Delay-independent stability via a reset loop

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L2S Paris, 20-22 November 2012

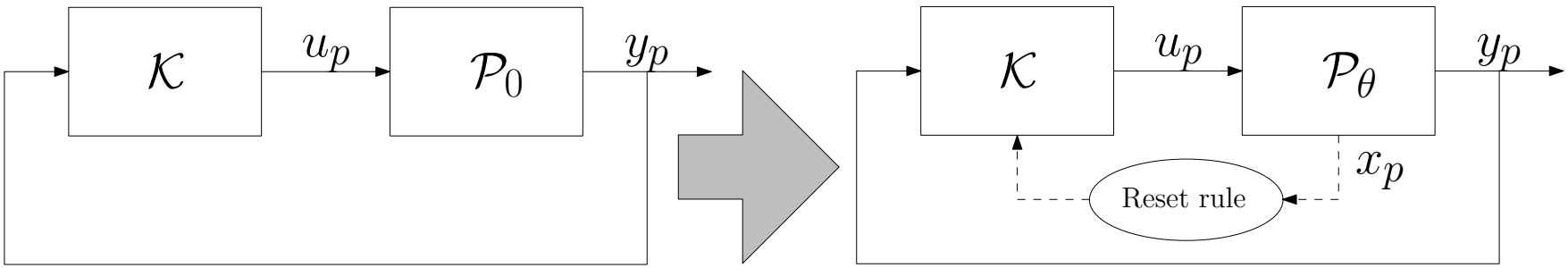
Outline

1. Overview of dynamical hybrid systems framework of Teel et al.
2. Problem statement and proposed solution
3. Numerical example
4. Concluding remarks

Goal of this work

- Context. Employ tools from **Hybrid dynamical systems** within control of **Linear time-delay plants**

  Not much work done in this context – recent work [Liu, Teel CDC2012]
- Objective. Use *hybrid loops or resets* for stability **recovery**:
  - ▷ start from a *linear closed-loop* where controller $\mathcal{K}$ stabilizes the plant $\mathcal{P}_0$ without any time delay;
  - ▷ recover (delay-independent) *global exponential stability* with time-delay plant $\mathcal{P}_\theta$ by enforcing suitable *resets* in the controller.



Hybrid techniques reach beyond limits of classical control

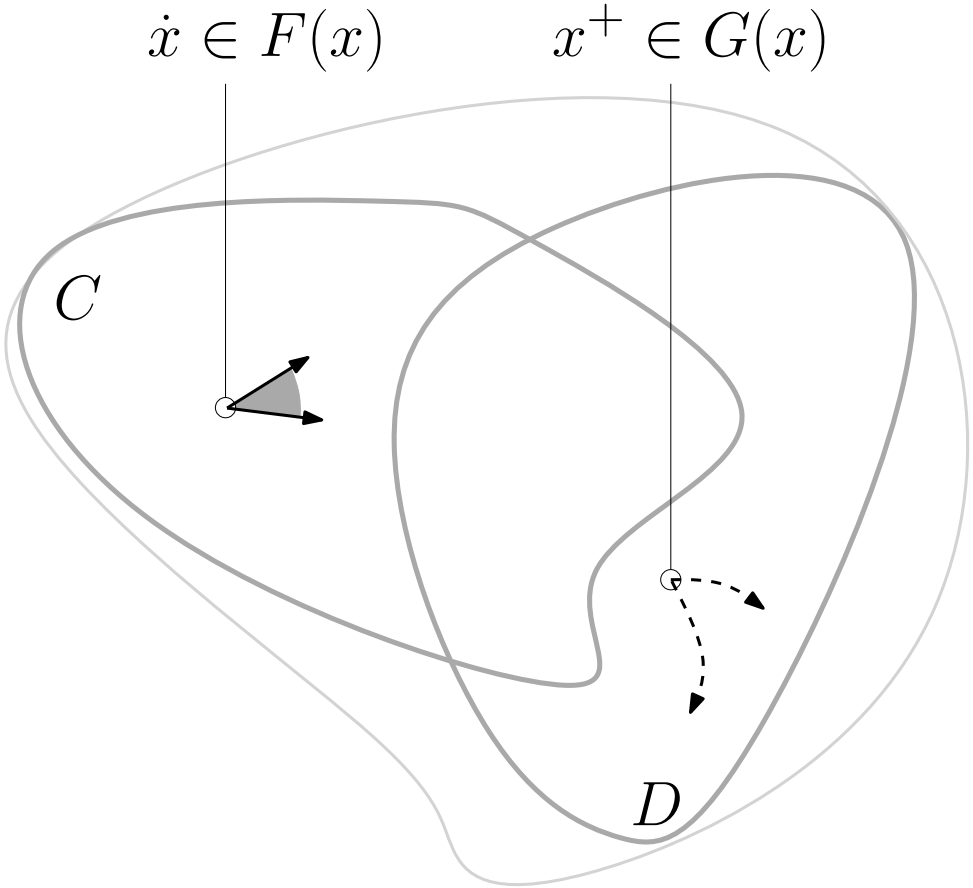
- We will design a hybrid loop to augment to the continuous-time controller by jump rules that recover stability **lost due to the delay**
  - ▷ For this we use reset control systems formalism or, more generally hybrid dynamical systems formalism [Teel, Goebel, Sanfelice 2011]
- The use of hybrid systems is desirable due to their capability to:
  - ▷ provide global asymptotic stability of closed-loops not stabilizable by continuous feedback (see e.g. [Hespanha et al, 1999, 2003]).
  - ▷ guarantee a robustness with respect to small errors in the loop, which cannot be obtained using classical (i.e. with a continuous dynamics) controllers (see e.g. [Prieur 2005, Goebel and Teel, 2009]).
  - ▷ improve the performance of linear systems in presence of disturbances [Becker et al, 2004, Nesic et al, 2008, Witvoet et al, 2007]
  - ▷ More generally, hybrid systems can model a wider range of physical problems and provide improved observers [Allgower et al. 2007, Prieur et al. 2012]

Hybrid dynamical systems review: **dynamics**

$\mathcal{H} = (C, D, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $C \subseteq \mathbb{R}^2$ (flow set)
- $D \subseteq \mathbb{R}^2$ (jump set)
- $F : C \rightrightarrows \mathbb{R}^2$ (flow map)
- $G : D \rightrightarrows \mathbb{R}^2$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}$$



Hybrid dynamical systems review: **continuous dynamics**

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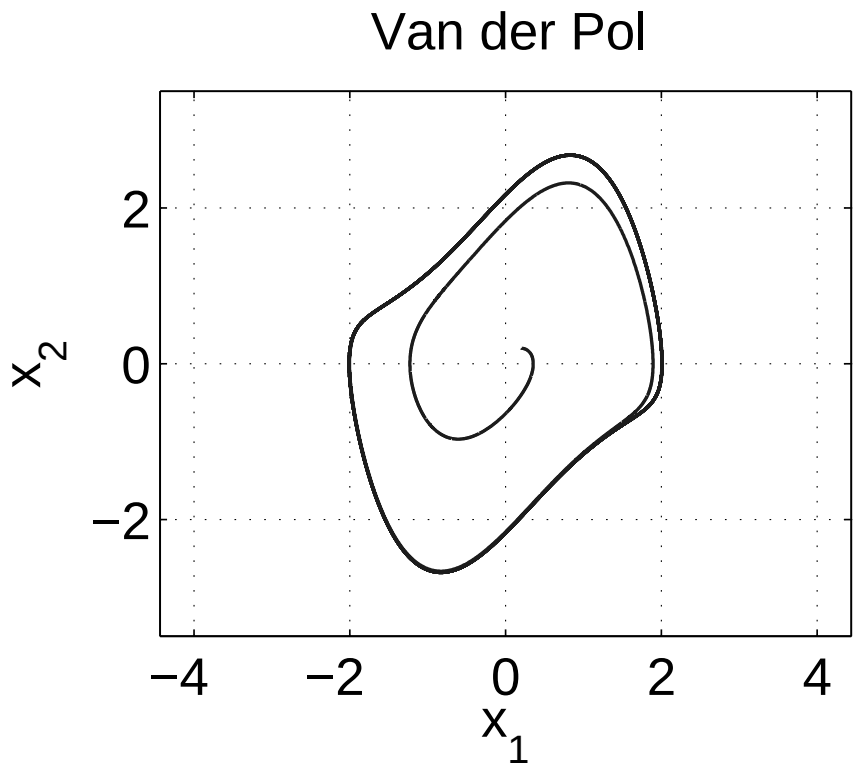
- $F : C \rightrightarrows \mathbb{R}^2$ (flow map)
- $G : D \rightrightarrows \mathbb{R}^2$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}$$

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2) \end{cases}$$



Hybrid dynamical systems review: **discrete dynamics**

$\mathcal{H} = (C, D, F, G)$

- $n \in \mathbb{N}$ (state dimension)
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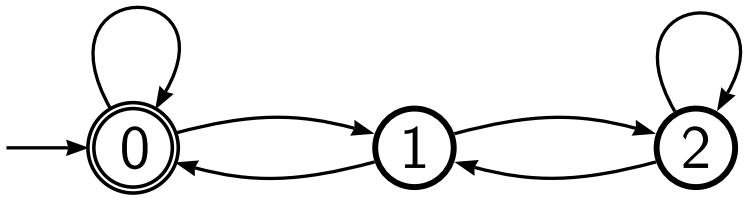
- $F : C \rightrightarrows \mathbb{R}^2$ flow map)
- $G : D \rightrightarrows \mathbb{R}^2$ (jump map)

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}$$

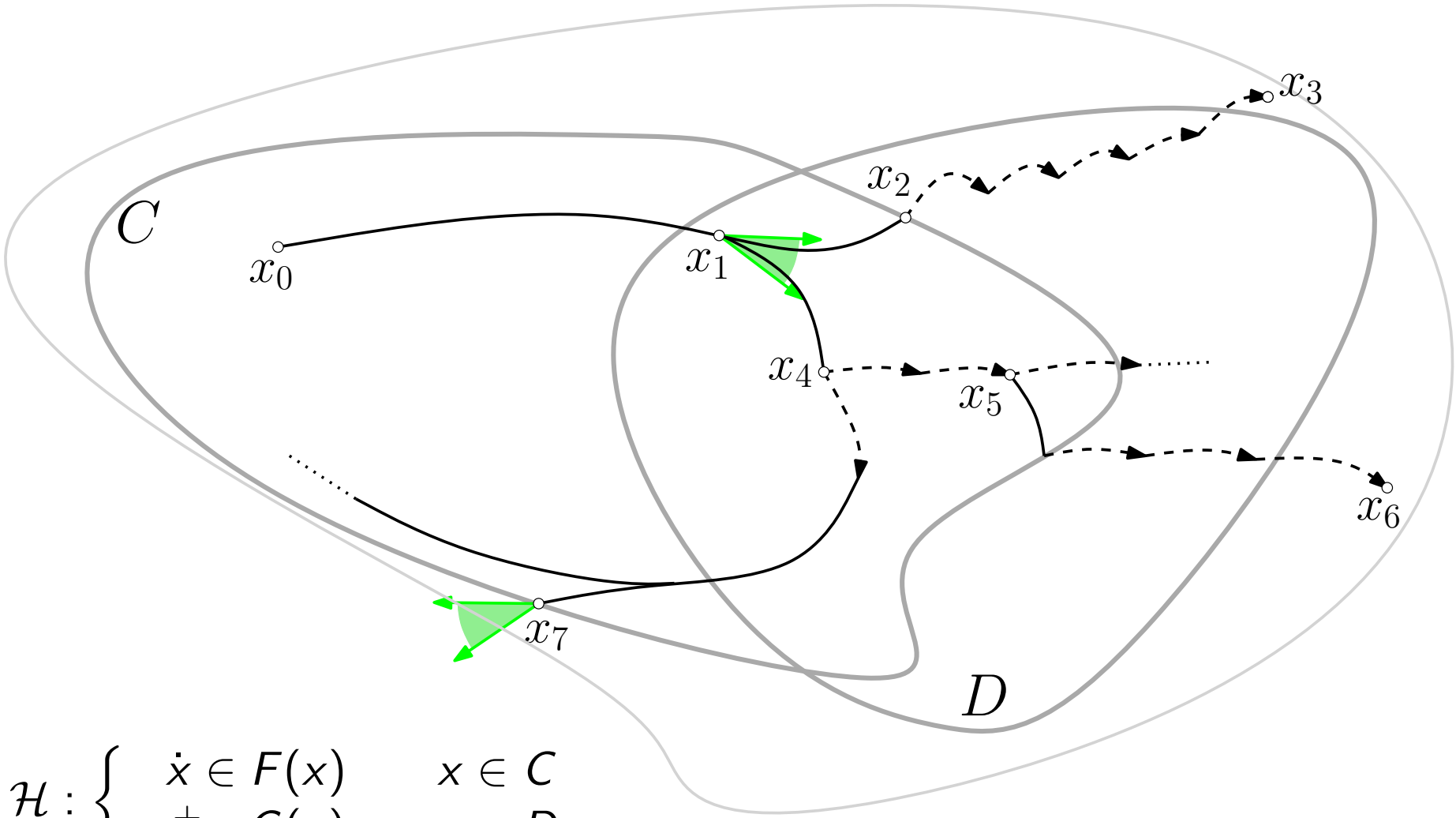
$$x^+ \in \begin{cases} \{0,1\} & \text{if } x = 0 \\ \{0,2\} & \text{if } x = 1 \\ \{1,2\} & \text{if } x = 2 \end{cases}$$



A possible sequence of states from $x_0 = 0$ is:

$$(0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in N$$

Hybrid dynamical systems review: **trajectories**

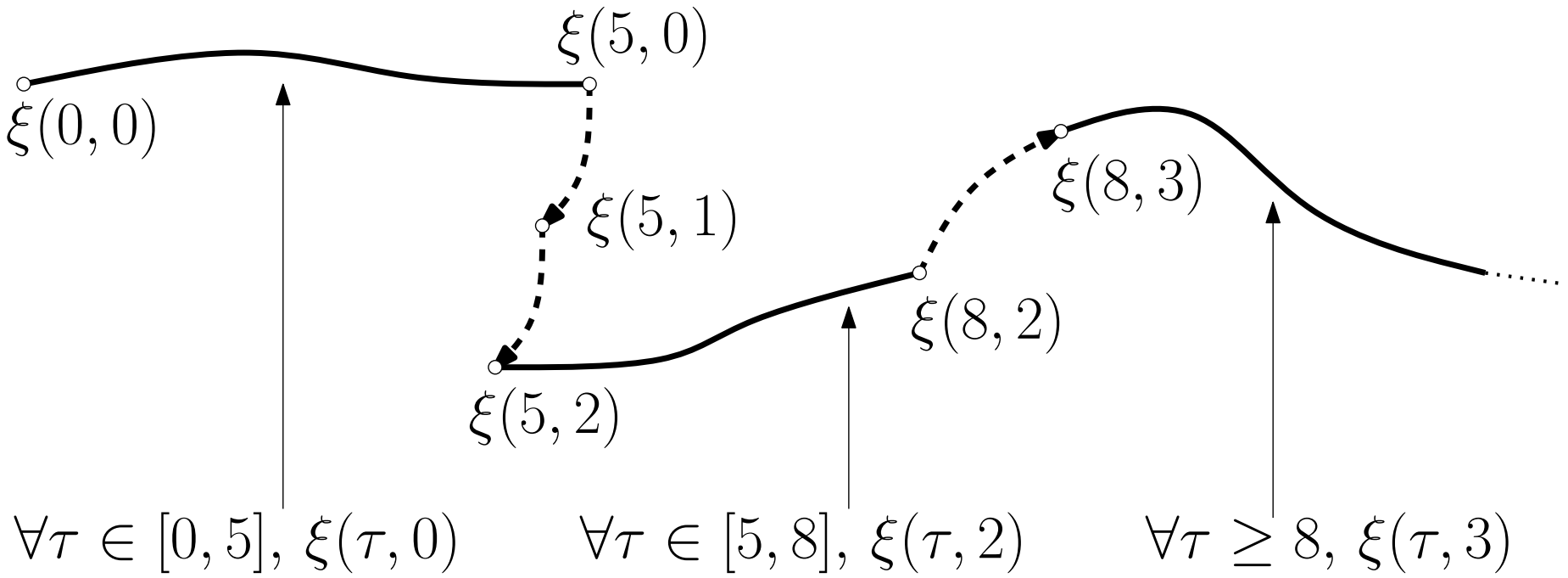


$$\mathcal{H}: \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}$$

Hybrid dynamical systems review: **hybrid time**

The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.



Hybrid dynamical systems review: **hybrid time**

$E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

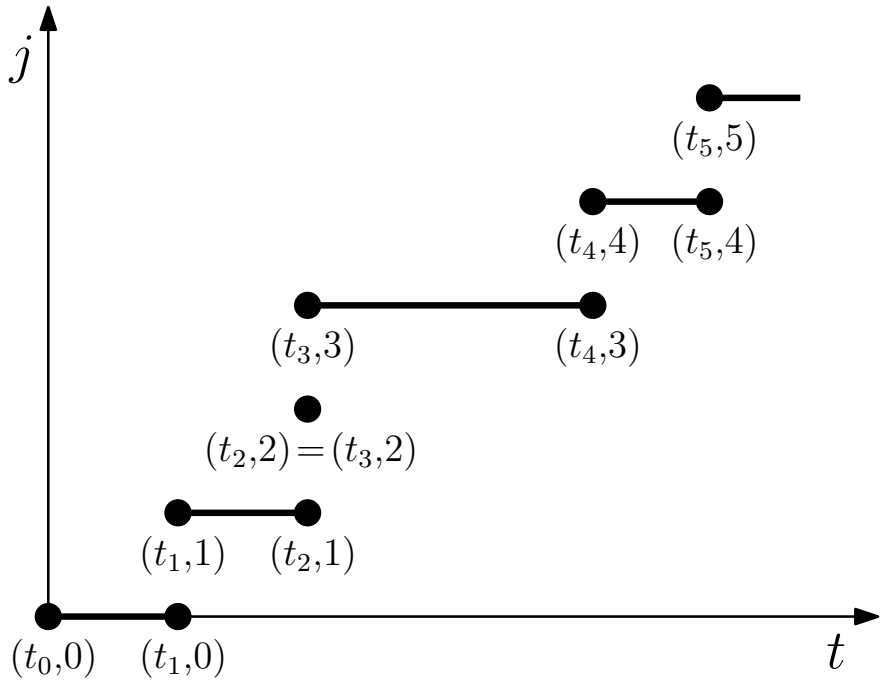
$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\})$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

E is a **hybrid time domain** if for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

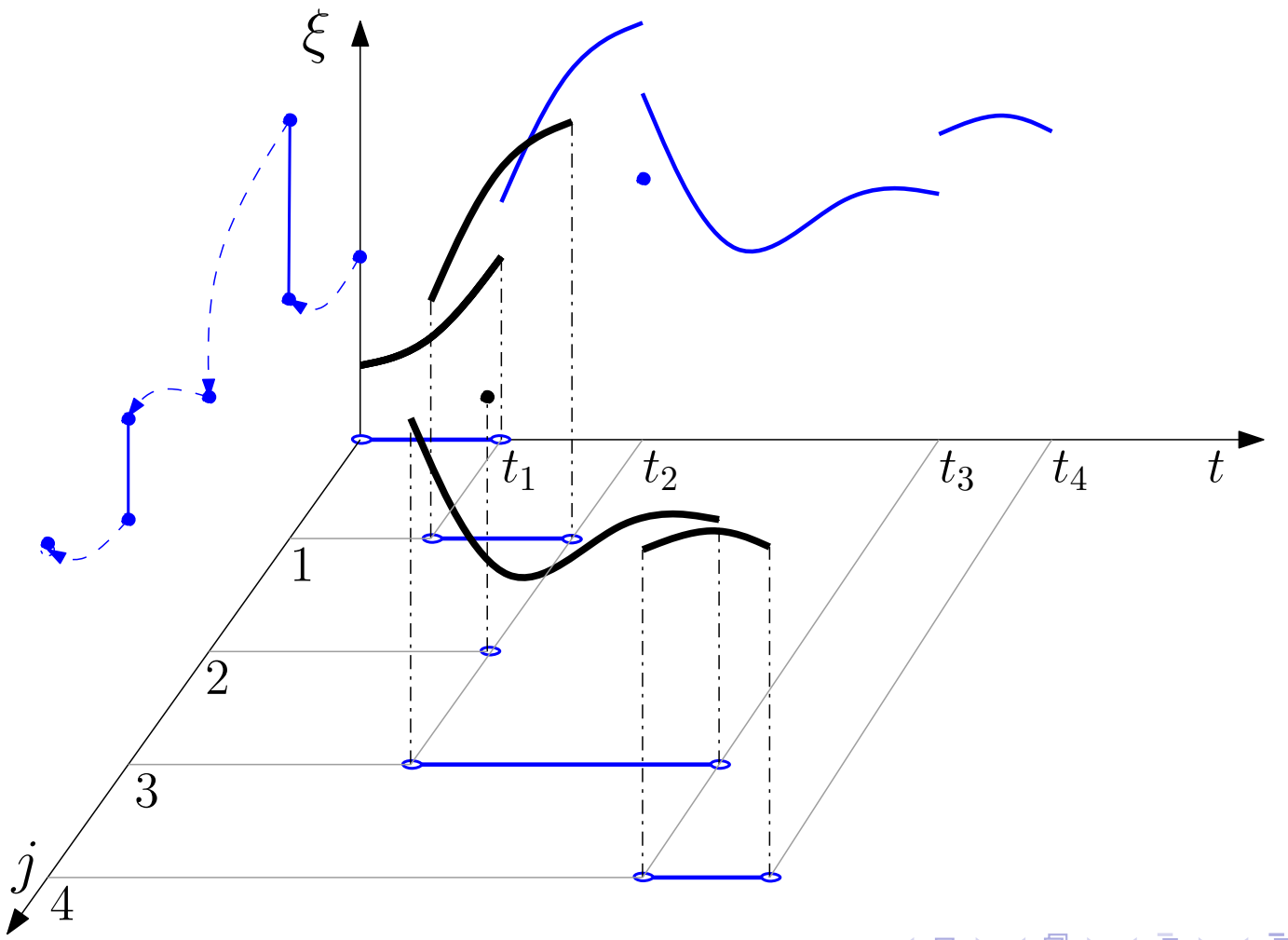
$$E \cap ([0, T] \times \{0, 1, \ldots, J\})$$

is a compact hybrid time domain.



Hybrid dynamical systems review: **solution**

- Formally, a solution satisfies the flow dynamics when flowing and satisfies the jump dynamics when jumping



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- Objective. Use *hybrid loops or resets* for stability **recovery**:
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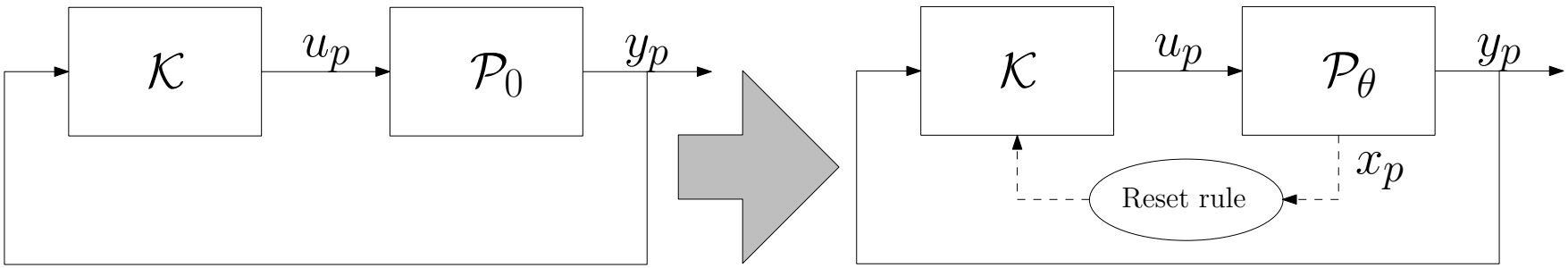


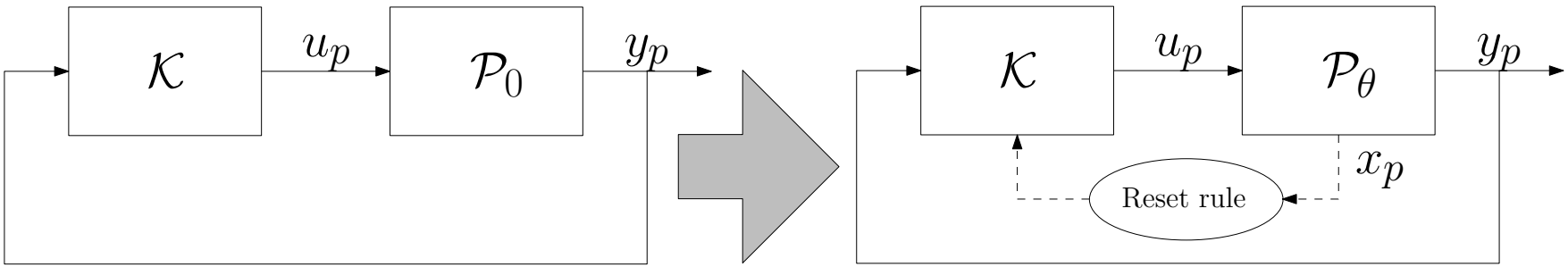
Plant and controller dynamics is linear

- Continuous-time linear plant with time-delay θ in the state:

$$\mathcal{P}_\theta \begin{cases} \dot{x}_p = A_p x_p + A_{pd} x_p(t-\theta) + B_p u_p \\ y_p = C_p x_p \end{cases}$$

- Continuous-time controller stabilizing $\mathcal{P}_0$ is:

$$\mathcal{K} \begin{cases} \dot{x}_c = A_c x_p + B_c y_p \\ u_p = C_c x_c \end{cases}$$



Problem Formulation

- Continuous-time closed-loop with delay without resets **may be unstable**:

$$\begin{cases} \dot{x}_p = A_p x_p + A_{pd} x_p(t-\theta) + B_p C_c x_c \\ \dot{x}_c = A_c x_c + B_c C_p x_p \end{cases}$$

- Define the augmented state $x = \begin{bmatrix} x_p \\ x_c \\ x_p(t-\theta) = x_{pd} \end{bmatrix}$
- **Problem:** Determine K_p, $\mathcal{C}$, $\mathcal{D}$ augmenting the above closed loop as

$$\left.\begin{array}{rcl} \dot{x}_p &=& A_p x_p + A_{pd} x_{pd} + B_p C_c x_c \\ \dot{x}_c &=& A_c x_c + B_c C_p x_p \end{array}\right\} \quad x \in \mathcal{C}$$

$$\left.\begin{array}{rcl} x_p^+ &=& x_p \\ x_c^+ &=& K_p x_p \end{array}\right\} \quad x \in \mathcal{D}$$

  with the goal to **recover global asymptotic stability** of the origin.
- Recall: $\mathcal{C}$, $\mathcal{D}$ are the flow and jump sets.

Delay-independent results require assumption on $\mathcal{P}_\theta$

- $\mathcal{P}_\theta$ must be good enough to allow for Lyapunov-Krasovskii functionals
- Assumption LK. Given the matrices in $\mathcal{P}_\theta$, there exist two symmetric positive definite matrices P_p and Q, a positive scalar ϵ_p and a "hybrid gain" K_p such that

$$\begin{bmatrix} (A_p + B_p C_c K_p)' P_p + P_p (A_p + B_p C_c K_p) + Q & P_p A_{pd} \\ A_{pd}' P_p & -Q \end{bmatrix} \leq -2\epsilon_p \begin{bmatrix} P_p & 0 \\ 0 & Q \end{bmatrix}$$

- Meaning of Assumption LK: delay-independent stability of the linear time-delay plant stabilized by K_p:

$$\dot{z}(t) = (A_p + B_p C_c K_p) z(t) + A_{pd} z(t-\theta) = A_K z(t) + A_{pd} z(t-\theta)$$

- z dynamics coincides with the closed loop dynamics with delay when clamping $x_c = K_p x_p$

Convex formulation helps optimizing P_p, K_p, Q, ϵ_p

- Lemma LK Assumption LK holds with $|K_p| \leq \kappa_{max}$ if and only if there exist two symmetric positive definite matrices Q_p and S, a positive scalar ϵ_p and a matrix X such that:

$$|X| \leq \kappa_{max}, \qquad Q_p \geq I$$

$$\begin{bmatrix} Q_p A_p' + X' C_c' B_p' + A_p Q_p + B_p C_c X + S & A_{pd} Q_p \\ Q_p A_{pd}' & -S \end{bmatrix} \leq -2\epsilon_p \begin{bmatrix} Q_p & 0 \\ 0 & S \end{bmatrix}$$

- One then can build the following solution to Assumption LK

$$K_p = X Q_p^{-1}, \quad Q = Q_p^{-1} S Q_p^{-1}, \quad P_p = Q_p^{-1}$$

- Maximizing ϵ_p is a Generalized Eigenvalue Problem (qausi-convex)
- Recall that $x_c \equiv K_p x_p$ induces exponentially stable dynamics $\dot{z}(t) = (A_p + B_p C_c K_p) z(t) + A_{pd} z(t-\theta) = A_K z(t) + A_{pd} z(t-\theta)$

Problem solution: use resets to go back to z dynamics

- **Solution**: define the coordinate $\xi = [x_p \;\; x_{pd} \;\; x_c - K_p x_p]^T$ and select $K_p = K_p$ from Lemma LK and for any $\epsilon_c > 0$ fix

$$\mathcal{C} = \left\{ \xi' \mathrm{He} \begin{bmatrix} P_p A_K + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \leq \xi' \begin{bmatrix} -\epsilon_p P_p & 0 & 0 \\ 0 & -\epsilon_p Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \right\}$$

$$\mathcal{D} = \left\{ \xi' \mathrm{He} \begin{bmatrix} P_p A_K + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \geq \xi' \begin{bmatrix} -\epsilon_p P_p & 0 & 0 \\ 0 & -\epsilon_p Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \right\}$$

- Recall hybrid scheme where $x = [x_p \; x_c \; x_{pd}]^T$

$$\left.\begin{array}{rcl} \dot{x}_p &=& A_p x_p + A_{pd} x_{pd} + B_p C_c x_c \\ \dot{x}_c &=& A_c x_c + B_c C_p x_p \end{array}\right\} \quad x \in \mathcal{C}$$

$$\left.\begin{array}{rcl} x_p^+ &=& x_p \\ x_c^+ &=& K_p x_p \end{array}\right\} \quad x \in \mathcal{D}$$

- **Interpretation**: if $x \in \mathcal{D}$, jumps ensure $x_c^+ = K_p x_p$ (i.e., back to the "good" z dynamics: $\dot{z} = (A_p + B_p C_c K_p) z + A_{pd} z_d$

Idea behind proof of (hybrid) global exponential stability

- Recall definition of flow $\mathcal{C}$ and jump $\mathcal{D}$ sets:

$$\mathcal{C} = \left\{ \xi' \mathrm{He} \begin{bmatrix} P_p A_K + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \leq \xi' \begin{bmatrix} -\epsilon_p P_p & 0 & 0 \\ 0 & -\epsilon_p Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \right\}$$

$$\mathcal{D} = \overline{(\mathbb{R}^n \times \mathbb{R}^{n_c} \times \mathbb{R}^n) \setminus \mathcal{C}}$$

- Consider the Lyapunov-Krasovskii functional (with suitable $\lambda > 0$):

$$V(x_t) = x_p' P_p x_p + \int_{t-\theta}^{t} x_p'(\tau) Q x_p(\tau) d\tau + \lambda (x_c - K_p x_p)'(x_c - K_p x_p)$$

- During **flows**, since $x \in \mathcal{C}$, from Assumption KL (for small $\lambda > 0$):

$$\dot{V}(x_t) \quad \leq \quad -\epsilon |x|^2, \quad \text{for some } \epsilon > 0$$

- Across **jumps**, since $x_p^+ = x_p$, $x_c^+ = K_p x_p$, we get from the blue guy):

$$V(x_t^+) - V(x_t) \quad \leq \quad 0$$

- Stability proof uses **Hybrid Lyapuonv Krasovskii theorem** (next slide)

Lyapunov-Krasovskii thm for a class of hybrid systems

- For a state $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and
$$\|x_t\|_\theta := \max_{s\in[0,\theta]} |x(t-s)|$$
- Theorem LK *Assume that* there exists a functional $V(x_t)$, class $\mathcal{K}_\infty$ functions α_1, α_2 and a positive definite function ρ such that

$$\begin{aligned}
\alpha_1(|x|) \;&\leq\; V(x_t) \;\leq\; \alpha_2(\|x\|_\theta) \\
\dot{V}(x_t) \;&\leq\; -\rho(|x|), \quad \forall x \in \mathcal{C} \\
V(x_t^+) - V(x_t) \;&\leq\; 0, \quad \forall x \in \mathcal{D}
\end{aligned}$$

  and assume that the solutions exhibit *persistent flow*.
  Then the origin of the hybrid time-delay system is **GAS**.
- Alternative formulations appeared in [Banos et al. 2009, Guo et al. 2012]
- Exponential stability holds for **homogeneous** dynamics

Numerical example: problem data

- The plant is defined by the following data

$$A_p = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_{pd} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- The controller is defined by the following data

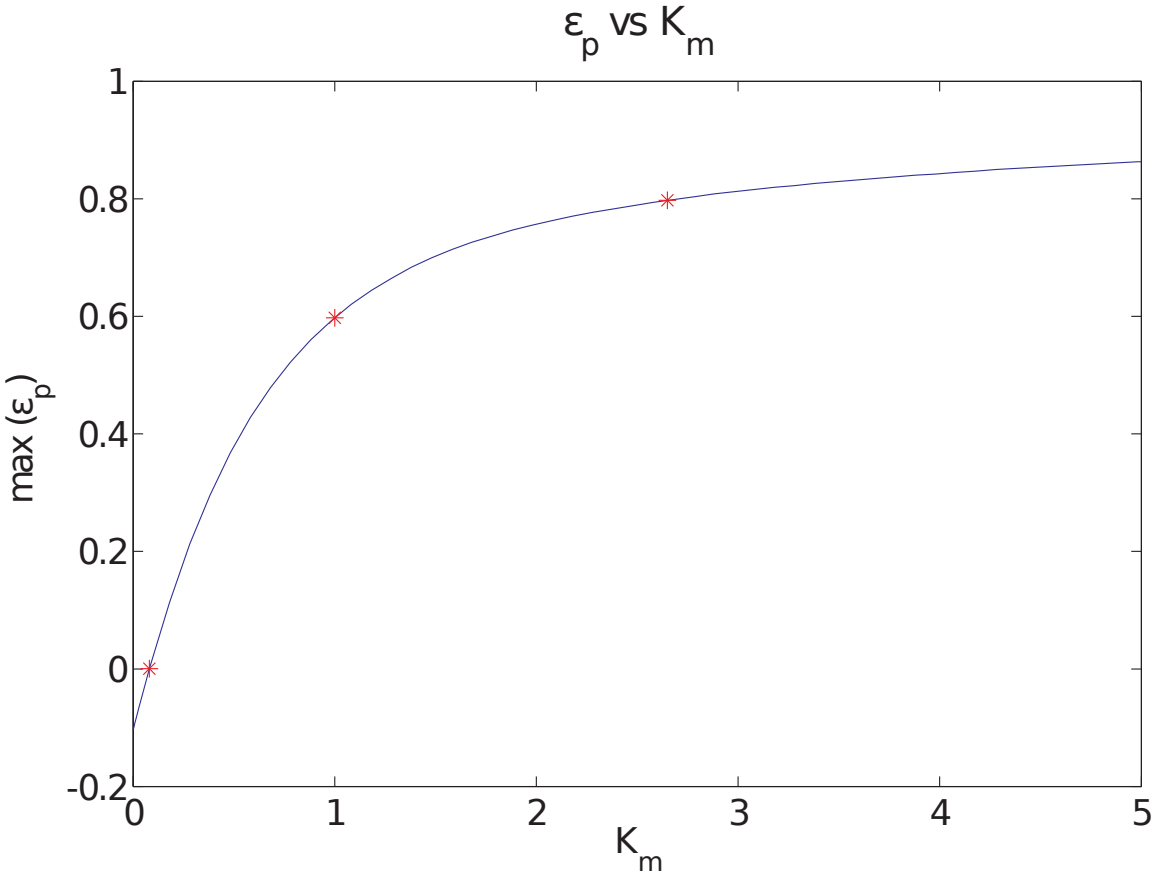
$$A_c = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- The continuous-time closed-loop system with no delay is **exponentially stable** with $\theta = 0$.
- The continuous-time closed-loop system with delay system is **unstable** for $\theta > 1.6$

Numerical example: selection of the reset parameters

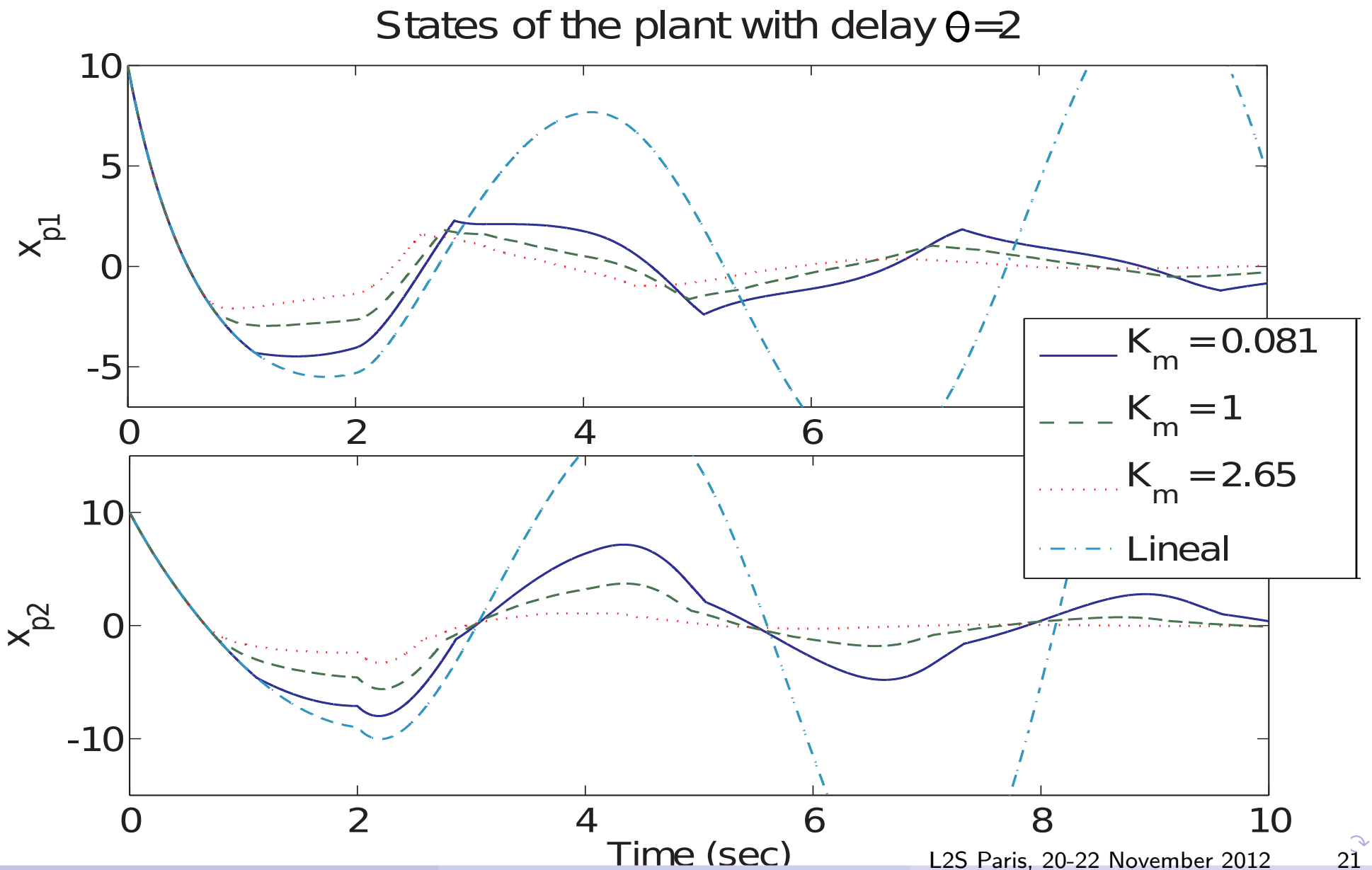
- Plot of the maximized ϵ_p versus the bound κ_m imposed on K_p. Three simulations cases $(*)$.

κ_m	ϵ_p	K_p
0.081	$3.6 \cdot 10^{-4}$	0.0809
1	0.5976	0.9974
2.65	0.7977	2.6474

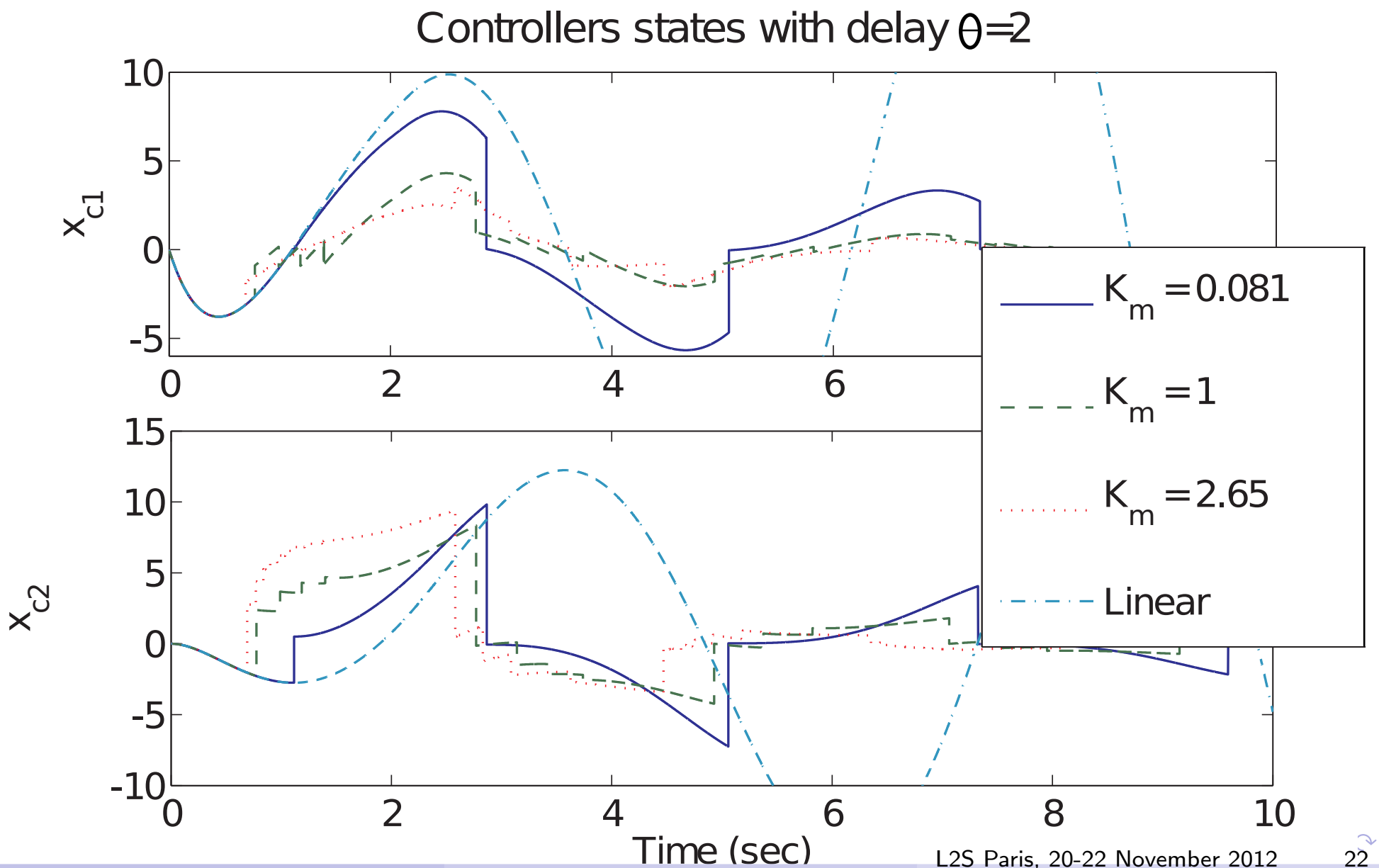


- Need K_p sufficiently large to ensure $\epsilon_p > 0$, namely GAS of the origin of the hybrid closed loop

Example: Plant state response with $\theta = 2$



Example: Controller state response with $\theta = 2$



Concluding Remarks

- Summary:
  - ▷ This work combines concepts from the recent framework for hybrid dynamical systems with ideas from time-delay systems
  - ▷ Use of hybrid loops (controller jumps) allows to recover stability which may be destroyed by time delay
  - ▷ A hybrid Lyapunov Krasovskii theorem is instrumental for this goal
  - ▷ Academic example illustrated the proposed approach
- Perspectives:
  - ▷ Extend to nonlinear systems (using Lyapunov)
  - ▷ Determine delay-dependent compensation schemes using Lyapunov Razumikin approach

References

- Delay-Independent Stability of Reset Systems, IEEE Trans. Aut. Contr. 54(2) 341–346, 2009.
- O. Beker, C.V. Hollot and Y. Chait. Fundamental properties of reset control systems. Automatica 40(6), 905-915, 2004.
- F. Fichera, C. Prieur, S. Tarbouriech, L. Zaccarian. Improving The Performance of Linear Systems by Adding a Hybrid Loop: The Output Feedback Case, American Control Conference, Montreal, Canada, June 2012.
- R. Goebel, R. Sanfelice and A. R. Teel. Hybrid dynamical systems. IEEE Control Systems Magazine 29(2), 28-93, 2009.
- D. Nesic, L. Zaccarian and A.R. Teel. Stability properties of reset systems. Automatica 44(8), 2019-2026, 2008.
- C. Prieur, S. Tarbouriech, L. Zaccarian. Guaranteed stability for nonlinear systems by means of a hybrid loop, 8th IFAC Symposium on Nonlinear Control Systems, Bologna (Italy), September 2010.
- C. Prieur, S. Tarbouriech, L. Zaccarian. Lyapunov-based hybrid loops for stability and performance of continuous-time control systems, Automatica, to appear.