $\tau_a := \dfrac{L_a}{R_a}$ is the **rotor time constant**.

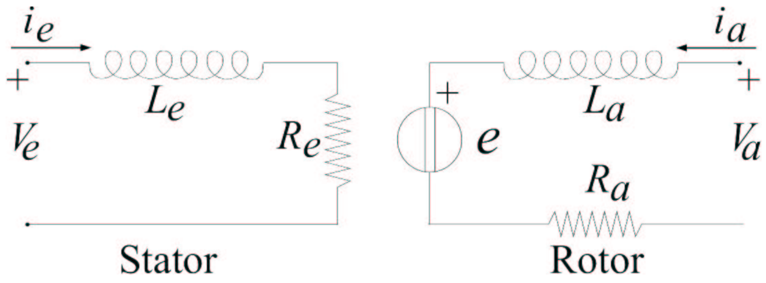

Figure 7: Electrical equivalent scheme of a DC motor.

Taking into account the results of Section 1.2 (in particular, equations (11) and (13)), the following two equations hold for the back EMF e and the torque exerted by the motor T_M:

$$T_M = K_\Phi \, \Phi \, i_a, \tag{18}$$
$$e = K_\Phi \, \Phi \, \omega. \tag{19}$$

As regards to the flux flowing in the motor, the flux Φ is generated by the coils located on the stator. Since, as remarked in Section 1.1, the flux is proportional to the current i_e, by equation (4), it can be written

$$T_M = K \, i_e \, i_a, \tag{20}$$
$$e = K \, i_e \, \omega, \tag{21}$$

where $K := K_\Phi \, K_0 \, N$.

2.2 The mechanical equations

Let us now deal with the mechanical representation of the motor. It has been shown in Section 1.2 that the motor exerts a torque, while supplied by voltages on the stator and on the rotor. This torque acts on the mechanical structure, which is characterized by the rotor inertia J and the viscous friction coefficient F. It has also to be taken into account that in any operating environment a load torque is exerted on the motor; then, if T_L is the load torque, the following equation may be written:

$$T_M - T_L = J \, \frac{\mathrm{d}\,\omega}{\mathrm{d}t} + F \, \omega. \tag{22}$$

As for the electrical case, also for the mechanical equations, a linear transfer function may be associated to equation (22):

$$\frac{\omega(s)}{T_M(s) - T_L(s)} = \frac{K_m}{1 + \tau_m \, s}, \tag{23}$$

where $K_m := \dfrac{1}{F}$ is the **mechanical gain** and $\tau_m := \dfrac{J}{F}$ is the **mechanical time constant**.

2.3 Geared motors and direct drive motors

Often (*e.g.*, in robotic applications), the speed required by the load is too low as compared to the nominal speed of the motor. [2] In this cases, gears are introduced between the motor and the load, thus reducing by a factor n the angular velocity of the load itself.

Besides the increase of damping and inertia due to the presence of the additional rotating cogwheels of the gear, the mechanical coupling between the load and the motor is altered by the gear itself. To correctly understand the effects of the gear, the fist thing to remark is that damping and inertia are not the same if measured at the input or at the output of the gear. Since we are interested in the complete characterization of the motor block, let's refer to the output quantities, and denote by F_G the internal damping of the gear and by J_G the internal inertia of the gear.

Then, notice that, since the power exerted by the motor is the same at the input and at the output of the gear, denoting by T'_M and ω' the torque and the speed at the output of the gear, it can be written

$$T_M\,\omega = T'_M\,\omega',$$

and, since $\omega' = \omega/n$, then $T'_M = n\,T_M$.

Substituting the above equations in equation (22), and taking into account the increase of damping and inertia due to the cogwheels of the gear, it turns out [3]

$$T'_M - T_L = (J_G + n^2\,J)\,\frac{\mathrm{d}\,\omega'}{\mathrm{d}t} + (F_G + n^2\,F)\,\omega'. \tag{24}$$

By a comparison between equations (22) and (24), it is stressed that the presence of the gear highly increases the inertia and the damping of the motor from the point of view of the load.

In addition, when a gear is inserted in an actuator system, *backlash* is experienced on its output due to the coupling between the cogwheels of the gear. This gives rise to nonlinearities that may lead to instability effects. For this reason, especially in high precision systems, direct-drive motors are used. Such motors may exert reasonable torques at low speeds, whence they do not need gears to drive the load. However, these motors may not be adopted for high power tasks, since the maximum torque exerted has physical limitations.

2.4 Block diagram of the DC motor

By implementing equations (14), (16), (20), (21), and (22) in a nonlinear block diagram, the result shown in Figure 8 is obtained. In the block diagram, the variable θ represents the rotor angular position (whence, $\omega = \dot{\theta}$).

The nonlinear model results in a two-input, one-output map, having a disturbance input T_L and with four state variables, related to

- the energy stored in the inductance L_e;
- the energy stored in the inductance L_a;

[2] The nominal speed of the motor is the speed corresponding to the efficiency maximum.

[3] Note that T_L is exerted by the load, whence it should not be scaled.

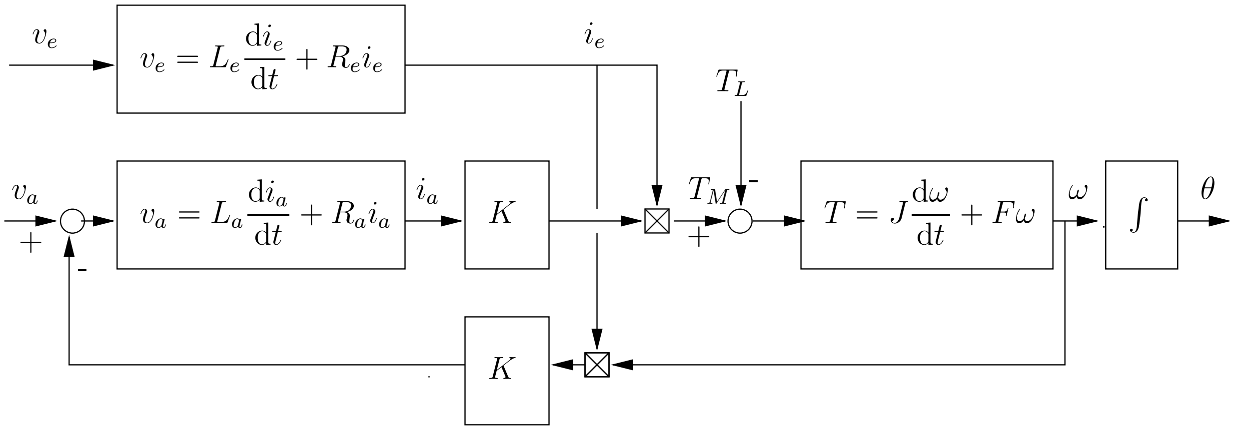

Figure 8: Nonlinear block diagram of a DC motor.

- the kinetic energy of the rotor (related to J);
- the position θ of the rotor.

Remark 2.1 Note that the nonlinear model of the motor is indeed constituted by three linear relationships between physical quantities, constituted by the transfer functions (15), (17) and (23), and two multipliers, which represent the system nonlinearities. Various control techniques performed on the motor, aim to linearize such a block diagram, by suitably controlling the system by means of the two inputs v_a and v_e. ◦

Remark 2.2 When a geared motor is considered, a constant gain block equal to $1/n$ should be added in the block diagram of Figure 8 right before the integrator of ω (so that ω' will be integrated instead) and divides by a factor n the disturbance input T_L. Note that the feedback branch is related to ω and not to ω', since the gear does not change the electrical properties of the motor. ◦

To simplify the nonlinear block diagram in Figure 8, three main control techniques are introduced in the following sections, showing the performance of the system in each case and giving the simplified block diagram related to each technique.

3 Stator voltage control with constant armature current

Assume that a constant current supply is available, regardless of the voltage absorbed by the load; then, supplying the armature circuit with such a device, a stator voltage control configuration is obtained. It should be noticed that, since the armature current is constant, the nonlinear block diagram in Figure 8 becomes linear; as a matter of fact, the whole feedback branch is erased because the rotor current is imposed by the current supply.

The main problem associated with this control is that a current generator is quite expensive, as far as it works for high power applications. The drawbacks of this control technique, on the other hand, are several.

- Since the rotor current is constant, a main danger of DC motors is overcame: if the back EMF in the rotor drops down to zero, [4] the rotor current raises to very high values, thus damaging the armature coils. This control technique directly controls the current, possibly decreasing the armature voltage supply if the back EMF is very low.
- The power input is constant and the control input drives a low power device (*i.e.*, the stator voltage v_e). This is useful because a low power reference signal is required to control the system, instead of a high precision, high gain amplifier.
- In some applications, if a high power current generator is available, many motors could be connected in series, each one with its own low power control signal, thus reducing the number of high power devices involved in the control scheme.

3.1 Steady state behavior

Consider equations (14), (20) and (22), and suppose the motor reaches its steady state condition (if it exists). Then, the following relations may be written:

$$T_M = \frac{K}{R_e} v_e \tag{25a}$$

$$T_M - T_L = F\omega. \tag{25b}$$

From equation (25a), it can be seen that the dependence between the torque exerted by the motor T_M and the stator voltage v_e is linear. This relationship corresponds in Figure 9 to the horizontal lines: each line corresponds to a value of v_e. Equation (25b), instead, corresponds to the dotted diagonal lines, whose slope is dependent on F and whose vertical offset is dependent on T_L. Since the load torque T_L is usually an increasing function of the speed, the solid curves may be used instead of the dotted lines, namely we may assume T_L to be an increasing function of the speed ω. The actual operating point of the system corresponds to the intersection between the two curves determined by the values of v_e, T_L and F (see Figure 9).

Note that, the more the motor friction F is low, the more the lines related to equation (25b) become horizontal. In the ideal situation when $F = 0$, such lines are horizontal and the steady-state speed approaches infinity (this corresponds to the case when the block related to the mechanical system is an integrator, thus when a constant torque is applied at the input, an increasing speed is experienced at the output).

3.2 Transfer function

The linear block diagram of the system can be easily computed by specializing the block diagram in Figure 8 to the case when the armature current is constant (note that, since the whole system has become linear, the linear relations (15) and (23) have been used). It can be easily seen that the system shrinks to the only above branch; as a matter of fact, the constant current i_a yields a constant transfer function (see Figure 10).

[4] This happens when the rotor is stuck, either because the speed is at an inversion point, or because the load torque is equal to the motor torque, or otherwise, because the flux drops down to zero.

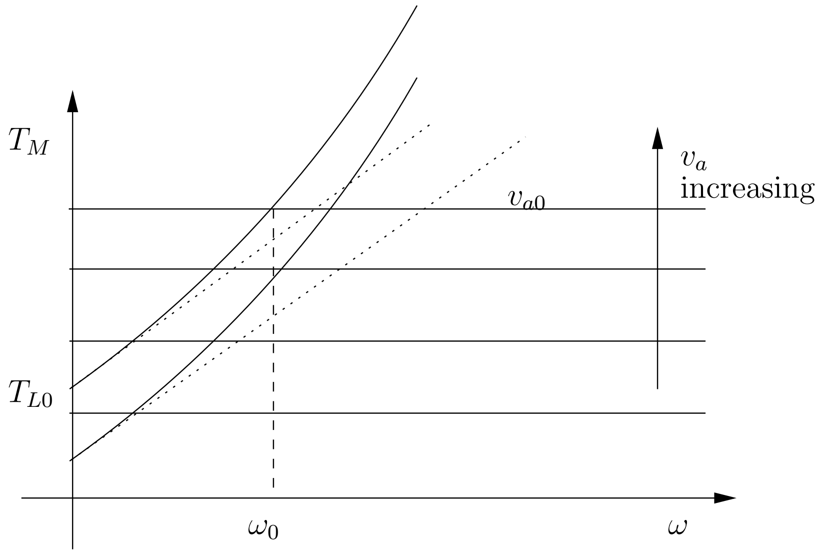

Figure 9: Steady state relationship between torque and speed in the stator voltage control with constant armature current.

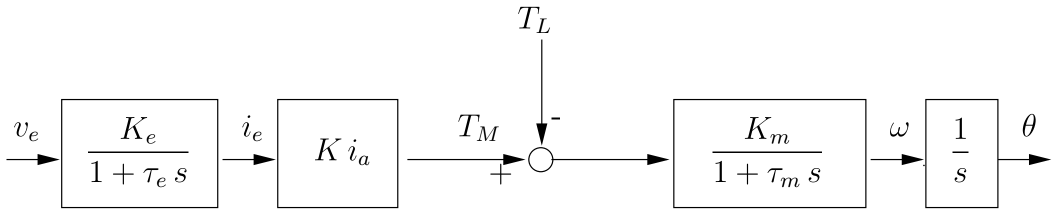

Figure 10: Block diagram of the DC motor with constant armature current.

The transfer matrix of the whole system is easily computed from the block diagram as:

$$\theta(s) = [W_1(s)\; W_2(s)] \begin{bmatrix} v_e(s) \\ T_L(s) \end{bmatrix},$$

where

$$\begin{aligned} W_1(s) &:= \frac{K_{AC}}{s\,(1+\tau_e\, s)\,(1+\tau_m\, s)}, \quad K_{AC} := K_e\, K\, K_m\, i_a. \\ W_2(s) &:= \frac{K_m}{s\,(1+\tau_m\, s)}. \end{aligned}$$

As remarked above, the motor damping may, in many cases, be negligible. So that the mechanical time constant approaches infinity. In such a case, it has been noticed that there is no steady state value for the speed ω. This can be here motivated by the fact that the transfer function related to the speed $\dfrac{\omega(s)}{v_e(s)} = \dfrac{K_{AC}}{s\,(1+\tau_e\, s)}$ has a pole at the origin.

4 Stator voltage control with constant armature voltage

In this second control technique, the control is based on a constant voltage supply for the armature. It can be immediately seen from the non-linear block diagram in Figure 8, that the system thus controlled is still non-linear; however, there are several drawbacks also with this control configuration:

- the constant voltage generator is cheap, also for high power purposes;
- similarly to the control case of Section 3, the control signal is a low power signal, while the power is supplied to the system by means of the constant voltage generator;
- since this control scheme usually operates in the neighborhood of an operating point, the armature voltage is always balanced by a non null back EMF, thus limiting the armature current peaks.

4.1 Linearized equations

With reference to the general non-linear model, a linearization is here computed with respect to all the internal variables (except for v_a which is assumed to be constant) around an operating point.

First, from equations (14), (16), (20), (21) and (22), the relations between the values v_{e0}, i_{e0}, v_{a0}, i_{a0}, ω_0, T_{M0}, T_{L0} of the quantities v_e, i_e, v_a, i_a, ω, T_M and T_L, respectively, at the operating point, are written:

$$\begin{aligned} v_{e0} &= R_e\, i_{e0} \\ v_{a0} &= R_a\, i_{a0} + K\, i_{e0}\, \omega_0 \\ T_{M0} &= K\, i_{e0}\, i_{a0} \\ T_{M0} - T_{L0} &= F\, \omega_0; \end{aligned}$$

whence, the operating point must satisfy the above conditions.

The same equations may be written as the variations of the above quantities with respect to their nominal values, neglecting the second order terms:

$$\begin{aligned} \delta v_e &= R_e\, \delta i_e + L_e \dot{\delta i_e} && \text{(26a)} \\ 0 = \delta v_a &= R_a\, \delta i_a + L_a \dot{\delta i_a} + K\,(i_{e0}\, \delta\omega + \omega_0\, \delta i_e) && \text{(26b)} \\ \delta T_M &= K\,(i_{e0}\, \delta i_a + i_{a0}\, \delta i_e) && \text{(26c)} \\ \delta T_M - \delta T_L &= F\, \delta\omega + J\, \dot{\delta\omega}, && \text{(26d)} \end{aligned}$$

where the subscript 0 denotes the value of a quantity at the operating point and the δ symbol denotes its variation around that value.

4.2 Steady state behavior

Consider equations (26) when the motor is at a steady state. The static relationships between the torque variation δT_M, the stator voltage variation δv_e and the speed variation $\delta\omega$ are easily computed by putting to zero all the derivatives with respect to time:

$$\delta T_M = K\,\frac{R_a\,i_{a0} - K\,i_{e0}\,\omega_0}{R_e\,R_a}\,\delta v_e - \frac{(K\,i_{e0})^2}{R_a}\,\delta\omega. \tag{27}$$

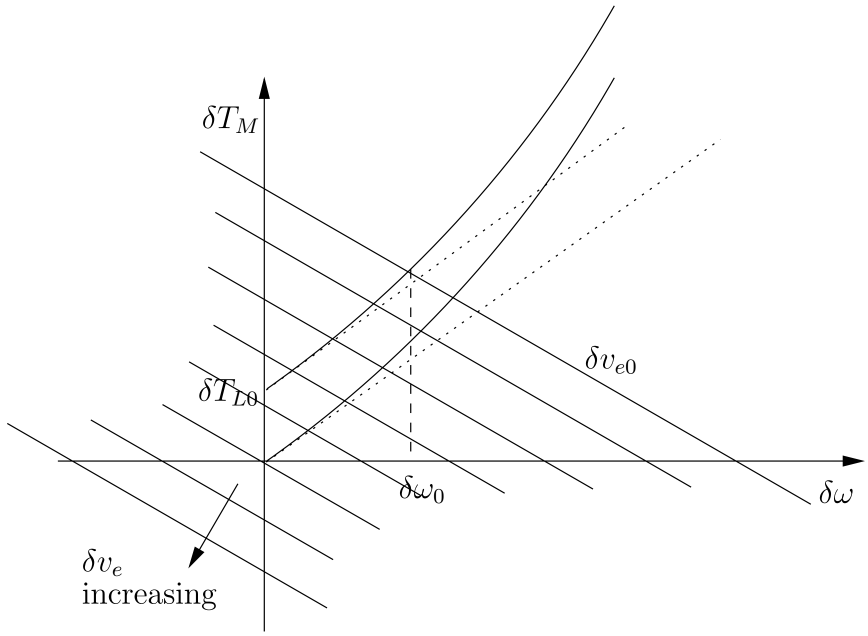

Figure 11: Steady state relationship between torque and speed in stator voltage control with constant armature voltage.

The curves corresponding to equation (27) are depicted in Figure 11, where it is shown how a decrease of the stator voltage variation δv_e leads to a torque increase. It can be also seen that the above relationships lead to a stable behavior of the motor; as a matter of fact, as the speed increases, the torque decreases, thus reducing the power supplied to the system and, consequently, reducing the speed as well.

4.3 Transfer function

Referring to the linearized equations (26), a block diagram of the system may be derived. It is interesting to compare such a block diagram with the nonlinear block diagram in Figure 8. In particular, note that after linearization, the multiplying nonlinear blocks are transformed as shown in Figure 12. Besides this transformation, the two block diagrams are exactly the same.

With reference to Figure 13, the relationship between the torque variation δT_M, the speed variation $\delta\omega$ and the variation of the stator voltage δv_e can be computed. The resulting

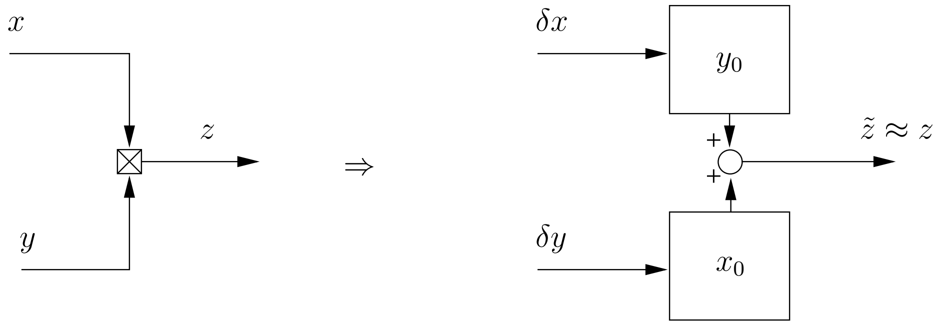

Figure 12: Transformation of a multiplier after linearization.

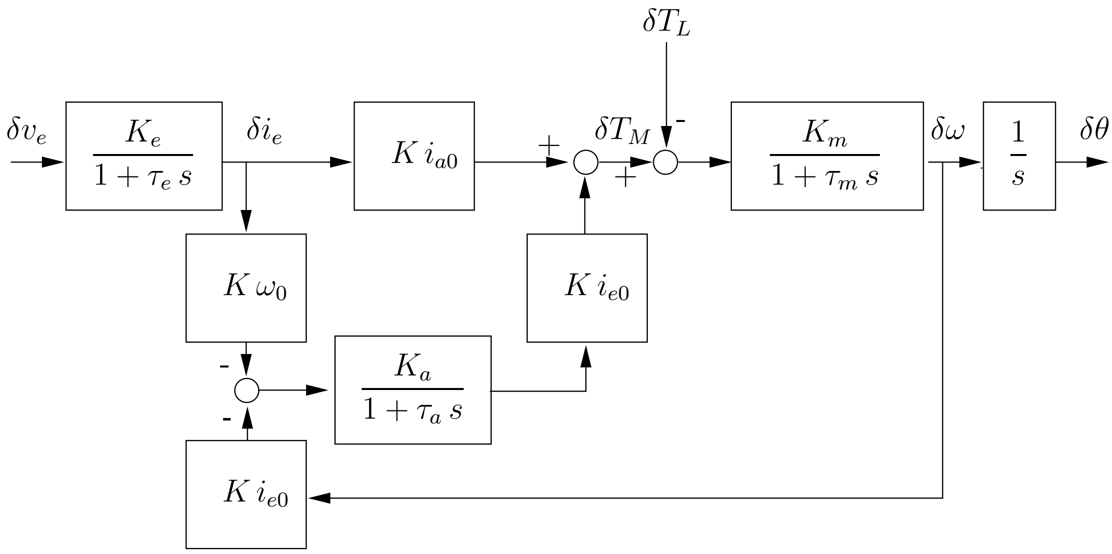

Figure 13: Block diagram of the linearization of the DC motor under armature voltage control.

equation is

$$\delta T_M = K\, K_e\, \frac{(1+\tau_a\, s)\, i_{a0} - K\, K_a\, i_{e0}\, \omega_0}{(1+\tau_a\, s)(1+\tau_e\, s)}\, \delta v_e - K_a\, \frac{(K\, i_{e0})^2}{1+\tau_a\, s}\, \delta\omega. \tag{28}$$

Considering the relationship between the torque and the speed established by equation (23), the two equations may be rearranged to compute the transfer function between the inputs δv_e, δT_L and the output $\delta\omega$ (or, equivalently $\delta\theta$ if a pole in the origin is added):

$$\frac{\delta\omega(s)}{\delta v_e(s)} = \frac{K_m\, K_a\, K\, ((1+\tau_a\, s)\, i_{a0} - K\, K_a\, i_{e0}\, \omega_0)}{(1+\tau_e\, s)\, ((1+\tau_m\, s)(1+\tau_a\, s) + K_m\, K_a\, (K\, i_{e0})^2)}$$

$$\frac{\delta\omega(s)}{\delta T_L(s)} = -K_m\, \frac{1+\tau_a\, s}{((1+\tau_m\, s)(1+\tau_a\, s) + K_m\, K_a\, (K\, i_{e0})^2)}$$

5 Armature-Current control

The most common control technique for DC motors is the armature-current control. Such control is performed by keeping the flux constant inside the motor. To this aim, either the stator voltage is constant or the stator coils are replaced by a permanent magnet. In the latter case, the motor is said to be a **permanent magnet DC motor** and is driven by means of the only armature coils.

Permanent magnet DC motors are totally equivalent to armature-controlled DC motors; as a matter of fact, in both of them the flux Φ is constant. Besides this, what makes the difference between them is dependent on equation (11). It can be seen that, with standard DC motors, the desired torque may be obtained by increasing i_e (namely, the flux), i_a, or both of them. Conversely, permanent magnet motors may be controlled by means of the only available current i_a. This is bad from the heating point of view; as a matter of fact, the rotor coils are hard to be cooled (since they are inside the motor) and, indeed, in the permanent magnet motors, are the only power source of the motor itself.

Referring to Figure 8, note that keeping the flux constant, the block diagram becomes linear. In addition, the motor has an intrinsic negative feedback structure, whence at the steady state, the speed ω is proportional to the reference input v_a. This two facts, in addition to the cheaper price of a permanent magnet motor with respect to a standard DC motor (as a matter of fact only the rotor coils need to be winded), are the main reasons why armature controlled motors are widely used.

However, several disadvantages arise from this control technique.

- Although the flux is constant (hence the back EMF never goes to zero when the motor is running), the rotor current could take, in several cases, high values, thus bringing the motor into dangerous operating conditions. In particular, the speed of the motor could decrease to zero due to an equilibrium between the load and the motor torque, or, equivalently, the current could take high values during the transient, after a step has been applied at the input. In this latter case, the mechanical time constant will delay the increase of speed $\Delta\omega$ corresponding to the increase of armature voltage Δv_a and during this delay the rotor current will raise at high values.

- The reference input and the power input of the motor are the same. This often leads to a trade off, between accuracy with respect to a desired reference value and maximum power exerted. As a matter of fact, the fidelity of a linear power amplifier is inversely proportional to the maximum power exerted by such power amplifier.

5.1 Steady state behavior

Consider equations (16), (18), (19), and (22); the steady state behavior of the armature controlled motor may be easily determined by putting to zero the time derivatives of the variables. The resulting equations are:

$$T_M = \frac{K_\Phi \, \Phi}{R_a} v_a - \frac{(K_\Phi \, \Phi)^2}{R_a} \omega \tag{29a}$$

$$T_M - T_L = F \, \omega. \tag{29b}$$

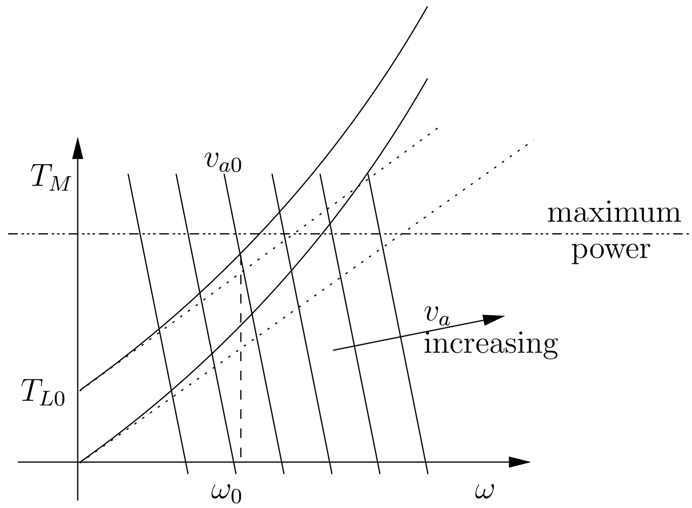

Figure 14: Steady state relationship between torque and speed in armature-current control.

In Figure 14 the curves corresponding to equations (29) are depicted. In particular, the negative slope lines are the lines satisfying equation (29a) for different values of v_a. It has to be stressed that, since usually $\dfrac{(K_\Phi\,\Phi)^2}{R_a} >> 1$, the slope is close to $-\infty$, whence, once assigned a value for the input voltage v_a, the speed variation of the load torque is very low. This is due to the fact that an intrinsic negative feedback action is present in the motor.

The mechanical curves, related to equation (29b), are traced considering a load torque increasing with the speed ω (as it has been done in the previous sections). The intersection of two curves gives the steady state values of torque and speed.

Since the power dissipated in the rotor is equal to $R_a\, i_a^2$, there is a maximum continuous current allowed in the rotor, which determines a maximum torque T_M that the motor may exert at the steady state. This maximum power line is generally the main constraint to the maximum power exerted by the motor; as a matter of fact, the heat generated on the rotor coil is hard to be transferred outside the motor for mechanical reasons.

5.2 Transfer function

Since, as remarked above, the armature-current control renders linear the nonlinear block diagram in Figure 8, it is of interest to compute the transfer function of the motor between the input v_a and the output ω (or, equivalently, θ) and the transfer function between the disturbance input T_L and the same output.

The resulting diagram, obtained by imposing Φ to be constant in equations (18) and (19) is depicted in Figure 15.

It is once more stressed that, due to the feedback action of the back EMF, the system is a velocity control system, as a matter of fact, besides the action of the disturbance T_L, the speed ω, at the steady state, is proportional to the input signal v_a with a steady-state error depending on the armature resistance R_a.

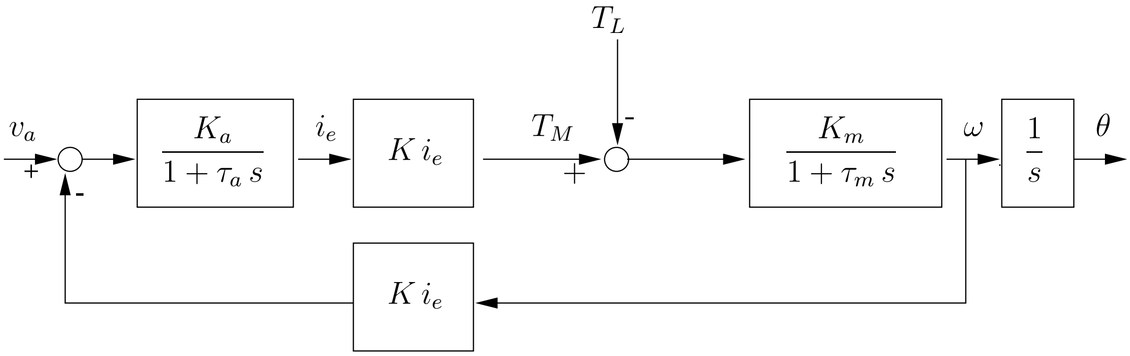

Figure 15: Block diagram of the linearization of the DC motor under armature-current control.

The transfer matrix of the system may be written as

$$\omega(s) = [W_1(s)\; W_2(s)] \begin{bmatrix} v_a(s) \\ T_L(s) \end{bmatrix},$$

where

$$\begin{aligned} W_1(s) &:= \frac{K_a\, K_\Phi\, \Phi\, K_m}{(1+\tau_a\, s)\,(1+\tau_m\, s) + K_a\, K_m\, (K_\Phi\, \Phi)^2}, \\ W_2(s) &:= \frac{K_m\,(1+\tau_a\, s)}{(1+\tau_a\, s)\,(1+\tau_m\, s) + K_a\, K_m\, (K_\Phi\, \Phi)^2} \end{aligned}$$

5.3 Speed control system

As remarked in Section 5.1, the armature current current controlled DC motor is intrinsically a velocity control scheme. In particular, consider the steady-state equations (29) and the dependence of the speed ω upon the values of the inputs v_a and T_L. Considering, for the sake of simplicity, the case $F = 0$ (the general case is similar), it turns out:

$$\omega \approx \frac{1}{K_\Phi\, \Phi}\, v_a - \frac{R_a}{(K_\Phi\, \Phi)^2}\, T_L. \tag{30}$$

It can be seen that if the load torque T_L is different from zero, a steady state error will be experienced on the system. Now, it can be seen that, the more $K_\Phi\, \Phi$ is high, the less the steady state error is large. Whence, it seems obvious to perform a control technique in which an additional speed feedback is performed.

By means of a tachometer (*i.e.*, a sensor for the measurement of the speed of the motor), an additional feedback loop is closed over the block diagram in Figure 15. Suppose the tachometer has a constant transfer function with a gain K_T, then placing a gain K_A in the direct branch, the control loop in Figure 16 is obtained.

Now, with reference to equation (30), noticing that for the control scheme in Figure 16, it holds $v_a = K_A\,(v_r - K_T\,\omega)$, then the following equation is obtained:

$$\omega \;\approx\; \frac{K_A}{K_\Phi\, \Phi + K_T\, K_A}\, v_r - \frac{R_a}{(K_\Phi\, \Phi)\,(K_\Phi\, \Phi + K_T\, K_A)}\, T_L.$$

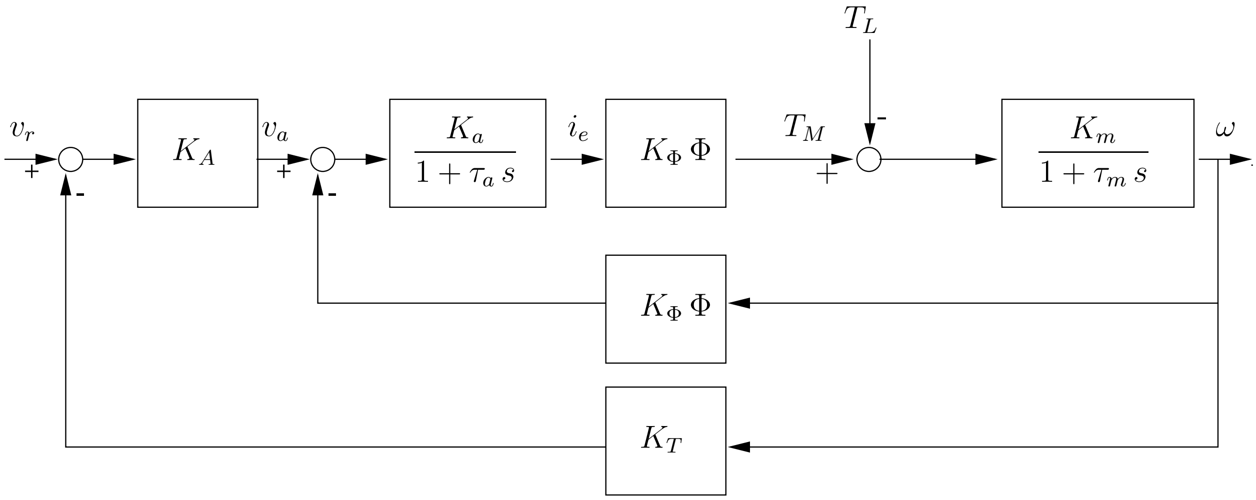

Figure 16: Block diagram of the speed control system for an armature controlled DC motor.

From the above equation, for sufficiently high values of K_A, it turns out:

$$\omega \approx \frac{1}{K_T} v_r,$$

as desired.

It should be noticed that, with this control scheme, a steady state error is always found, although this error may be decreased as much as necessary by the choice of the gain K_A.

Since the two feedback loops in Figure 16 may be represented as a unique feedback loop with suitable feedback and direct branch gains, the root locus of the system having the two poles related to τ_m and τ_a, may be traced to analyze the behavior of the closed loop system for different values of K_A.

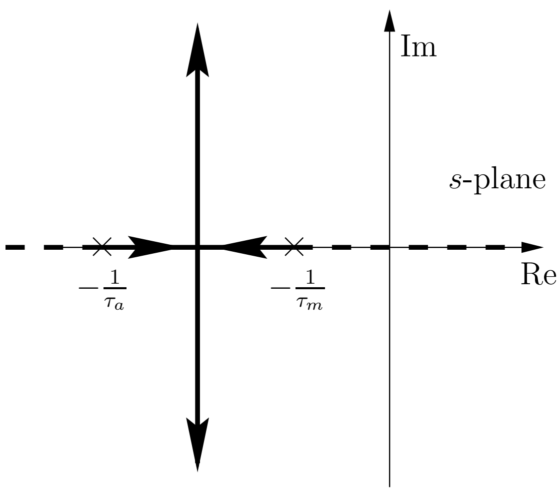

Figure 17: Root locus of the system in Figure 16, as K_A varies.

The resulting diagram, reported in Figure 17 shows that, as K_A increases, the system step response gets faster and overshooted.

5.4 Position control system

With reference to the linear block diagram in Figure 15, if a feedback loop is closed between the input and the position output θ, then a position regulation is performed on the system.

By virtue of the fact that, as remarked above, the back EMF effect is equivalent to a velocity feedback, denote by K_ω the feedback gain from ω which refers to both the back EMF and the tachometer action. In addition, define A as the product of the motor static gain and the controller gain. Then, the diagram depicted in Figure 18 correctly represents the behavior of the system having two feedback loops: one with respect to the motor speed and one with respect to its position.

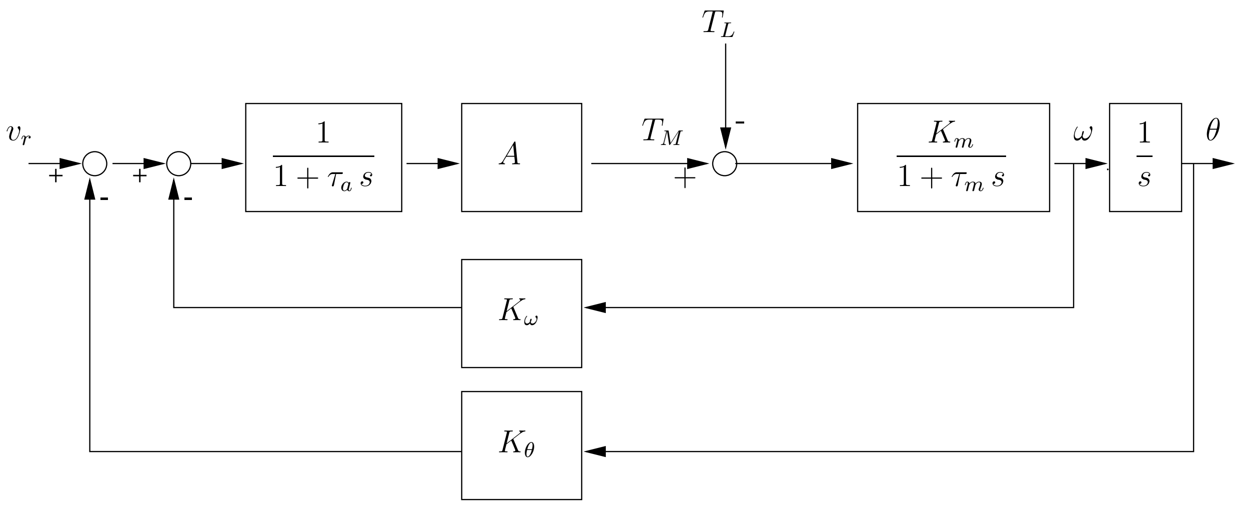

Figure 18: Block diagram of the position control system for an armature controlled DC motor.

The transfer function between the reference input v_a and the output θ is given by

$$W(s) := \frac{\theta(s)}{v_a(s)} = \frac{\dfrac{1}{s}\dfrac{A}{(1+\tau_a\, s)}\dfrac{K_m}{(1+\tau_m\, s)}}{1+\dfrac{1}{s}\dfrac{A}{(1+\tau_a\, s)}\dfrac{K_m}{(1+\tau_m\, s)}(K_\theta + K_\omega\, s)}.$$

The root locus of the feedback system may be traced with respect to the gain A. Consider first the case when $K_\omega = 0$. The result is depicted in Figure 19. It can be seen that, since the relative degree of the transfer function is equal to 3, then three branches reach infinity, hence the closed loop system becomes unstable for high values of the gain K_θ.

Consider now the more general case in which the velocity feedback is non-zero. In this case, the root locus changes, as a matter of fact a zero is added to the open loop transfer function. The value of the zero is $z = -\dfrac{K_\omega}{K_\theta}$; whence, the more the ratio between the two gains is high, the more the closed loop poles move away from the abscissa $s = 0$ (see

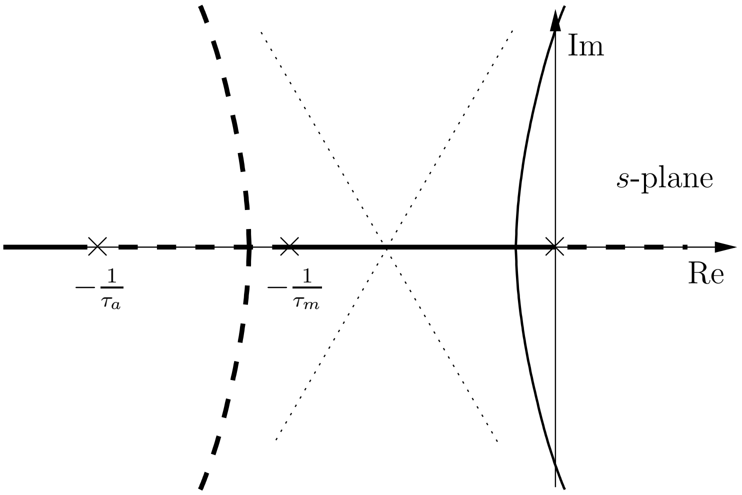

Figure 19: Root locus of the position control system when the velocity feedback is zero.

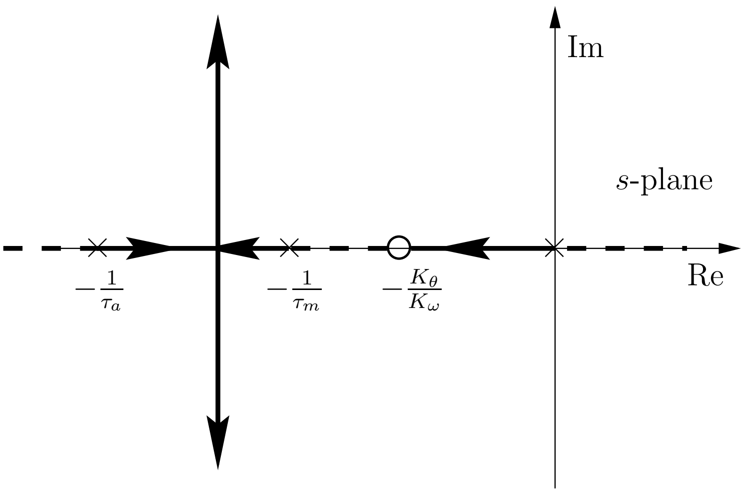

Figure 20: Root locus of the position control system when the velocity feedback is non-null.

Figure 19). Note that, since the branch approaching infinity is vertical, in this second case, the closed loop system is asymptotically stable for any value of the gain A.

From Figure 19, it is clear that, as the position feedback gain increases with respect to the velocity feedback gain, the situation depicted in Figure 19 is reached; as a matter of fact, the branch approaching infinity moves on the right direction as the zero moves in the left direction, whence destabilizing the system. In the limit case in which $z = -\infty$, the exact situation of Figure 19 is reached. On the other hand, if the velocity feedback gain is increased with respect to the position feedback gain, the zero approaches the origin and the vertical branch moves left. The limit case coincides with the root locus of Figure 17, as a matter of fact, the zero cancels the pole in the origin and the only two poles related to the electrical and mechanical time constants of the motor remain.

5.5 Torque control system

Due to the fact that, by equation (18), the torque is proportional to the armature current, a simple hardware feedback loop is often implemented to keep the current proportional to a reference voltage value, by means of an operational amplifier.

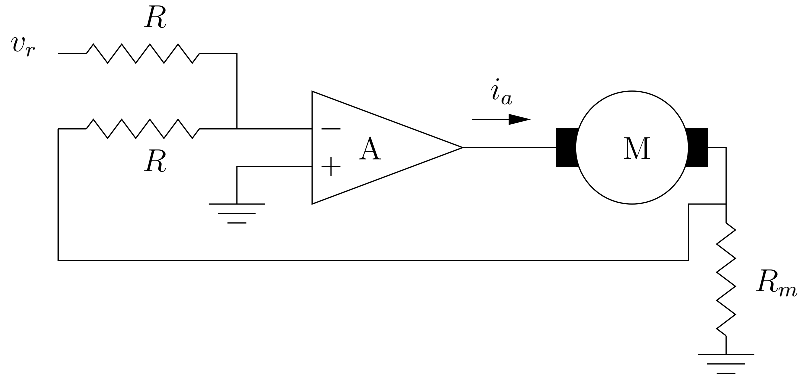

Figure 21: Torque control system of an armature controlled DC motor.

This last control scheme is usually integrated with PWM power amplifiers, and is obtained by a high power resistor with low resistance R_m, connected in series to the rotor windings (see Figure 21). If the reference voltage is chosen as

$$v_a = -R_m\, i_d,$$

where i_d is the desired current, the steady state value of the current will be the desired one. The time constant of the circuit is dependent on R and the high gain of A; however, a very fast response is usually obtained due to the very high gain of the commercial operational amplifiers.

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