Hybrid reset rules for peaking avoidance of a class of high-gain observers

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Outline

Peaking and high-gain observers

- Introduced in the early '90s:
  1. F. Esfandiari and H.K. Khalil. Output feedback stabilization of fully linearizable systems. International Journal of Control, 56(5):1007-1037, 1992.
  2. P.V. Kokotovic. The joy of feedback: nonlinear and adaptive. IEEE Control Systems Magazine, 12(3):7-17, 1992.
  3. S. Nicosia, P. Tomei, and A. Tornambè . An approximate observer for a class of nonlinear systems. Systems and Control Letters, 13(1):43-51, 1989.
- Peaking corresponds to large transients of the error dynamics caused by high-gain
- Typically it is resolved by saturating the plant input
- In this talk we will use hybrid techniques to reduce peaking for a restricted class of high gain observers

High-gain injection dominates Lipschitz nonlinearities

- Multiplanar case: $x_i = (p_i, v_i)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$
  $\psi = (\psi_1, \ldots, \psi_n)$, $\phi = (\phi_1, \ldots, \phi_n)$
- Interconnection of n two-dimensional systems: $i \in \{1, \ldots, n\}$,

$$\begin{array}{c} \text{GAS} \\ \text{plant} \end{array} \left\{ \begin{array}{rcl} \dot{p}_i & = & v_i + \psi_i(y) \\ \dot{v}_i & = & \phi_i(x, u = \gamma(x)) \end{array} \right.$$

Available output:

$$y_i = p_i,$$

- Lipschitz-type assumption on nonlinearity ϕ: $\exists L_\delta > 0$ s.t.

$$|\phi_i(x, \gamma(\hat{x})) - \phi_i(\hat{x}, \gamma(\hat{x}))| \leq L_\delta |x - \hat{x}|, \qquad \forall i \in \{1, \ldots, n\}$$

- Output feedback scheme using a high-gain observer: $u = \gamma(\hat{x})$:

$$\begin{array}{rcl} \dot{\hat{p}}_i & = & \hat{v}_i + \psi_i(y) + \ell k_p(y_i - \hat{p}_i) \\ \dot{\hat{v}}_i & = & \phi_i(\hat{x}, \gamma(\hat{x})) + \ell^2 k_v(y_i - \hat{p}_i), \end{array} \qquad i = 1, \ldots, n$$

induces GAS as long as the high gain ℓ is "high" (large) enough (+ extra properties of cascades).

Error dynamics in scaled coordinates is linearly dominant

- Scaled error $e_i := \begin{bmatrix} e_{pi} \\ e_{vi} \end{bmatrix} = \begin{bmatrix} p_i - \hat{p}_i \\ \ell^{-1}(v_i - \hat{v}_i) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \ell^{-1} \end{bmatrix} \begin{bmatrix} p_i - \hat{p}_i \\ v_i - \hat{v}_i \end{bmatrix}$
- Scaled error dynamics corresponds to:

$$\dot{e}_i = \ell \underbrace{\begin{bmatrix} -k_p & 1 \\ -k_v & 0 \end{bmatrix}}_{A_e} e_i + \begin{bmatrix} 0 \\ \underbrace{\dfrac{\phi_i(x, \gamma(\hat{x})) - \phi_i(\hat{x}, \gamma(\hat{x}))}{\ell}}_{\delta_i(x,\hat{x})} \end{bmatrix}, \quad i = 1, \ldots, n$$

- Useful property exploiting Lipschitz assumption:

$$|\delta_i(x, \hat{x})| \leq \frac{L_\delta}{\ell} |x - \hat{x}| \leq L_\delta |e|.$$

- **Proposition** "High" gain ℓ is large enough if $\exists\, P > 0$, $\vartheta > 0$ s.t.

$$\begin{bmatrix} \ell({A_e}^T P + P A_e) + n\vartheta L_\delta^2 I_2 & P \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \\ \left(P \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)^T & -\vartheta \end{bmatrix} < 0.$$

Error Lyapunov function V is quadratic in e

- **Proposition** "High" gain ℓ is large enough if $\exists\, P > 0$, $\vartheta > 0$ s.t.

$$\begin{bmatrix} (\ell(A_e{}^T P + P A_e) + n\vartheta L_\delta^2 I_2) \otimes I_n & P\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \otimes I_n \\ (P\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])^T \otimes I_n & -\vartheta \otimes I_n \end{bmatrix} < 0.$$

- Candidate Lyapunov function is $V(e) := \sum_{i=1}^{n} e_i^T P e_i = e^T (P \otimes I_n) e$
- Proof uses the following facts:

$$\begin{aligned} \dot{e} &= \ell(A_e \otimes I_n)e + (\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \otimes I_n)\,\delta(x,\hat{x}), \\ |\delta(x,\hat{x})| &\leq \sqrt{n} L_\delta |e|, \end{aligned}$$

which imply

$$\begin{aligned} \dot{V} &= 2e^T (P \otimes I_n)\Big(\ell(A_e \otimes I_n)e + (\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \otimes I_n)\,\delta(x,\hat{x})\Big) \\ &< -\epsilon_V(|e|^2 + |\delta(x,\hat{x})|^2) \end{aligned}$$

- Condition $A_e{}^T P + P A_e < 0$ is necessary (linear dynamics dominant)

Peaking phenomenon due to scaling of $e \neq$ actual error

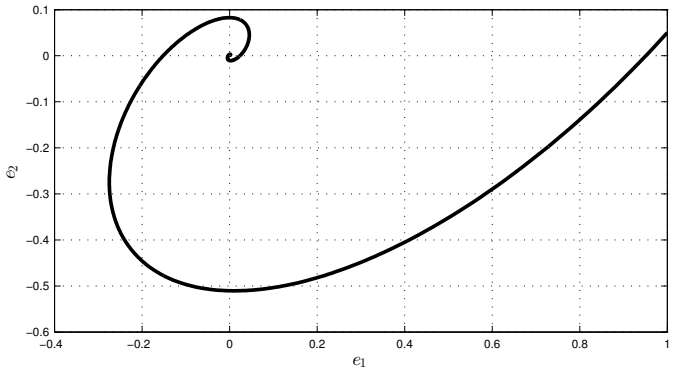
- Example from Esfandiari/Khalil '92
- Closed-loop trajectory with

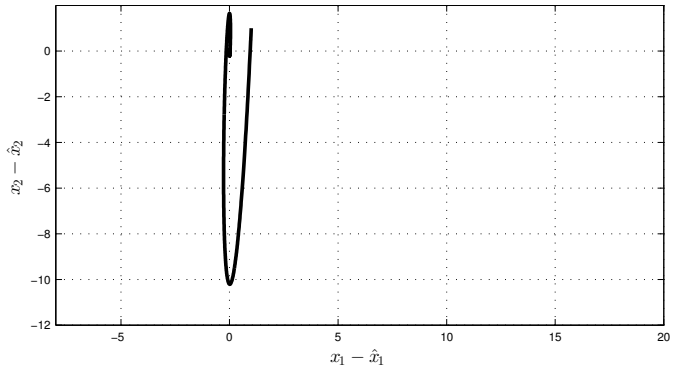
$$\dot{e} = \underbrace{\ell A_e e}_{\text{dominant}} + \begin{bmatrix} 0 \\ \underbrace{\delta(x, \hat{x})}_{\text{small}} \end{bmatrix}$$

- Actual error variables

$$\begin{bmatrix} p - \hat{p} \\ v - \hat{v} \end{bmatrix} = \begin{bmatrix} e_1 \\ \ell e_2 \end{bmatrix}$$

exhibit peaking of e_2 (recall: ℓ is large!)





- Peaking is inevitable due to aggressive action of output injection:

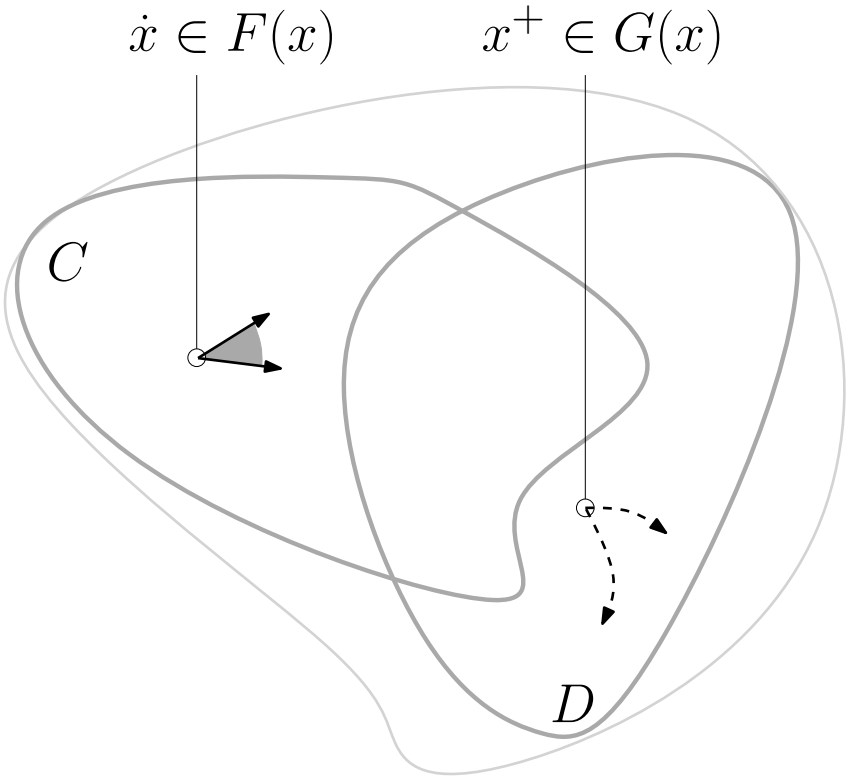
$$\begin{aligned} \dot{\hat{p}} &= \hat{v} + \psi(y) + \ell k_p (y - \hat{p}) \\ \dot{\hat{v}} &= \phi_i(\hat{x}, \gamma(\hat{x})) + \ell^2 k_v (y - \hat{p}) \end{aligned}$$

Hybrid dynamical systems review: **dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ (flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D} \end{cases}$$



Hybrid dynamical systems review: **continuous dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
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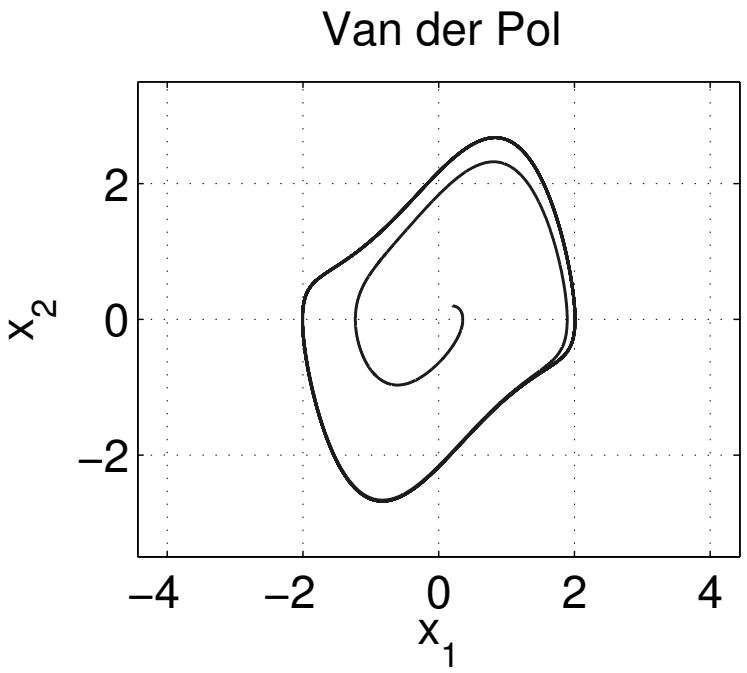
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ (flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D} \end{cases}$$

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2) \end{cases}$$



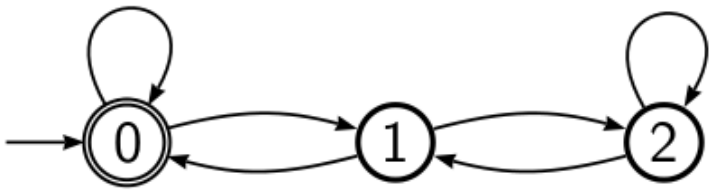
Hybrid dynamical systems review: **discrete dynamics**

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $\mathcal{C} \subseteq \mathbb{R}^n$ (flow set)
- $\mathcal{D} \subseteq \mathbb{R}^n$ (jump set)
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ flow map)
- $G : \mathcal{D} \rightrightarrows \mathbb{R}^n$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D} \end{cases}$$

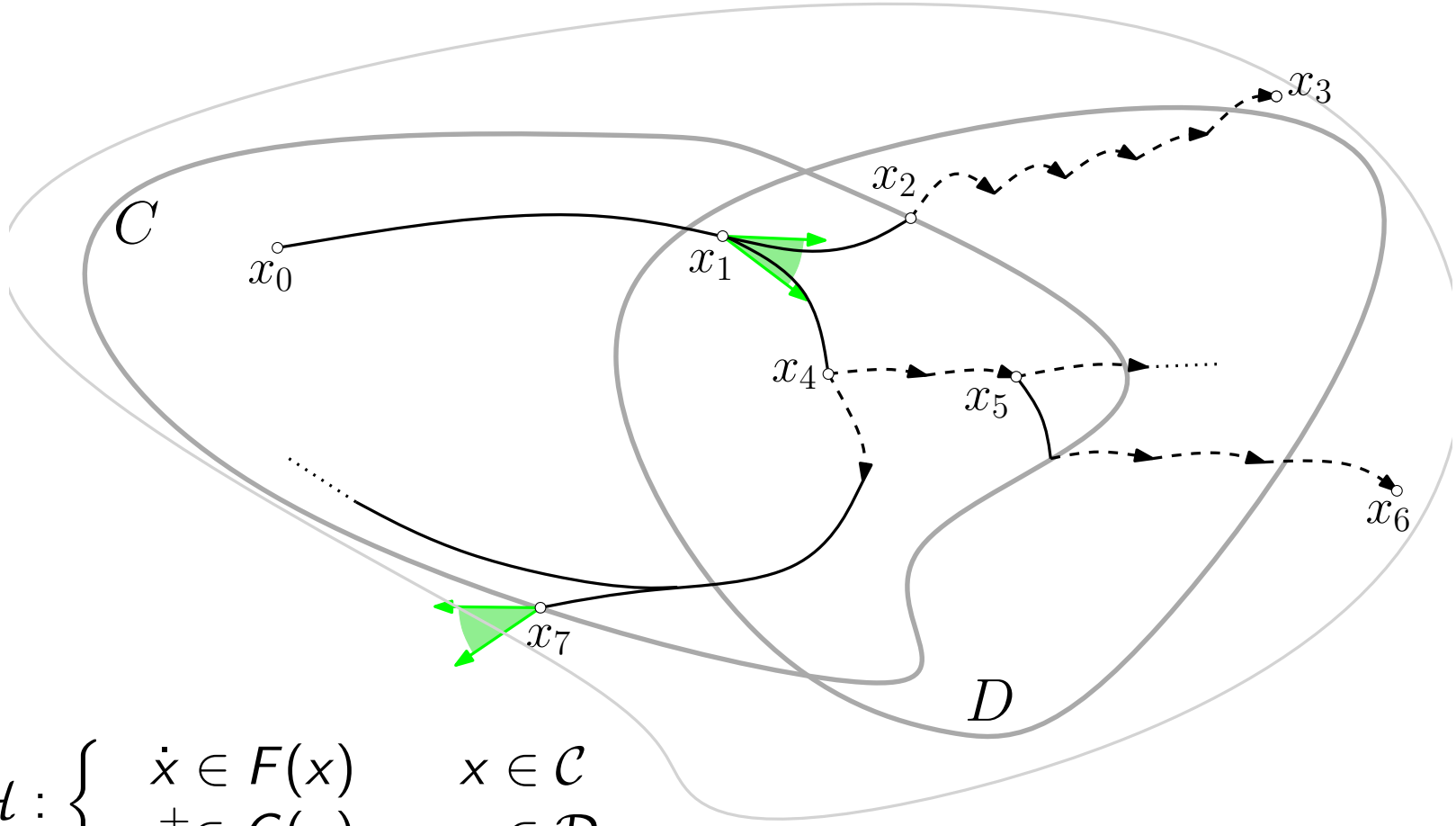
$$x^+ \in \begin{cases} \{0,1\} & \text{if } x = 0 \\ \{0,2\} & \text{if } x = 1 \\ \{1,2\} & \text{if } x = 2 \end{cases}$$



A possible sequence of states from $x_0 = 0$ is:

$$(0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in N$$

Hybrid dynamical systems review: **trajectories**

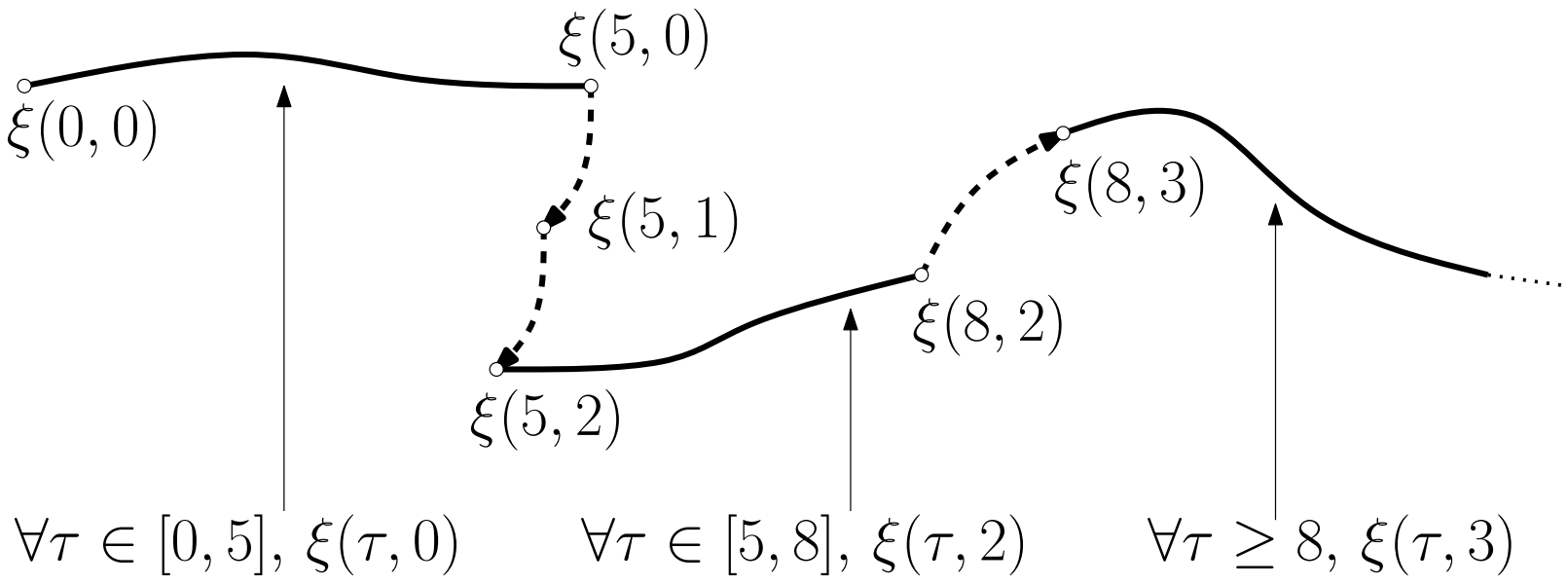


$$\mathcal{H}: \begin{cases} \dot{x} \in F(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D} \end{cases}$$

Hybrid dynamical systems review: **hybrid time**

The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.



Hybrid dynamical systems review: **hybrid time**

$E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

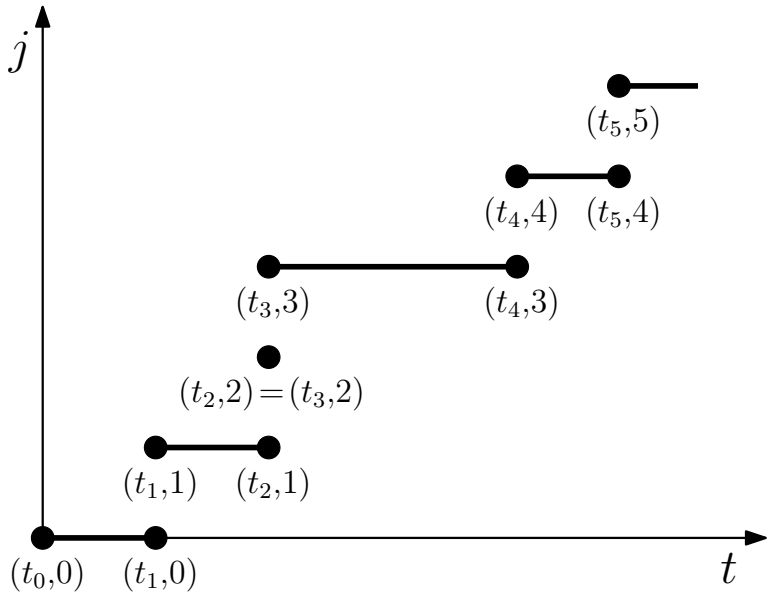
$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\})$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

E is a **hybrid time domain** if for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

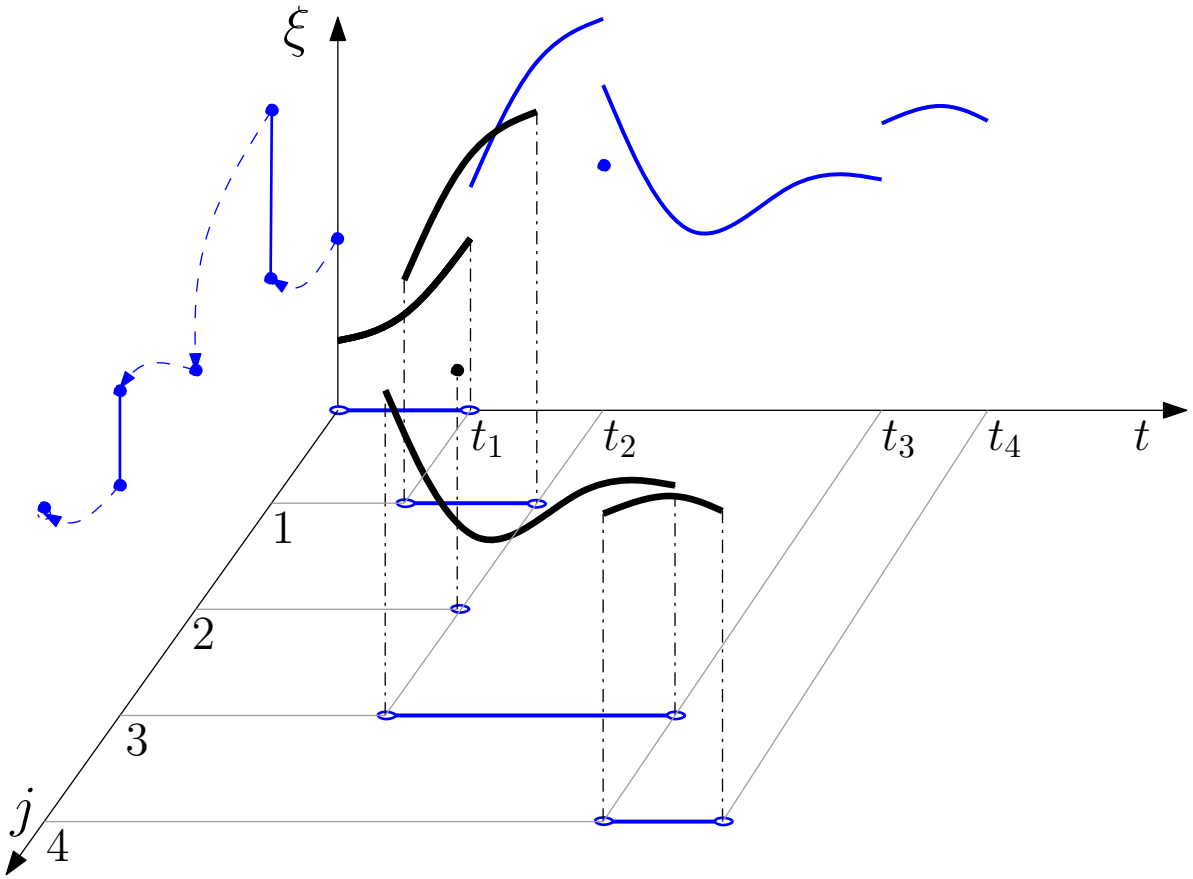
$$E \cap ([0, T] \times \{0, 1, \ldots, J\})$$

is a compact hybrid time domain.



Hybrid dynamical systems review: **solution**

- Formally, a solution satisfies the flow dynamics when flowing and satisfies the jump dynamics when jumping



Hybrid dynamical systems review: **Lyapunov theorem**

Theorem Lyap Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D}, \end{cases}$$

aassume that function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies for some α_1, α_2 $\in \mathcal{K}_\infty$ and ρ positive definite:

$$\begin{aligned} &\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}), \quad \forall x \in \mathbb{R}^n \\ &\langle \nabla V(x), f \rangle \leq -\rho(|x|_\mathcal{A}), \quad \forall x \in \mathcal{C}, f \in F(x), \\ &V(g) - V(x) \leq -\rho(|x|_\mathcal{A}), \quad \forall x \in \mathcal{D}, g \in G(x) \end{aligned}$$

then $\mathcal{A}$ is unifromly globally asymptotically stable (UGAS) for $\mathcal{H}$, namely there exists $\beta \in \mathcal{KL}$ such that all solutions satisfy

$$|\xi(t,j)| \leq \beta(|\xi(0,0)|, t+j), \quad \forall (t,j) \in \operatorname{dom} \xi$$

<u>Note</u>: Lyapunov conditions comprise **flow** and **jump** conditions.
<u>Note</u>: UGAS is characterized in terms of hybrid time (t, j)

Peaking phenomenon due to scaling of e (recall)

- Example from Esfandiari/Khalil '92
- Closed-loop trajectory with

$$\dot{e} = \underbrace{\ell A_e e}_{\text{dominant}} + \begin{bmatrix} 0 \\ \underbrace{\delta(x, \hat{x})}_{\text{small}} \end{bmatrix}$$

- Actual error variables

$$\begin{bmatrix} p - \hat{p} \\ v - \hat{v} \end{bmatrix} = \begin{bmatrix} e_1 \\ \ell e_2 \end{bmatrix}$$

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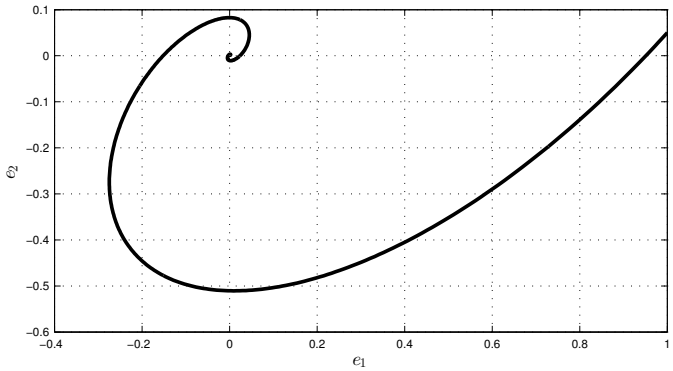
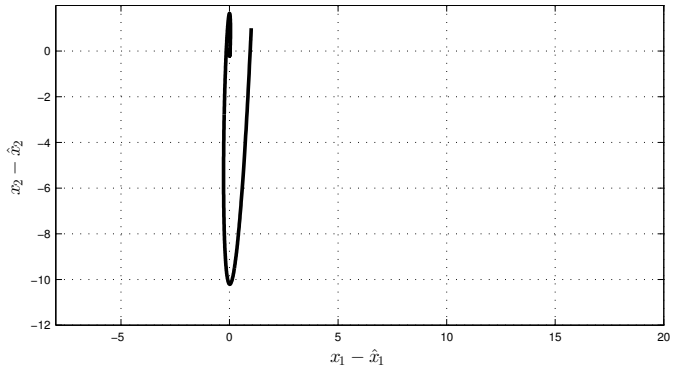




- Peaking is inevitable due to aggressive action of output injection:

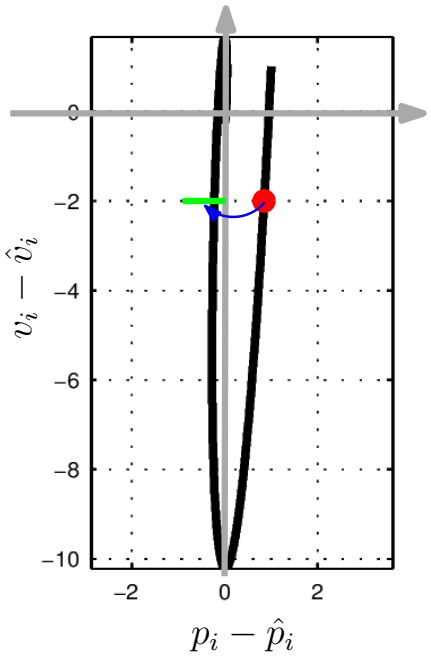
$$\begin{aligned} \dot{\hat{p}} &= \hat{v} + \psi(y) + \ell k_p (y - \hat{p}) \\ \dot{\hat{v}} &= \phi_i(\hat{x}, \gamma(\hat{x})) + \ell^2 k_v (y - \hat{p}) \end{aligned}$$

Unviable sol'n: avoid peaking by jumping to the other side

- Clean (but unviable) solution:

$$\begin{cases} \dot{\hat{p}}_i &= \hat{v}_i + \psi_i(y) + \ell k_p e_{pi} \\ \dot{\hat{v}}_i &= \phi_i(\hat{x}, \gamma(\hat{x})) + \ell^2 k_v e_{pi}, \end{cases} \qquad e_{pi} e_{vi} \leq 0,$$

$$\underbrace{\begin{cases} \hat{p}_i^+ - p_i \\ \hat{v}_i^+ - v_i \end{cases}}_{e_i^+} \underbrace{\begin{cases} = -\alpha(\hat{p}_i - p_i) \\ = \hat{v}_i - v_i, \end{cases}}_{g_\alpha(e_i)} \qquad e_{pi} e_{vi} \geq 0$$



- **Lemma U** Consider any P such that $A_e^T P + P A_e < 0$ and the corresponding $V(e) = e^T (P \otimes I_n) e$ for the error dynamics, then

$$e_{pi} e_{vi} \geq 0 \Rightarrow V(e^+) = (e^+)^T (P \otimes I_n) e^+ \leq e^T (P \otimes I_n) e = V(e),$$

for any $\alpha \in [0, 1]$

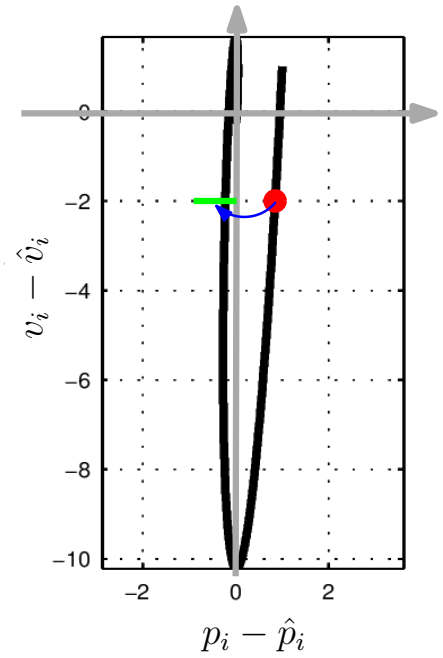
- Proof: Uses $P A_e = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -k_p & 1 \\ -k_v & 0 \end{bmatrix} = \begin{bmatrix} \star & \star \\ \star & p_2 \end{bmatrix} < 0$
- Problem: Knowledge of $\mathrm{sign}(e_{pi} e_{vi})$ is required

Unviable sol'n: avoid peaking by jumping to the other side

- Clean (but unviable) solution:

$$\left\{ \dot{\hat{x}}_i = \begin{bmatrix} \dot{\hat{p}}_i \\ \dot{\hat{v}}_i \end{bmatrix} = f_i(\hat{x}, y) \qquad e_{pi} e_{vi} \leq 0, \right.$$

$$\left\{ \underbrace{\hat{x}_i^+ = \begin{bmatrix} \hat{p}_i^+ \\ \hat{v}_i^+ \end{bmatrix}}_{e_i^+} \underbrace{= g_{\alpha i}(\hat{x}_i, y_i)}_{g_\alpha(e_i)} \qquad e_{pi} e_{vi} \geq 0 \right.$$



- **Lemma U** Consider any P such that $A_e^T P + P A_e < 0$ and the corresponding $V(e) = e^T (P \otimes I_n) e$ for the error dynamics, then

$$e_{pi} e_{vi} \geq 0 \Rightarrow V(e^+) = (e^+)^T P_e e^+ \leq e^T P_e e = V(e),$$

for any $\alpha \in [0, 1]$

- Proof: Uses $P A_e = \left[\begin{smallmatrix} p_1 & p_2 \\ p_2 & p_3 \end{smallmatrix}\right] \begin{bmatrix} -k_p & 1 \\ -k_v & 0 \end{bmatrix} = \left[\begin{smallmatrix} \star & \star \\ \star & p_2 \end{smallmatrix}\right] < 0$
- Problem: Knowledge of $\mathrm{sign}(e_{pi} e_{vi})$ is required

Introduce hybrid dynamics to compute the integral of e_{pi}^2

- Add extra hybrid states and use $\xi = (x, \hat{x}, \zeta, \eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^n \times \mathbb{R}^n$

$$\xi \in \mathcal{D}_i^-, \begin{cases} \hat{x}_i^+ = g_{\alpha i}(\hat{x}_i, y_i) \\ \zeta_i^+ = \alpha^2 (y_i - \hat{y}_i)^2 \\ \eta_i^+ = 0, \end{cases} ; \quad \xi \in \mathcal{D}_i^+, \begin{cases} \hat{x}_i^+ = \hat{x}_i \\ \zeta_i^+ = (y_i - \hat{y}_i)^2 \\ \eta_i^+ = 0, \end{cases}$$

$$\xi \in \mathcal{C}, \begin{cases} \dot{\hat{x}}_i = f_i(\hat{x}, y) \\ \dot{\zeta}_i = 0 \\ \dot{\eta}_i = (y_i - \hat{y}_i)^2, \end{cases}$$

Result:

$$\zeta_i(t,j) = e_{pi}^2(t_j, j),$$
$$\eta_i(t,j) = \int_{t_j}^{t} e_{pi}^2(\tau, j) d\tau$$

for all $i = 1, \ldots, n$, where, for a fixed $\Delta > 0$,

$$\begin{aligned} \mathcal{C} &:= \{\xi \in \mathbb{R}^{6n} : -\Delta \leq e_{pi}^2 - \zeta_i + 2\ell k_p \eta_i \leq \Delta, \forall i = 1, \ldots, n\}, \\ \mathcal{D} &:= \bigcup_{i=1}^{n} \mathcal{D}_i^+ \cup \mathcal{D}_i^-, \\ \mathcal{D}_i^+ &:= \{\xi \in \mathbb{R}^{6n} : e_{pi}^2 - \zeta_i + 2\ell k_p \eta_i \geq \Delta\}, \\ \mathcal{D}_i^- &:= \{\xi \in \mathbb{R}^{6n} : e_{pi}^2 - \zeta_i + 2\ell k_p \eta_i \leq -\Delta\}, \end{aligned}$$

Sign of $e_{pi}e_{vi}$ estimated from the integral of e_{pi}^2

- Flow set evaluated along solutions: $\xi(t,j) \in \mathcal{C}$ if

$$-\Delta \leq e_{pi}^2(t,j) - \zeta_i(t,j) + 2\ell k_p \eta_i(t,j) \leq \Delta, \forall i = 1, \ldots, n$$

- First flow equation of error dynamics: $\dot{e}_{pi} = -\ell k_p e_{pi} + \ell e_{vi}$ yields

$$\int_{t_j}^{t} e_{pi}(\tau,j)\dot{e}_{pi}(\tau,j)d\tau = -\ell k_p \int_{t_j}^{t} e_{pi}^2(\tau,j)d\tau + \ell \int_{t_j}^{t} e_{pi}(\tau,j)e_{vi}(\tau,j)d\tau,$$

$$\Rightarrow 2\ell \int_{t_j}^{t} e_{pi}(\tau,j)e_{vi}(\tau,j)d\tau = 2\ell k_p \eta_i(t,j) + e_{pi}^2(t,j) - e_{pi}^2(t_j,j)$$

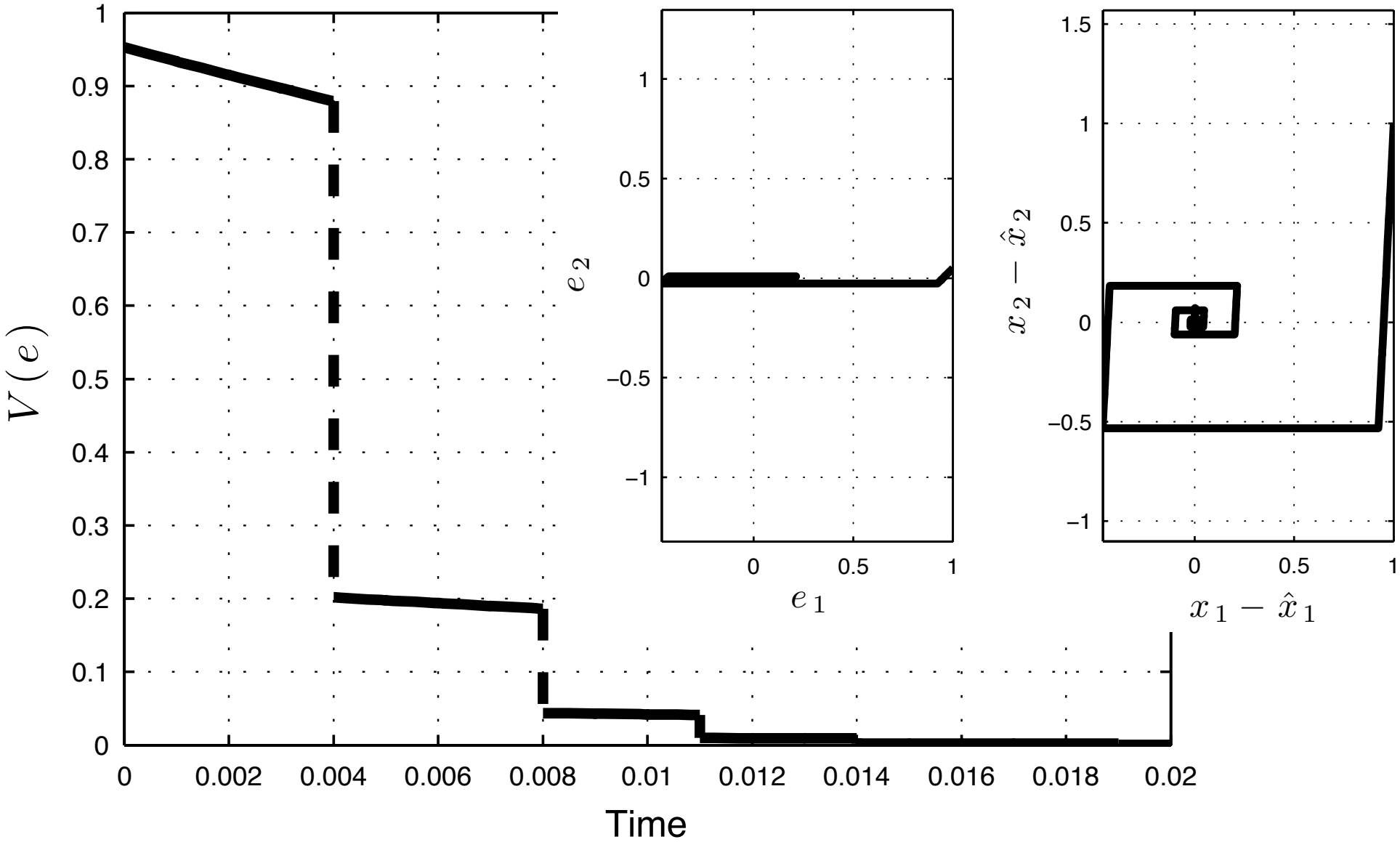
- **Lemma S** Given any hybrid solution ξ and $(t,j) \in \operatorname{dom} \xi$,

$$\xi(t,j) \in \mathcal{D}_i^- \quad \Rightarrow \quad e_{pi}(t,j)e_{vi}(t,j) \leq 0$$

- **Lemma U + S** Given $V(e) = e^T(P \otimes I_n)e$ and any solution ξ

$$\dot{V}(\xi(t,j)) \leq -\epsilon_V |e(t,j)|^2, \quad V(e(t_j,j)) - V(e(t_j,j-1)) \leq 0$$

Esfandiari's example: response with $\alpha = 0.5$



Average dwell time despite multiple jumps in $\mathcal{D}_i^{+/-}$

- **Lemma ADT** Each solution ξ satisfies an *average dwell-time condition*, namely there exist N, σ such that for each pair $(t, j) \geq (s, k)$ in $\mathrm{dom}\ \xi$,

$$j - k \leq \sigma(t - s) + N.$$

Proof

- e *is bounded* because V is nonincreasing along both flows and jumps (*id est* $\dot{V} \leq 0$, $\Delta V \leq 0$ from Lemma U+S)

$$\Rightarrow |e(t, j)| \leq M_e |e(0, 0)|, \quad \forall (t, j) \in \mathrm{dom}\ \xi.$$

- $\dot{e}$ *is bounded* from the error dynamics and the L_δ assumption
- After jump of $\hat{x}_i$ we have $|e_{pi}^2(t_j, j) - \zeta_i(t_j, j) + 2\ell k_p \eta_i(t_j, j)| = 0$
- Then at least ρ ordinary time elapses before $\xi \in \mathcal{D}_i^- \cup \mathcal{D}_i^+$:

$$|e_{pi}^2(\tau, j) - \zeta_i(\tau, j) + 2\ell k_p \eta_i(\tau, j)| < \Delta, \quad \forall \tau \in [t_j, t_j + \rho)$$

- Solutions may however jump multiple times at each ordinary time t

Persistent flow implies UGAS of origin of error dynamics

Theorem PF (persistent flow) Assume that, for a closed attractor $\mathcal{A}$,

1. Lyapunov: V satisfies for some positive scalars a_1, a_2, a_3:

$$a_1|\xi|_{\mathcal{A}} \leq V(\xi) \leq a_2|\xi|_{\mathcal{A}}$$
$$\dot{V}(\xi) \leq -a_3 V(\xi), \quad \forall \xi \in \mathcal{C},$$
$$V(\xi^+) - V(\xi) \leq 0, \quad \forall \xi \in \mathcal{D}.$$

2. ADT: for each $r > 0$, $|\xi(0,0)|_{\mathcal{A}} \leq r$ implies that all solutions satisfy an average dwell-time constraint.

THEN the set $\mathcal{A}$ is UGAS and ULES. In particular, for each $r > 0$, there exist $M > 0, \lambda > 0$, such that

$$|\xi(0,0)|_{\mathcal{A}} \leq r \quad \Rightarrow \quad |\xi(t,j)|_{\mathcal{A}} \leq \mathrm{e}^{-\lambda t}|\xi(0,0)|_{\mathcal{A}}, \ \forall (t,j) \in \mathrm{dom}\ \xi$$

- Theorem PF can be applied to the closed set

$$\mathcal{A} = \{\xi = (x, \hat{x}, \zeta, \eta) \in \mathbb{R}^{6n} : x - \hat{x} = 0\}$$

to show that the estimation error satisfies an exponential bound.

Cascaded results on hybrid systems allow to tackle GAS

Theorem C (cascaded hybrid systems) Given a hybrid dynamical system on $\mathbb{R}^m$, consider a closed set $\mathcal{A} \subset \mathbb{R}^m$ and a compact set $\mathcal{A}_\circ \subset \mathcal{A}$. If

1. the set $\mathcal{A}$ is UGAS for the hybrid dynamics;
2. the set $\mathcal{A}_\circ$ is UGAS for the hybrid dynamics restricted to $\mathcal{A}$;

THEN the set $\mathcal{A}_\circ$ is ULAS for the hybrid dynamics with domain of attraction $\equiv$ to the set from which all solutions bounded.

- Theorem C can be applied with

$$\begin{aligned} \mathcal{A} &= \{\xi = (x, \hat{x}, \zeta, \eta) \in \mathbb{R}^{6n} : x - \hat{x} = 0\} \\ \mathcal{A}_\circ &= \{0\}, \end{aligned}$$

to conclude UGAS of the hybrid observer closed loop

Regularity of flow/jump sets and maps gives robustness

- The proposed solution $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$ satisfies the *basic conditions*:
  1. set $\mathcal{C}$ is closed;
  2. set $\mathcal{D}$ is closed;
  3. map F is (outer semi-)continuous, convex, nonempty and locally bounded in $\mathcal{C}$;
  4. map G is (outer semi-)continuous, nonempty and locally bounded in $\mathcal{D}$.
- Under these conditions, the hybrid system is *well posed*, namely:
  1. UGAS of compact $\mathcal{A}_\circ$ is robust "in the small"
  2. its UAS is semiglobally practically robust "in the large"
  3. All solutions coincide with Hermes and Krasovskii solutions
- This requirement would still hold if we use $\mathcal{C} = \overline{\mathbb{R}^{6n} \setminus \mathcal{D}}$ and

$$\begin{aligned} \mathcal{D}_i^+ &:= \{\xi \in \mathbb{R}^{6n} : \; e_{pi}^2 - \zeta_i + 2\ell k_p \eta_i \geq \Delta \max(\zeta_i, \Delta)\}, \\ \mathcal{D}_i^- &:= \{\xi \in \mathbb{R}^{6n} : \; e_{pi}^2 - \zeta_i + 2\ell k_p \eta_i \leq -\Delta \max(\zeta_i, \Delta)\}, \end{aligned}$$

to get some homogeneity of solutions

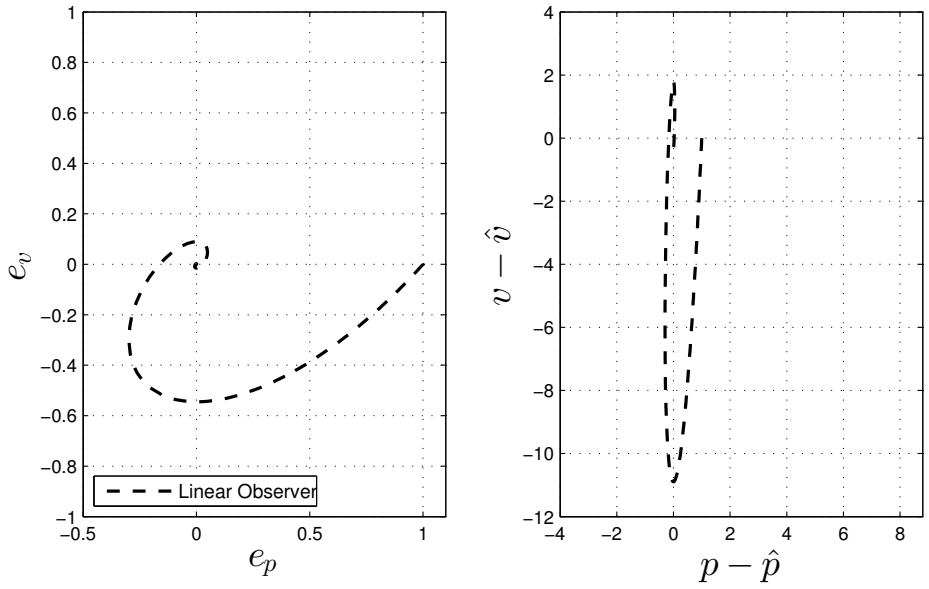
Example from Esfandiari/Khalil 1992

- System dynamics (no uncertainty)

$$\begin{cases} \dot{p} &= v \\ \dot{v} &= 1.4\sin(p) + 0.8u \end{cases}$$

- Auxiliary function

$$\mu(w) = \begin{cases} \frac{w}{|w|}, & \text{if } |w| \geq 1 \ , \\ w, & \text{if } |w| < 1 \ . \end{cases}$$



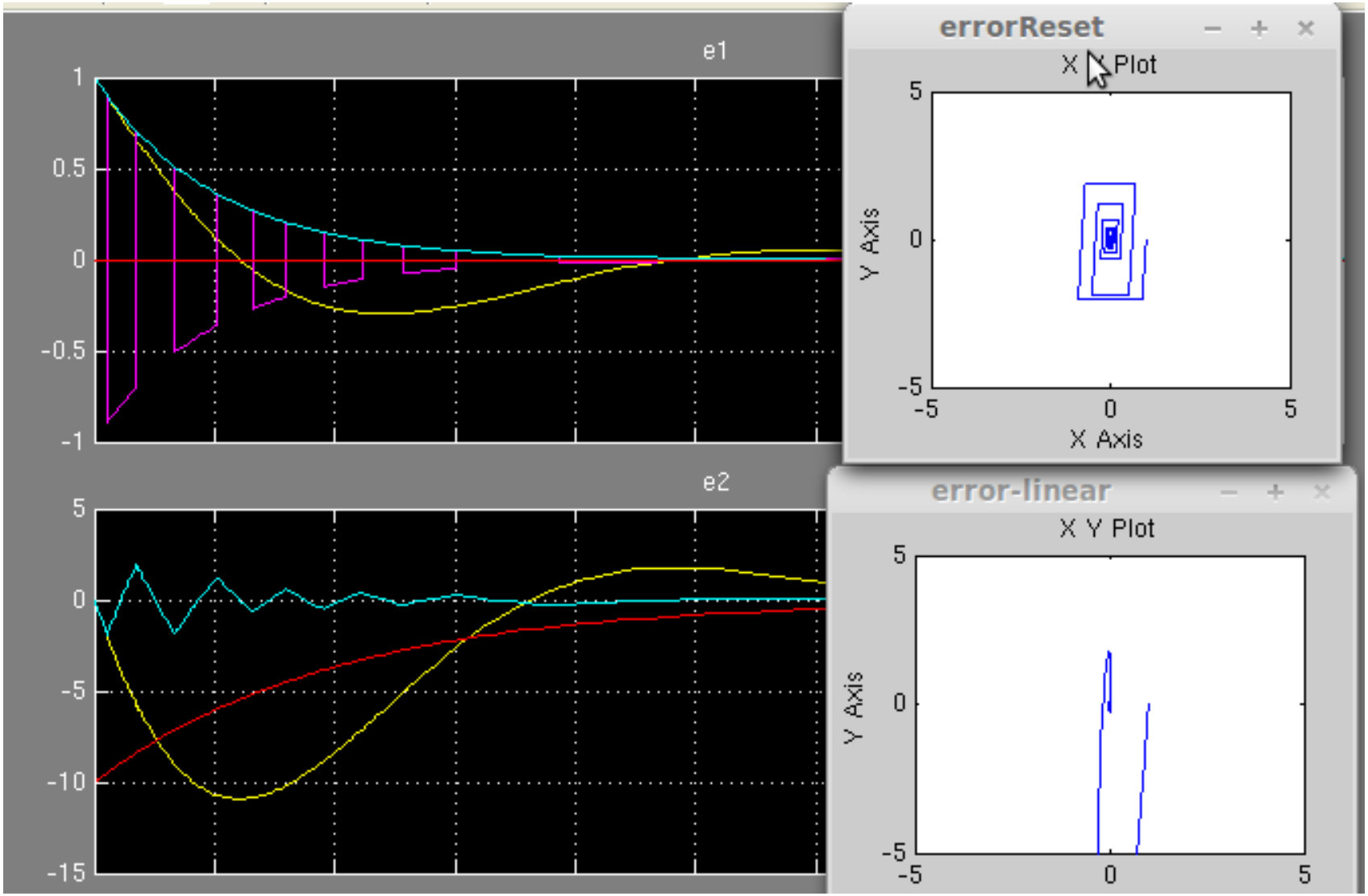
- Control input

$$u = \gamma(x) = -\sin(p) - p - v - \frac{1.6}{3}(2 + |x|)\mu(p + 2v)$$

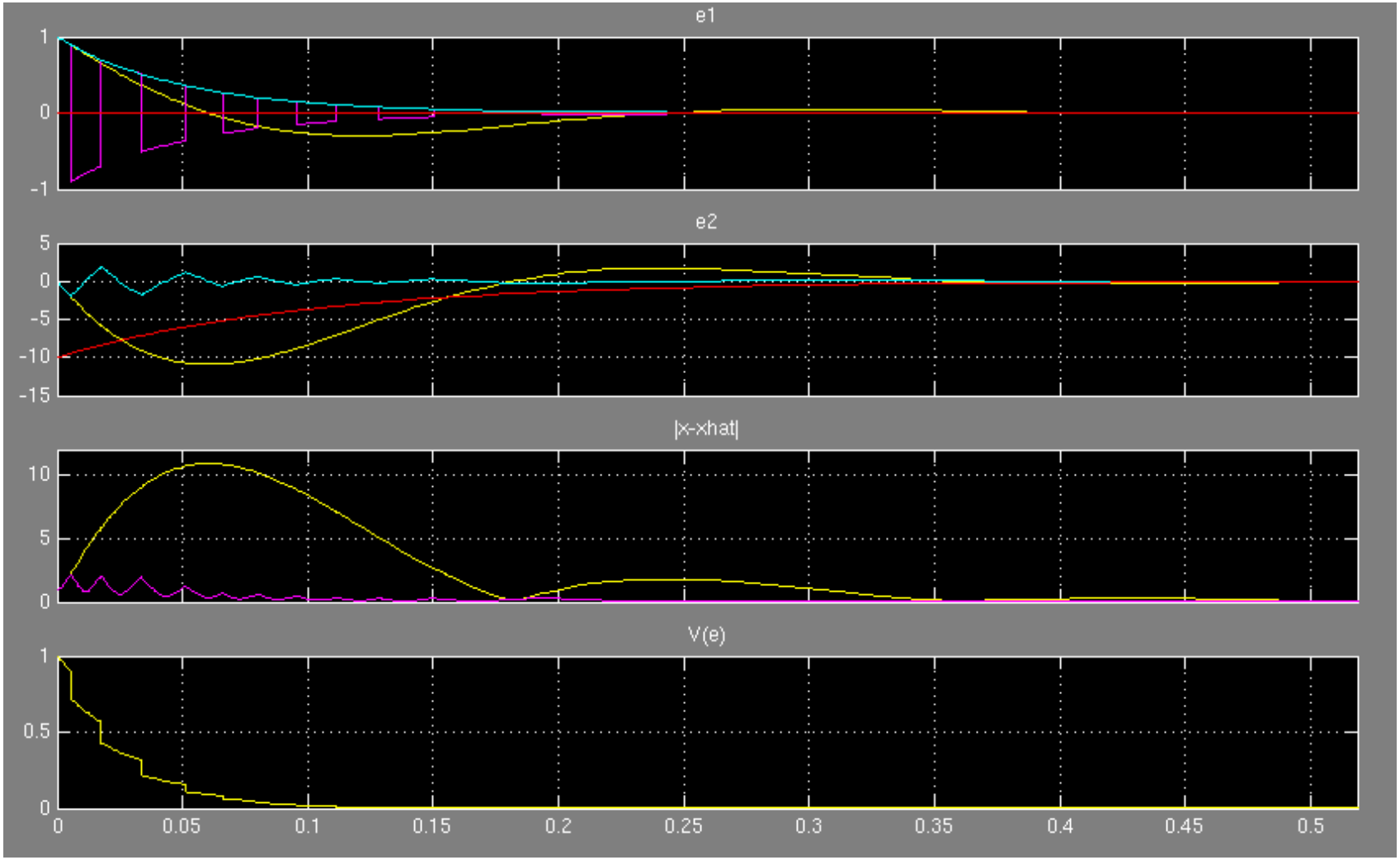
- Parameteres in proposition are $P = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$, $\vartheta = 2$, $\ell = 20$

Various responses with $\alpha = 1$

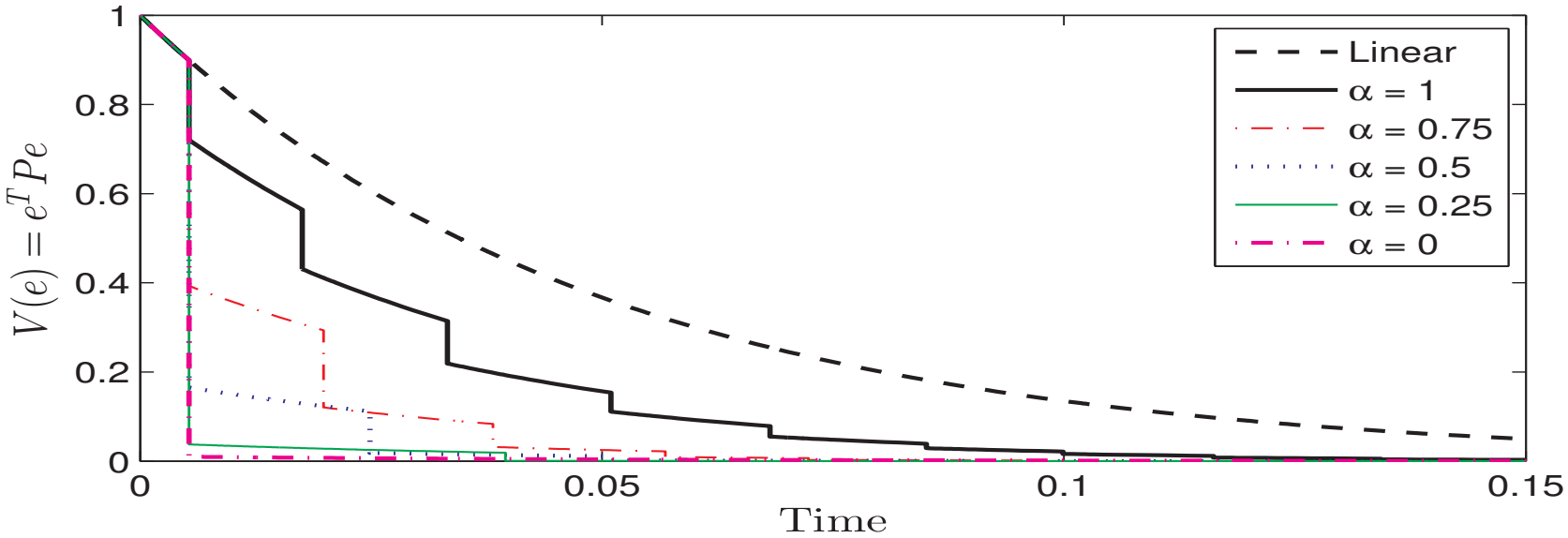
- Responses: Linear, Hybrid, Hybrid Continuous, Laurent

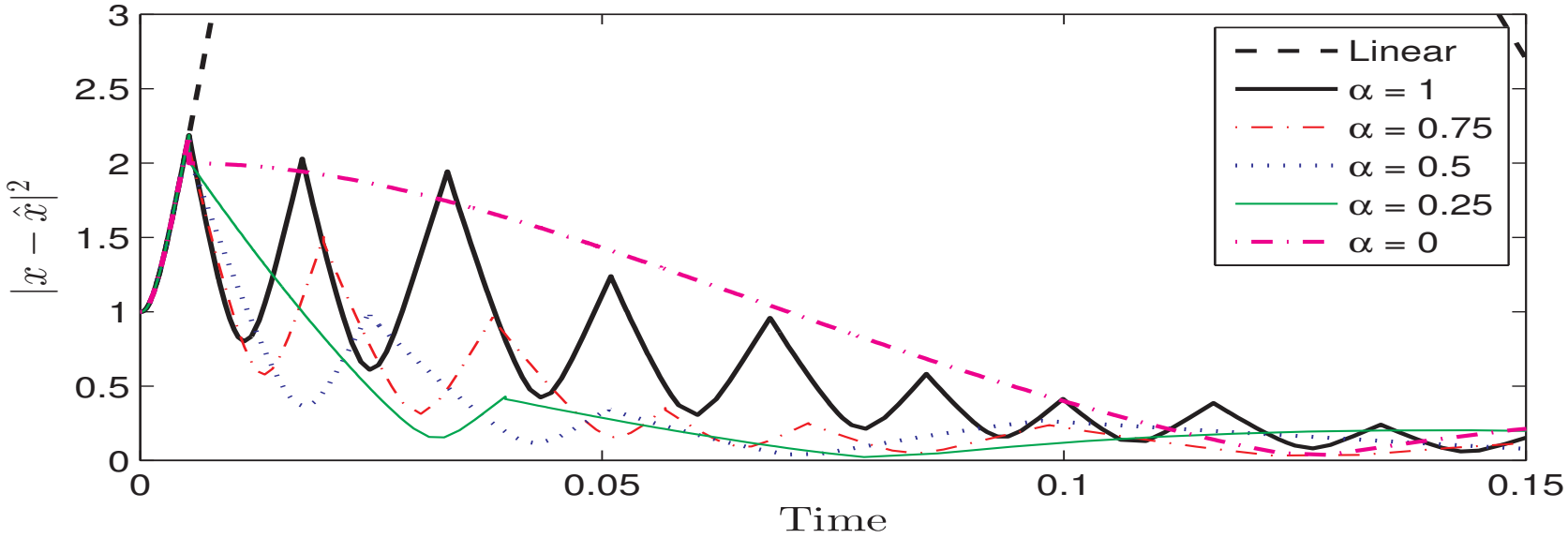


Norm of the error $|x - \hat{x}| \neq |e|$ and value of $V(e)$



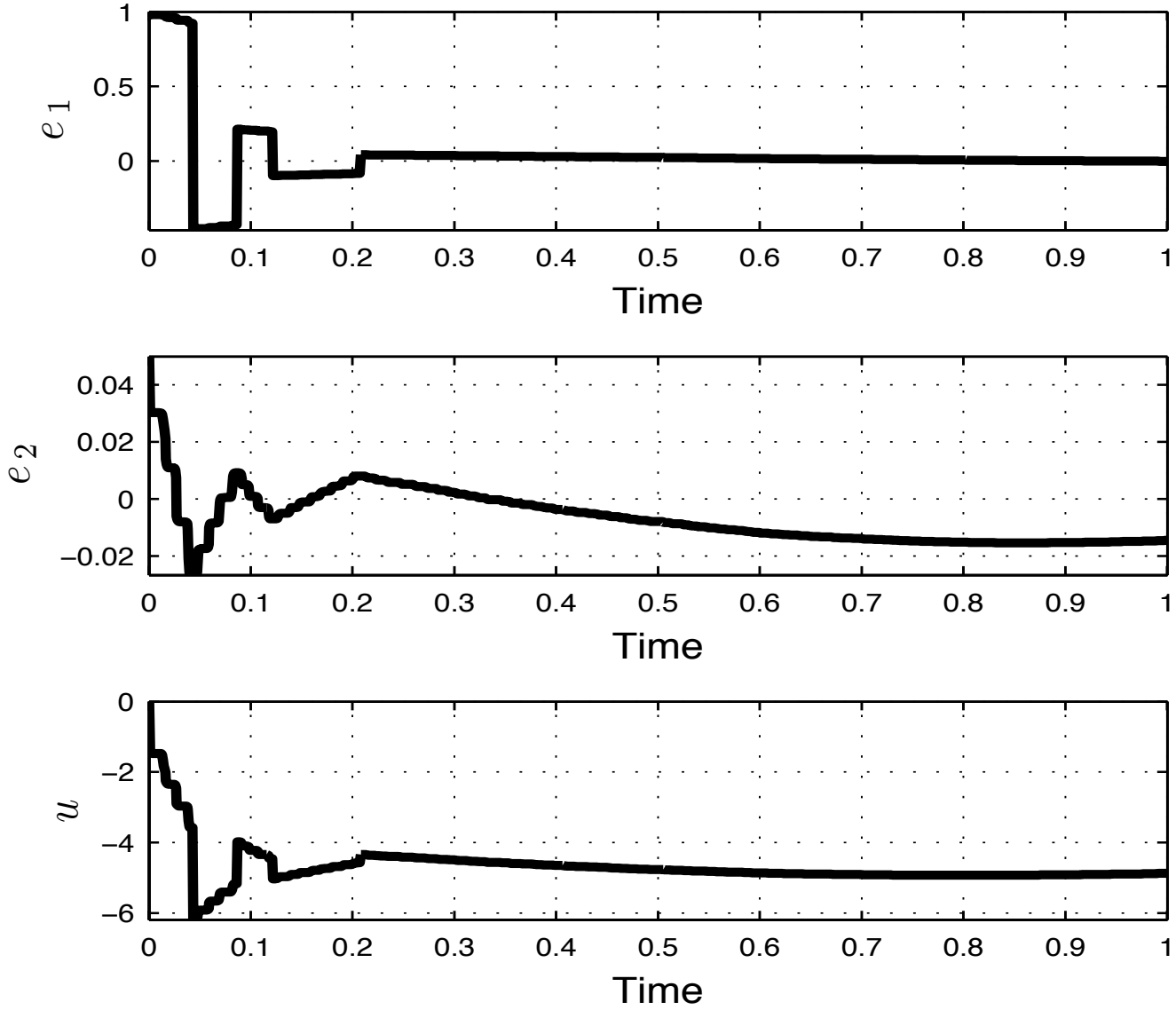
Hybrid response obtained with different α's





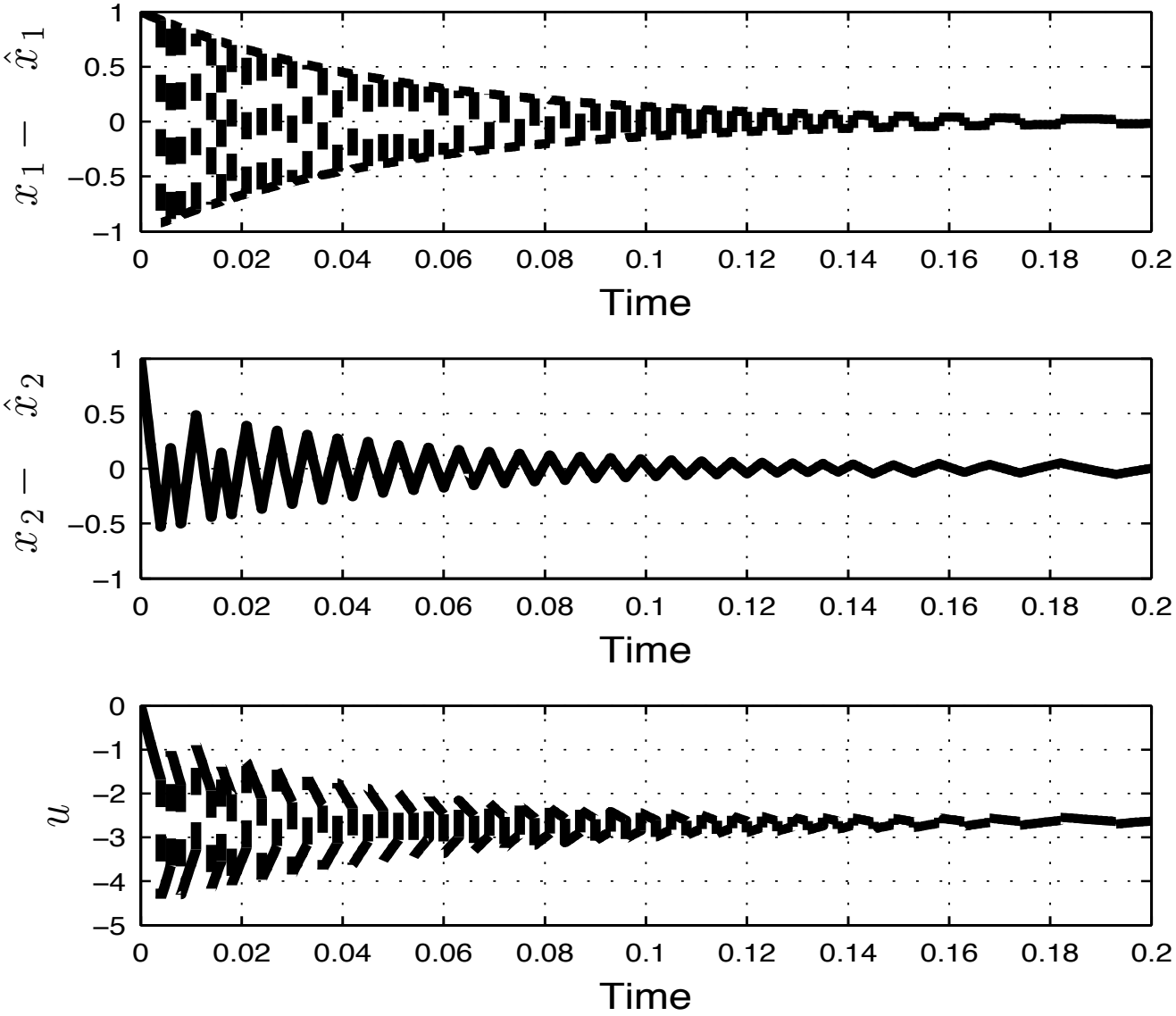
Solution gives discontinuous state estimate ($\alpha = 0.5$)

- This causes a discontinuous plant input u



Solution gives discontinuous state estimate ($\alpha = 1$)

- This causes a discontinuous plant input u



For $\alpha = 1$ can recover continuity of estimate and of input

- New logic variables $q_i \in \{-1, 1\}$, $i = 1, \ldots, n$ toggling at each jump
- New overall state $\widetilde{\xi}$ comprises old state plus $q = (q_1, \ldots, q_n)$:

$$\widetilde{\xi} = (\xi, q) = (x, \hat{x}, \zeta, \eta, q) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

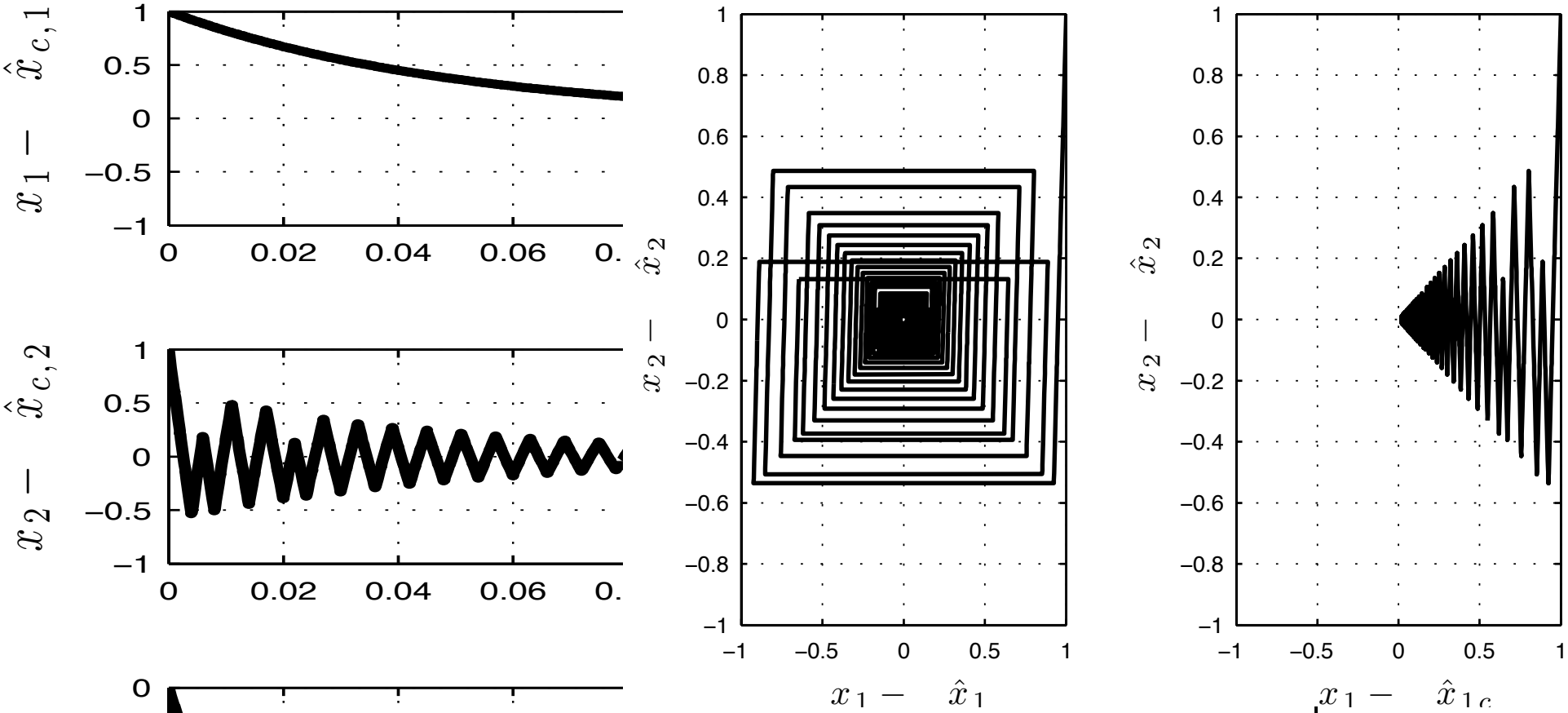
$$\widetilde{\xi} \in \widetilde{\mathcal{D}}_i^-, \begin{cases} \hat{x}_i^+ = g_{\alpha i}(\hat{x}_i, y_i) \\ \zeta_i^+ = \alpha^2 (y_i - \hat{y}_i)^2 \\ \eta_i^+ = 0, \\ q_i^+ = -q_i, \end{cases} ; \quad \widetilde{\xi} \in \widetilde{\mathcal{D}}_i^+, \begin{cases} \hat{x}_i^+ = \hat{x}_i \\ \zeta_i^+ = (y_i - \hat{y}_i)^2 \\ \eta_i^+ = 0, \\ q_i^+ = q_i, \end{cases}$$

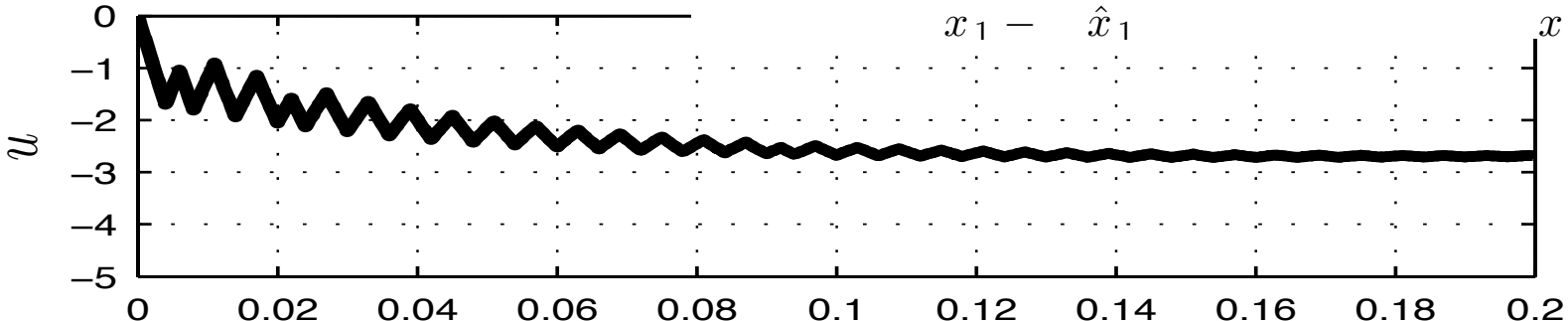
$$\widetilde{\xi} \in \widetilde{\mathcal{C}}, \begin{cases} \dot{\hat{x}}_i = f_i(\hat{x}, y) \\ \dot{\zeta}_i = 0 \\ \dot{\eta}_i = (y_i - \hat{y}_i)^2, \\ \dot{q}_i = 0, \end{cases} \qquad \begin{aligned} \widetilde{\mathcal{C}} &= \mathcal{C} \times \{-1, 1\}^n \\ \widetilde{\mathcal{D}}_i^+ &= \mathcal{D}_i^+ \times \{-1, 1\} \\ \widetilde{\mathcal{D}}_i^- &= \mathcal{D}_i^- \times \{-1, 1\} \end{aligned}$$

- Select "continuous" output $\hat{x}_{ic} = \begin{bmatrix} q_i \hat{p}_i + (1 - q_i) y_i \\ \hat{v}_i \end{bmatrix}$
- Continuity (wrt ordinary time t) only makes sense because of ADT

For $\alpha = 1$ can recover continuity of estimate and of input

- Use logic variable $q \in \{-1, 1\}$ toggling at each jump





Laurent's reduced order observer provides elegant solution

Theorem Consider plant and continuous-time observer:

$$\begin{cases} \dot{y} = f_y(x,y), \\ \dot{x} = f_x(x,y) \end{cases} \qquad \begin{cases} \dot{\hat{y}} = f_y(\hat{x},\hat{y}) - K_y e_y, \\ \dot{\hat{x}} = f_x(\hat{x},\hat{y}) - K_x e_y, \end{cases}$$

and assume that for some $\bar{P} = \begin{bmatrix} P_y & P_{xy}^T \\ P_{xy} & P_x \end{bmatrix} > 0$ and $V(e) = e^T \bar{P} e$:

$$\dot{V} = \begin{bmatrix} e_y \\ e_x \end{bmatrix}^T \begin{bmatrix} P_y & P_{xy}^T \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} f_y(x,y) - f_y(\hat{x},\hat{y}) + k_y e_y \\ f_x(x,y) - f_x(\hat{x},\hat{y}) + k_x e_y \end{bmatrix} < 0$$

for all $\hat{x}, \hat{y}, x, y$ such that $e = (e_x, e_y) \neq 0$.
Then using $\omega = x + P_x^{-1} P_{xy} y$, the reduced order observer

$$\dot{\hat{\omega}} := f_x(\hat{\omega} - P_x^{-1} P_{xy} y, y) + P_x^{-1} P_{xy} f_y(\hat{\omega} - P_x^{-1} P_{xy} y, y),$$

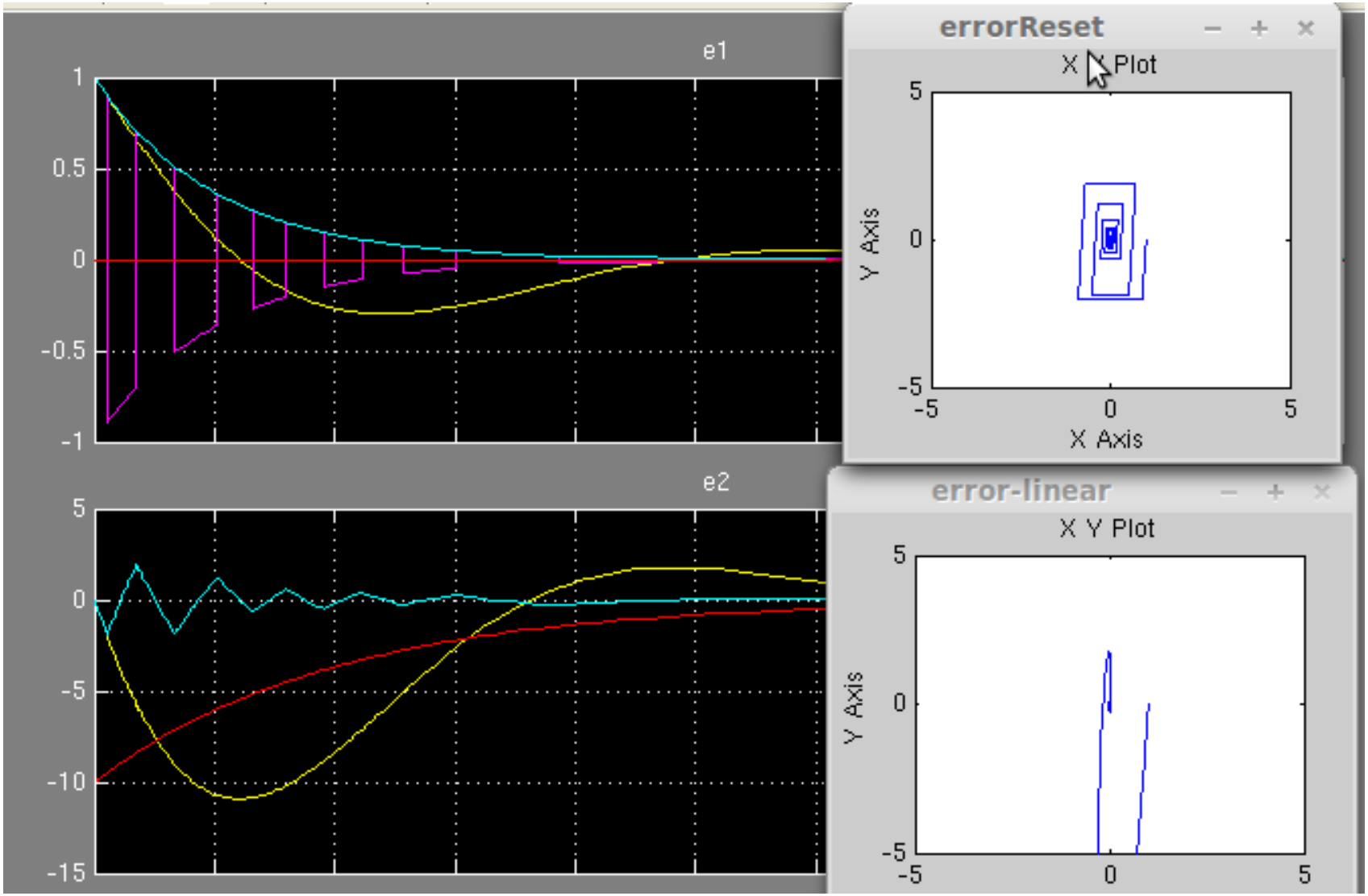
provides an asymptotic estimate $\begin{bmatrix} \hat{y} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} y \\ \hat{\omega} - P_x^{-1} P_{xy} y \end{bmatrix}$.

<u>Note</u>: $\hat{x} - x = \hat{\omega} - P_x^{-1} P_{xy} y - \omega + P_x^{-1} P_{xy} y = \hat{\omega} - \omega$

<u>Note</u>: $V_\omega = (\hat{\omega} - \omega)^T P_x (\hat{\omega} - \omega) = (\hat{x} - x)^T P_x (\hat{x} - x)$ decreases

Various responses with $\alpha = 1$ (revisited)

- Responses: Linear, Hybrid, Hybrid Continuous, Laurent

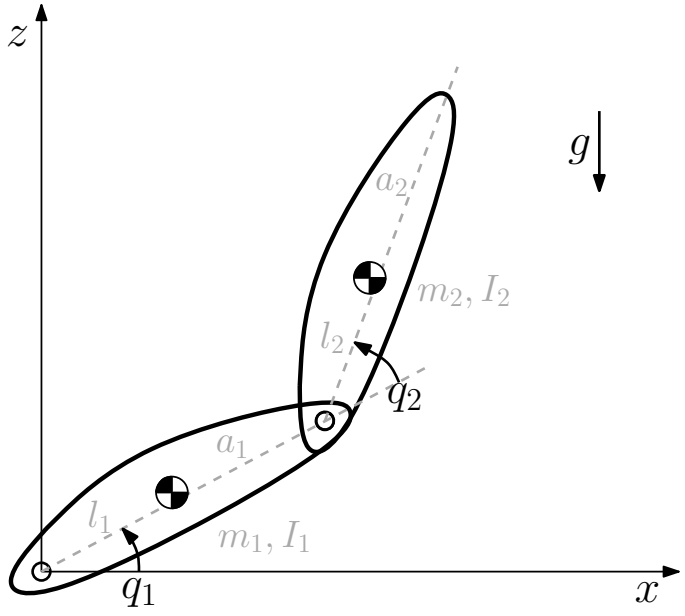


Example with $n = 2$: two link planar manipulator

- Dynamic equations:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + h(q) = u$$

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix} \ddot{q} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \dot{q} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = u,$$



$$\begin{aligned}
c_{11} &= -m_2 a_1 l_2 \sin(q_2)\dot{q}_2, \\
c_{12} &= -m_2 a_1 l_2 \sin(q_2)(\dot{q}_1 + \dot{q}_2), \\
c_{21} &= m_2 a_1 l_2 \sin(q_2)\dot{q}_1, \\
d_{12} &= l_2 + m_2(l_2^2 + a_1 l_2 \cos(q_2)), \\
d_{22} &= l_2 + m_2 l_2^2, \\
d_{11} &= l_1 + m_1 l_2^2 + l_2 + m_2(a_1^2 + l_2^2 + 2a_1 l_2 \cos(q_2)), \\
h_1 &= g(m_1 l_1 + m_2 a_1)\cos(q_1) + g m_2 l_2 \cos(q_1 + q_2), \\
h_2 &= g m_2 l_2 \cos(q_1 + q_2).
\end{aligned}$$

Link	l_i $[m]$	m_i $[kg]$	I_i $[kgm^2]$	a_i $[m]$
1	0.5	6	0.2	1
2	0.25	5	0.1	0.5

Limiting speed satisfies Lipschitz assumption

- Controller is PD+gravity compensation: $\gamma(\hat{x}) = h(q) - k_p\hat{q} - k_d\dot{\hat{q}}$:

$$\begin{aligned}\phi(x, u = \gamma(x)) &= D(p)^{-1}(-C(q,\dot{q})\dot{q} + h(q) + u) \\ &= D(p)^{-1}(-C(q,\dot{q})\dot{q} - K_p q - K_d\dot{q})\end{aligned}$$

- Lipschitz bound on $\delta(x, \hat{x})$ using $|\dot{q}| \leq q_{MAX}$:

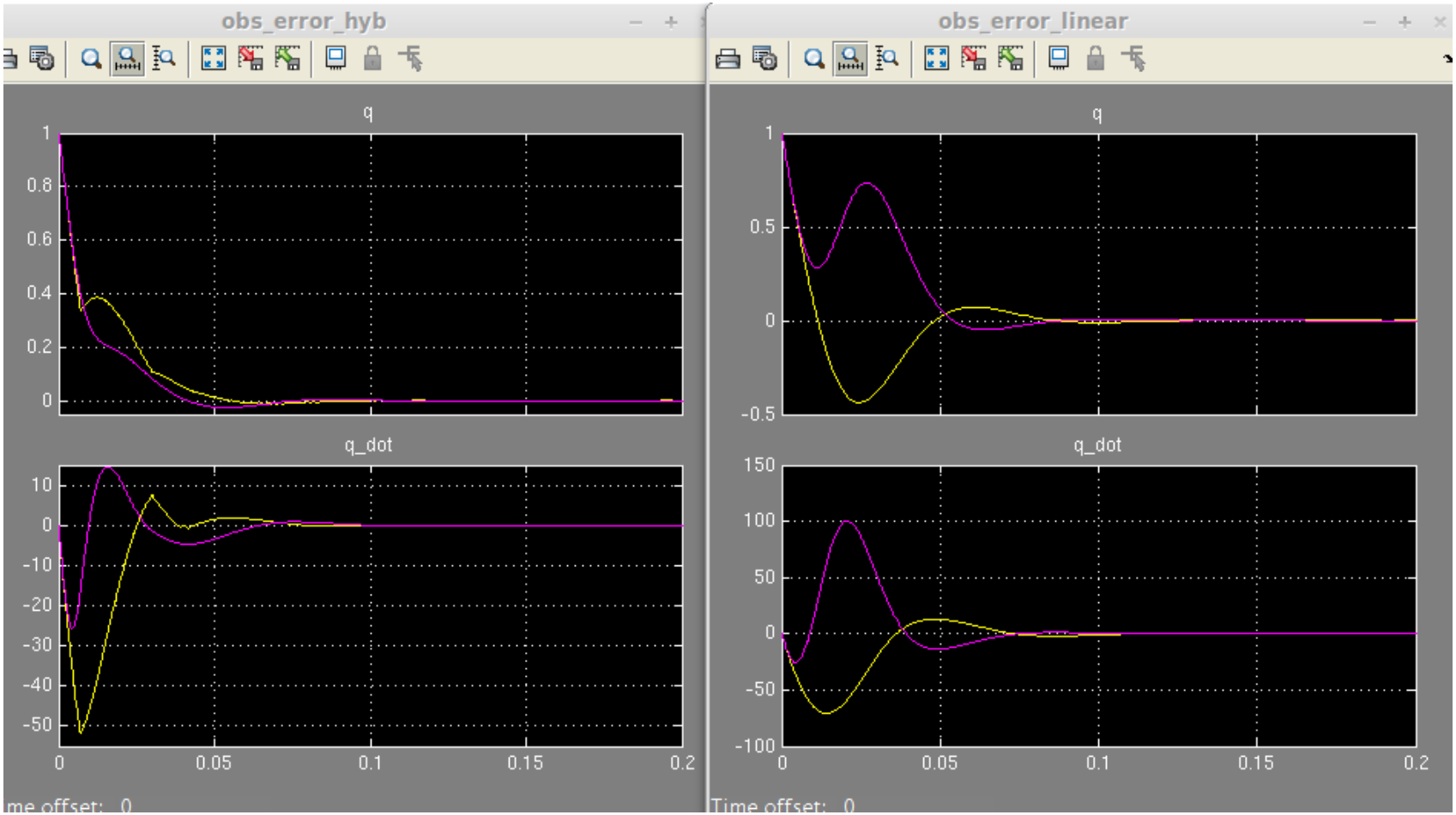
$$\begin{aligned}|\delta(x,\hat{x})| &\leq |D(p)^{-1} - D(\hat{p})^{-1}||K_p\hat{q} + K_d\dot{\hat{q}} + C(\hat{q},\dot{\hat{q}})\dot{\hat{q}}| \\ &\quad |D(\hat{p})^{-1}||C(\hat{q},\dot{\hat{q}})\dot{\hat{q}} - C(q,\dot{q})\dot{q}|\end{aligned}$$

- Using the mean value theorem on the red terms and setting $\dot{q}_{MAX} \approx 100$ deg/s, we get $L_\delta = 810$
- Selecting a feasible P and solving an LMI problem coming from Proposition, we get the super-high gain $\ell = 3600$
- We fix $\ell = 100$ to avoid crazy simulations
- With this selection Laurent's solution does not work (which P?)

Responses with $\alpha = 1$ and continuous estimate

- Responses: First Joint, Second joint

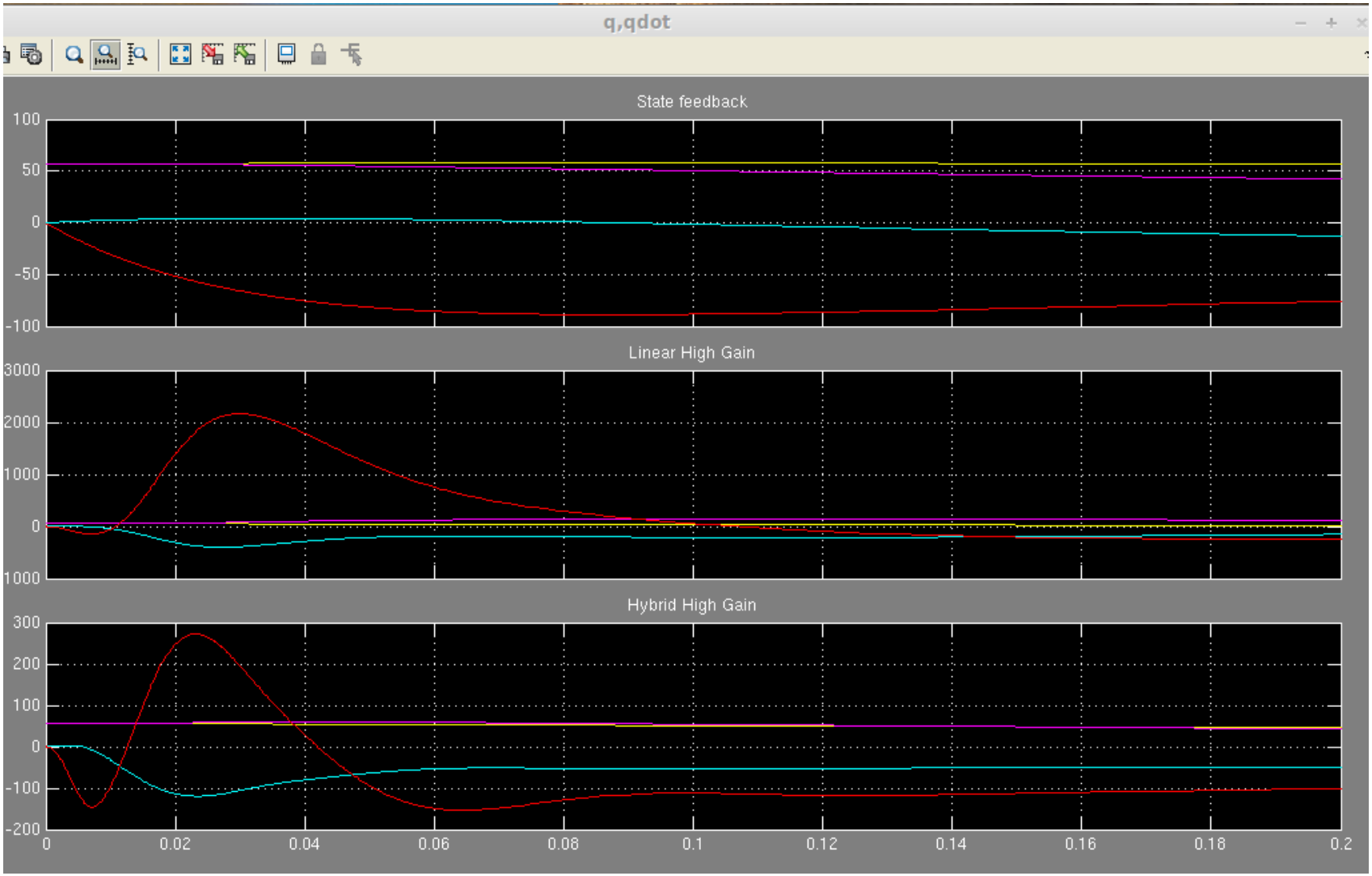
Hybrid | Linear



- Note the large peak in the second joint velocity

Responses with $\alpha = 1$ and continuous estimate

- Responses: Position 1, Position 2, Velocity 1, Velocity 2



Summary and outlook

Summary

- Hybrid jumps in observer states can reduce peaking:
  - Discontinuous estimates lead to faster convergence
  - For $\alpha = 1$ can produce continuous estimates
- A toy problem to play with hybrid dynamical systems

Possible extensions

- Establish a formal statement of "no peaking":
  - Perhaps show that e^2 does not increase along solutions
- Extend to higher dimensional systems
  - Given Hurwitz $A_e = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix}$ can we always find $P = \begin{bmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix}$?
- Test effectiveness of solution on larger dimensional systems
  - For example larger robot manipulators
- Perform more thorough comparison with Laurent's solution
  - Direct use of y may be undesirable?