Delay-independent stability via a reset loop

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Outline

1. Overview of dynamical hybrid systems framework of Teel et al.
2. Problem statement and proposed solution
3. Numerical example
4. Concluding remarks
Goal of this work

- **Context.** Employ tools from **Hybrid dynamical systems** within control of **Linear time-delay plants**
  Not much work done in this context – recent work [Liu, Teel CDC2012]

- **Objective.** Use *hybrid loops or resets* for stability **recovery**:
  - start from a *linear closed-loop* where controller $\mathcal{K}$ stabilizes the plant $\mathcal{P}_0$ without any time delay;
  - recover (delay-independent) *global exponential stability* with time-delay plant $\mathcal{P}_\theta$ by enforcing suitable *resets* in the controller.

![Diagram showing the flow of signals and control loops](image-url)
Hybrid techniques reach beyond limits of classical control

- We will design a hybrid loop to augment to the continuous-time controller by jump rules that recover stability lost due to the delay
  - For this we use reset control systems formalism or, more generally hybrid dynamical systems formalism [Teel, Goebel, Sanfelice 2011]

- The use of hybrid systems is desirable due to their capability to:
  - provide global asymptotic stability of closed-loops not stabilizable by continuous feedback (see e.g. [Hespanha et al, 1999, 2003]).
  - guarantee a robustness with respect to small errors in the loop, which cannot be obtained using classical (i.e. with a continuous dynamics) controllers (see e.g. [Prieur 2005, Goebel and Teel, 2009]).
  - More generally, hybrid systems can model a wider range of physical problems and provide improved observers [Allgower et al. 2007, Prieur et al. 2012]
Hybrid dynamical systems review: dynamics

\( \mathcal{H} = (C, D, F, G) \)

- \( n \in \mathbb{N} \) (state dimension)
- \( C \subseteq \mathbb{R}^2 \) (flow set)
- \( D \subseteq \mathbb{R}^2 \) (jump set)
- \( F : C \Rightarrow \mathbb{R}^2 \) (flow map)
- \( G : D \Rightarrow \mathbb{R}^2 \) (jump map)

\[ \mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \]
Hybrid dynamical systems review: continuous dynamics

\[ \mathcal{H} = (C, D, F, G) \]

- \( n \in \mathbb{N} \) (state dimension)
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\[ \mathcal{H} : \left\{ \begin{array}{c} \dot{x} \in F(x) \quad x \in C \\ x^+ \in G(x) \quad x \in D \end{array} \right\} \]

Van der Pol

\[ \left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + x_2(1 - x_1^2) \end{array} \right\} \]
Hybrid dynamical systems review: discrete dynamics

\( \mathcal{H} = (C, D, F, G) \)

- \( n \in \mathbb{N} \) (state dimension)
- \( C \subseteq \mathbb{R}^2 \) (flow set)
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\[ \begin{align*}
H & : \begin{cases}
\dot{x} \in F(x) & x \in C \\
x^+ \in G(x) & x \in D
\end{cases}
\end{align*} \]

Possible sequence of states from \( x_0 = 0 \) is:

\[ (0 \cdot 1 \cdot 2 \cdot 1)^i \quad i \in \mathbb{N} \]
Hybrid dynamical systems review: **trajectories**

\[ \mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \]
Hybrid dynamical systems review: *hybrid time*

The motion of the state is parameterized by two parameters:

- $t \in \mathbb{R}_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
- $j \in \mathbb{Z}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.

$\forall \tau \in [0, 5], \xi(\tau, 0)$

$\forall \tau \in [5, 8], \xi(\tau, 2)$

$\forall \tau \geq 8, \xi(\tau, 3)$
Hybrid dynamical systems review: **hybrid time**

$E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\})$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

$E$ is a **hybrid time domain** if for all $(T,J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

$$E \cap ([0, T] \times \{0,1,\ldots,J\})$$

is a compact hybrid time domain.
Hybrid dynamical systems review: solution

- Formally, a solution satisfies the flow dynamics when flowing and satisfies the jump dynamics when jumping.
Goal of this work (recall)

- **Context.** Employ tools from **Hybrid dynamical systems** within control of **Linear time-delay plants**
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- **Objective.** Use **hybrid loops or resets** for stability **recovery**:
  - start from a **linear closed-loop** where controller $\mathcal{K}$ stabilizes the plant $\mathcal{P}_0$ without any time delay;
  - recover (delay-independent) **global exponential stability** with time-delay plant $\mathcal{P}_\theta$ by enforcing suitable resets in the controller.
Plant and controller dynamics is linear

- Continuous-time linear plant with time-delay $\theta$ in the state:

$$\mathcal{P}_\theta \begin{cases} 
\dot{x}_p = A_p x_p + A_{pd} x_p (t - \theta) + B_p u_p \\
y_p = C_p x_p 
\end{cases}$$

- Continuous-time controller stabilizing $\mathcal{P}_0$ is:

$$\mathcal{K} \begin{cases} 
\dot{x}_c = A_c x_p + B_c y_p \\
u_p = C_c x_c 
\end{cases}$$
Problem Formulation

- Continuous-time closed-loop with delay without resets may be unstable:
  \[
  \begin{align*}
  \dot{x}_p &= A_p x_p + A_{pd} x_p(t - \theta) + B_p C_c x_c \\
  \dot{x}_c &= A_c x_c + B_c C_p x_p
  \end{align*}
  \]

- Define the augmented state \( x = \begin{bmatrix} x_p \\ x_c \\ x_p(t - \theta) = x_{pd} \end{bmatrix} \)

- **Problem:** Determine \( K_p, C, D \) augmenting the above closed loop as
  \[
  \begin{align*}
  \dot{x}_p &= A_p x_p + A_{pd} x_{pd} + B_p C_c x_c \\
  \dot{x}_c &= A_c x_c + B_c C_p x_p \\
  x_p^+ &= x_p \\
  x_c^+ &= K_p x_p
  \end{align*}
  \]
  \( x \in C \) \quad \( x \in D \)

  with the goal to **recover global asymptotic stability** of the origin.

- Recall: \( C, D \) are the flow and jump sets.
Delay-independent results require assumption on $P_\theta$

- $P_\theta$ must be good enough to allow for Lyapunov-Krasovskii functionals
- **Assumption LK.** Given the matrices in $P_\theta$, there exist two symmetric positive definite matrices $P_p$ and $Q$, a positive scalar $\epsilon_p$ and a “hybrid gain” $K_p$ such that

\[
\begin{bmatrix}
(A_p + B_p C_c K_p)' P_p + P_p (A_p + B_p C_c K_p) + Q & P_p A_{pd} \\
A_p' P_p & -Q
\end{bmatrix}
\leq -2\epsilon_p \begin{bmatrix} P_p & 0 \\ 0 & Q \end{bmatrix}
\]

- **Meaning of Assumption LK:** delay-independent stability of the linear time-delay plant stabilized by $K_p$:

\[\dot{z}(t) = (A_p + B_p C_c K_p) z(t) + A_{pd} z(t - \theta) = A_K z(t) + A_{pd} z(t - \theta)\]

- $z$ dynamics coincides with the closed loop dynamics with delay when clamping $x_c = K_p x_p$
Convex formulation helps optimizing $P_p, K_p, Q, \epsilon_p$

- **Lemma LK** Assumption LK holds with $|K_p| \leq \kappa_{\text{max}}$ if and only if there exist two symmetric positive definite matrices $Q_p$ and $S$, a positive scalar $\epsilon_p$ and a matrix $X$ such that:

  $$\begin{align*}
  |X| & \leq \kappa_{\text{max}}, \\
  Q_p & \geq I \\
  \begin{bmatrix}
  Q_p A'_p + X' C'_c B'_p + A_p Q_p + B_p C_c X + S & A_{pd} Q_p \\
  Q_p A'_{pd} & -S
  \end{bmatrix} & \leq -2\epsilon_p \begin{bmatrix}
  Q_p & 0 \\
  0 & S
  \end{bmatrix}
  \end{align*}$$

- One then can build the following solution to **Assumption LK**

  $$\begin{align*}
  K_p &= XQ^{-1}_p, \\
  Q &= Q^{-1}_p S Q^{-1}_p, \\
  P_p &= Q^{-1}_p
  \end{align*}$$

- Maximizing $\epsilon_p$ is a Generalized Eigenvalue Problem (quasi-convex)

- Recall that $x_c \equiv K_p x_p$ induces exponentially stable dynamics

  $$\dot{z}(t) = (A_p + B_p C_c K_p)z(t) + A_{pd} z(t - \theta) = A_K z(t) + A_{pd} z(t - \theta)$$
Problem solution: use resets to go back to $z$ dynamics

**Solution:** define the coordinate $\xi = [x_p \ x_{pd} \ x_c - K_p x_p]^T$ and select $K_p = K_p$ from Lemma LK and for any $\epsilon_c > 0$ fix

$$C = \left\{ \xi \in \mathbb{R}^3 \mid \begin{bmatrix} P_p A_K + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \leq \begin{bmatrix} -\epsilon_p P_p \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$D = \left\{ \xi \in \mathbb{R}^3 \mid \begin{bmatrix} P_p A_K + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \geq \begin{bmatrix} -\epsilon_p P_p \\ 0 \\ 0 \end{bmatrix} \right\}$$

Recall hybrid scheme where $x = [x_p \ x_c \ x_{pd}]^T$

$$\begin{align*}
\dot{x}_p &= A_p x_p + A_{pd} x_{pd} + B_p C_c x_c \\
\dot{x}_c &= A_c x_c + B_c C_p x_p \\
x_p^+ &= x_p \\
x_c^+ &= K_p x_p
\end{align*}$$

$x \in C$

$x \in D$

**Interpretation:** if $x \in D$, jumps ensure $x_c^+ = K_p x_p$ (i.e., back to the “good” $z$ dynamics: $\dot{z} = (A_p + B_p C_c K_p)z + A_{pd} z_d$}
Idea behind proof of (hybrid) global exponential stability

- Recall definition of flow $C$ and jump $D$ sets:

$$C = \left\{ \xi' \begin{bmatrix} P_pA_K + Q/2 & P_pA_{pd} & P_pB_pC_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & \epsilon_c I \end{bmatrix} \xi \leq \xi' \begin{bmatrix} -\epsilon_p P_p & 0 & 0 \\ 0 & -\epsilon_p Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \right\}$$

$$D = (\mathbb{R}^n \times \mathbb{R}^{nc} \times \mathbb{R}^n) \setminus C$$

- Consider the Lyapunov-Krasovskii functional (with suitable $\lambda > 0$):

$$V(x_t) = x_p'P_p x_p + \int_{t-\theta}^{t} x_p'(\tau) Q x_p(\tau) d\tau + \lambda (x_c - K_p x_p)'(x_c - K_p x_p)$$

- During flows, since $x \in C$, from Assumption KL (for small $\lambda > 0$):

$$\dot{V}(x_t) \leq -\epsilon |x|^2, \text{ for some } \epsilon > 0$$

- Across jumps, since $x_p^+ = x_p, x_c^+ = K_p x_p$, we get from the blue guy:

$$V(x_t^+) - V(x_t) \leq 0$$

- Stability proof uses Hybrid Lyapunov Krasovskii theorem (next slide)
Lyapunov-Krasovskii thm for a class of hybrid systems

- For a state $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of $x$ and
  $$\|x_t\|_{\theta} := \max_{s \in [0,\theta]} |x(t - s)|$$

- **Theorem LK** Assume that there exists a functional $V(x_t)$, class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$ and a positive definite function $\rho$ such that
  $$\alpha_1(|x|) \leq V(x_t) \leq \alpha_2(\|x\|_{\theta})$$
  $$\dot{V}(x_t) \leq -\rho(|x|), \quad \forall x \in C$$
  $$V(x_t^+) - V(x_t) \leq 0, \quad \forall x \in D$$

  and assume that the solutions exhibit persistent flow. Then the origin of the hybrid time-delay system is GAS.


- Exponential stability holds for homogeneous dynamics
Numerical example: problem data

The plant is defined by the following data

\[ A_p = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{pd} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 1 \end{bmatrix} \]

The controller is defined by the following data

\[ A_c = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

The continuous-time closed-loop system with no delay is **exponentially stable** with \( \theta = 0 \).

The continuous-time closed-loop system with delay system is **unstable** for \( \theta > 1.6 \).
Numerical example: selection of the reset parameters

- Plot of the maximized $\epsilon_p$ versus the bound $\kappa_m$ imposed on $K_p$. Three simulations cases (*).

<table>
<thead>
<tr>
<th>$\kappa_m$</th>
<th>$\epsilon_p$</th>
<th>$K_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.081</td>
<td>$3.6 \cdot 10^{-4}$</td>
<td>0.0809</td>
</tr>
<tr>
<td>1</td>
<td>0.5976</td>
<td>0.9974</td>
</tr>
<tr>
<td>2.65</td>
<td>0.7977</td>
<td>2.6474</td>
</tr>
</tbody>
</table>

- Need $K_p$ sufficiently large to ensure $\epsilon_p > 0$, namely GAS of the origin of the hybrid closed loop.
Example: Plant state response with $\theta = 2$
Example: Controller state response with $\theta = 2$

Controllers states with delay $\theta=2$

- $K_m = 0.081$
- $K_m = 1$
- $K_m = 2.65$
- Linear
Concluding Remarks

Summary:
- This work combines concepts from the recent framework for hybrid dynamical systems with ideas from time-delay systems.
- Use of hybrid loops (controller jumps) allows to recover stability which may be destroyed by time delay.
- A hybrid Lyapunov Krasovskii theorem is instrumental for this goal.
- Academic example illustrated the proposed approach.

Perspectives:
- Extend to nonlinear systems (using Lyapunov).
- Determine delay-dependent compensation schemes using Lyapunov Razumikhin approach.
References