

# A tutorial on modern anti-windup design

Sergio Galeani, Sophie Tarbouriech, Matthew Turner, Luca Zaccarian

## Abstract

In this paper, several constructive linear and nonlinear anti-windup techniques are presented and explained. Two approaches, namely direct linear anti-windup (DLAW) and model recovery anti-windup (MRAW), are described in an algorithmic way, in order to illustrate their main features. Hereafter, theoretical conditions ensuring stability and performance, their applicability, their accompanying guarantees, and their merits and deficiencies are given. The possible extensions to less standard problem settings are also briefly discussed.

## Index Terms

anti-windup, input saturation, saturated stability, saturated performance

## I. INTRODUCTION

Actuator saturation occurs when a controller demands a signal which is larger than the actuator is capable of delivering. In most cases, industrial control systems are designed by making the actuators large enough so that, during operation, the input effort commanded by the control law is well below the saturation levels. However this paradigm can, at times, be excessive in terms of costs and/or performance requirements. For example, in aerospace applications oversized actuators lead to increases in aircraft mass and therefore fuel costs. Similarly, in power systems small increases in performance can lead to huge advantages in efficiency and associated costs. Many other examples could be cited. In general, any high-tech application can potentially benefit from accounting for the saturation effects and exploiting the full power available in the actuators, rather than oversizing them, with the evident arising conservatism (see, [82] for an overview of relevant applications illustrating these aspects).

Historically, the industrial world started to face input saturation problems as early as the 1940s and it can probably be claimed that saturation effects were among the main phenomena inspiring the absolute stability literature. The need of techniques to deal with input saturations was already pointed out when control systems consisted of analog controllers [68]. In particular, [68] noted that it was unreasonable to sacrifice the small signal behavior of the control system in order to guarantee a suitable large signal behavior. Instead, a control system should be able to preserve any desirable small signal behavior, while also being able to resolve large signal issues arising from saturation. This type of observation was intimately connected with the fact that saturation is a strongly nonlinear phenomenon and that nonlinear techniques should be used to study it.

Some years later, when digital control systems became popular, researchers started looking for systematic solutions to the so-called “windup” problem, namely the performance, or even stability, loss experienced in some control systems after the saturation limits were reached by the actuators (see [17] for an early paper on this topic). While “windup” denoted the saturation-induced malfunctioning of the control system, “anti-windup” was the natural label for any controller augmentation strategy aimed at mitigating as much as possible the undesirable “windup” effects.

After the first properly documented anti-windup methodology proposed in [17], various works on intelligent integrators [64] and later the celebrated conditioning technique of Hanus et al [48], [49],

Work supported in part by ENEA-Euratom and MIUR under PRIN and FIRB projects.

S. Galeani and L. Zaccarian are with the Dipartimento di Informatica, Sistemi e Produzione, University of Rome, Tor Vergata, 00133 Rome, Italy [galeani@disp.uniroma2.it](mailto:galeani@disp.uniroma2.it)

S. Tarbouriech is with LAAS-CNRS, Université de Toulouse, 7 Avenue du Colonel Roche, 31077 Toulouse cedex 4, France [tarbour@laas.fr](mailto:tarbour@laas.fr)

M. Turner is with Department of Engineering, University of Leicester, Leicester, LE1 7RH, UK [mct6@leicester.ac.uk](mailto:mct6@leicester.ac.uk)

[111] emerged. This early literature heralded the beginnings of more formal studies of the saturation problem but, at this point, most papers still focused on developing ad hoc techniques which seemed to overcome certain practical problems, but provided no guarantees. Hence, roughly speaking anti-windup solutions were initially “tricks” and “fixes” typically proposed by industrial engineers, and aimed at particular applications, but perhaps with claims of being applicable to larger classes of systems. Until the 1980s, there were few approaches available to address saturation problems, and most of these lacked formal stability guarantees. This fact was pointed out at the 1987 American Control Conference (ACC) by Doyle and co-authors [15] and after this several academics started to look into the problem with more formality and detail (several papers were presented on this topic at ACC 1989). In particular, the 1990s saw researchers tackling the anti-windup problem from a constrained stabilization perspective and, by the end of the millennium, several constructive techniques with formal stability guarantees had been derived. Following on from this, techniques with optimal performance properties began to be published. These last results, which have been all produced in the past decade, are regarded here as “modern anti-windup approaches” and are the subject of this tutorial paper.

The purpose of this tutorial paper is to provide a fairly complete overview of the constructive anti-windup design algorithms that have been made available in the last decade. With this aim, we will guide the reader in the extensive literature aiming at clarifying what we believe the more interesting and relevant features of the various approaches. Moreover, we will present representative approaches in an algorithmic way, illustrating their suitability to addressing the different windup problems. Because of this, we shall not overview all the extensive literature on anti-windup designs (see, [101] for a survey of the area). All the results will be presented in continuous time, even though for most of them a discrete-time counterpart can be found too and we will comment on this, whenever relevant, throughout.

It should be recognized that several “modern” approaches will not be covered here. For example, we will focus on design techniques that lead to unified formulations for SISO and MIMO systems, while we will disregard results specifically tailored for SISO systems (even though these often can be extended to MIMO systems). Among these solutions that we shall not cover, several results can be found in the recent monographs [30], [53]. Moreover, modern anti-windup solutions have been also proposed in the context of the so-called “reference (or command, or error) governor” (see, [1], [29]) and references therein). These techniques actually correspond to a different viewpoint on the anti-windup problem and will not be treated here, even though they have been shown to be very effective in several important applications.

The paper is organized as follows. In Section II we discuss the fundamental issues in anti-windup design. We emphasize here what types of solutions are considered attractive and summarize the key peculiarities of their algorithms. Sections III and IV contain all the necessary descriptions of the two fundamental architectures illustrated in this tutorial, with details about the construction of the most relevant anti-windup design algorithms, together with indications of possible extensions to less standard problem settings.

**Notation.** For any vector  $x \in \mathfrak{R}^n$ ,  $x \succeq 0$  means that all the components of  $x$ , denoted  $x_{(i)}$ , are nonnegative. For two vectors  $x, y$  of  $\mathfrak{R}^n$ , the notation  $x \succeq y$  means that  $x_{(i)} - y_{(i)} \geq 0, \forall i = 1, \dots, n$ . The elements of a matrix  $A \in \mathfrak{R}^{m \times n}$  are denoted by  $A_{(i,l)}$ ,  $i = 1, \dots, m, l = 1, \dots, n$ .  $A_{(i)}$  denotes the  $i$ th row of matrix  $A$ . For two symmetric matrices,  $A$  and  $B$ ,  $A > B$  means that  $A - B$  is positive definite.  $A'$  denotes the transpose of  $A$ .  $diag(x)$  denotes a diagonal matrix obtained from vector  $x$ . When no confusion is possible,  $I$  denotes the identity matrix of appropriate dimensions.  $Co\{\cdot\}$  denotes a convex hull.  $He[A] = A + A'$ .

## II. THE ANTI-WINDUP PROBLEM

### A. Anti-windup in a nutshell

The word “windup” is motivated by the fact that many early (analog) control systems used PID-based controllers. In these controllers the negative effects of saturation were often experienced by

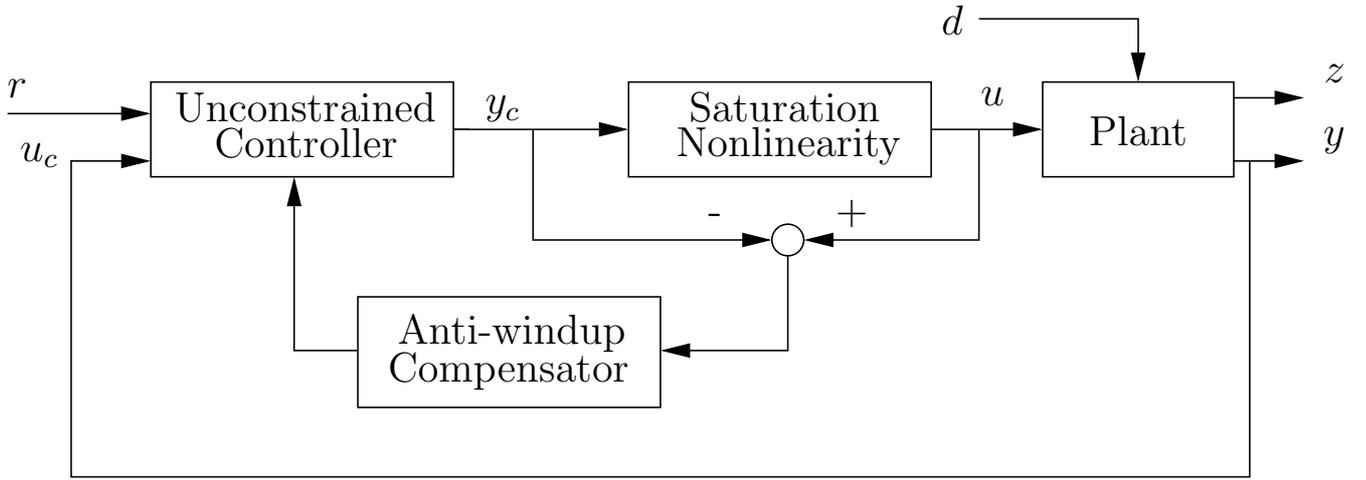


Fig. 1. The basic anti-windup architecture.

seeing the state associated with the integral action ramping up to very large values and then inducing large (sometimes diverging) oscillations in the closed-loop. Clearly, for the more complicated modern multivariable controllers, this *winding up* phenomenon is not as simple, but the word “windup” still represents the fact that a controller designed ignoring saturation can often become confused about the unexpected saturated plant response and induce closed-loop performance (and possibly stability) loss.

Windup has been always regarded from the industrialists’ perspective as a problem associated with performance deterioration after saturation has occurred. This hiddenly conveys the fact that whenever saturation does not occur, the underlying controller (which we will call the “unconstrained controller” in this paper) induces the most desirable (often linear) performance on the closed-loop. Due to this fact, the anti-windup goal is always stated in terms of deviation from what the response would have been if saturation had not been present. That ideal response is called the “unconstrained response” throughout this paper. Then, qualitatively, the goals of anti-windup augmentation are that for any selection of the initial conditions and of the external signals acting on the closed-loop (references, disturbances) the following holds:

- 1) (small signal preservation) if the unconstrained response is associated with a controller output ( $\equiv$  plant input) that does not exceed the saturation levels, then the response with anti-windup augmentation (that we will call “anti-windup response” hereafter) coincides with the unconstrained response;
- 2) (large signal recovery) in all other cases, the anti-windup response is as close as possible to the unconstrained response.

Clearly, item 2 above is qualitative. Mathematically what can be proven is global or nonglobal asymptotic (or even exponential) stability, as well as guaranteed performance measures that can be of various kinds. These measures of performance can be sometimes optimized and may apply to the whole state space or just to a subset of the state space. Several of these issues are discussed below in Section II-C.

### B. Basic anti-windup architecture

Item 1 in the previous section clearly hinges upon the property that the closed-loop without saturation is well behaved. In particular, in anti-windup design it is always assumed that the unconstrained controller guarantees closed-loop asymptotic stability in the absence of saturation. If this property was not satisfied, then satisfying item 1 above would prevent from obtaining closed-loop asymptotic stability of the anti-windup closed-loop system.

Since the saturation acts like the identity operator for small enough signals, it is clear that for the (nonlinear) closed-loop with saturation local asymptotic stability is automatically guaranteed because of item 1. Therefore, one could think about the saturation effects and the arising anti-windup correction actions as some sort of incremental action as compared to the non-saturated behavior. This observation motivates a well established architecture for anti-windup shown in Figure 1, where the linear closed-loop is “disturbed” by the dead-zone signal corresponding to  $dz(y_c) := \text{sat}(y_c) - y_c$  and where this same signal is exploited to activate the anti-windup filter action whenever necessary (namely whenever the saturation limits are exceeded or, equivalently, whenever the dead-zone function becomes different from zero).<sup>1</sup>

The architecture of Figure 1 is especially useful because 1) it allows isolation of the signals causing the undesired mismatch between the responses of the unconstrained closed-loop and of the actual closed-loop with saturation; 2) it allows the automatic satisfaction of the small signal preservation requirement of item 1 (as long as the initial condition of the anti-windup compensator is zero) because, whenever saturation is not activated, the anti-windup compensator input will stay at zero. It is also worth noticing that local asymptotic stability of the scheme in Figure 1 certainly implies local asymptotic stability of the anti-windup compensator. Indeed the compensator is driven by zero input for small signals and closed-loop stability could not possibly hold if the anti-windup compensator was not asymptotically stable itself.

In Figure 1, each block (controller, plant, anti-windup compensator) can be either linear or nonlinear, thus leading to different anti-windup problem formulations. However, most of the anti-windup literature deals with linear plants (before saturation) which highly simplifies the design problem. As for the unconstrained controller, it is typically assumed that it is linear too, but we will emphasize later that several of the approaches summarized here allow for nonlinear controllers as well, with very mild assumptions on their properties. Finally, the main distinction that will be made here is whether the anti-windup compensator block is linear or not. In particular, we will talk about *direct linear* anti-windup compensators when the dynamics driven by the dead-zone function are linear and we will talk about *fully nonlinear* or simply *nonlinear* anti-windup in all other cases. It has been emphasized already many years ago (see, e.g., [103]) that saturation effects are highly nonlinear and often require nonlinear compensation to achieve satisfactory performance (or even just stability – see the next section). Due to this fact, it is important to study and provide fully nonlinear constructive anti-windup schemes. Nevertheless, linear schemes are normally easier to implement and computationally simpler; hence they are often preferred to nonlinear solutions. Anti-windup design then becomes often a trade-off between computational complexity and stability/performance guarantees.

### C. Intrinsic limitations arising in AW

Following the two qualitative statements at the end of Section II-A, anti-windup can be seen as a bounded stabilization problem with extra constraints on the small- and medium-signal behavior. Therefore, it is relevant to summarize here what the intrinsic limitations of bounded stabilization are in order to clarify what are reasonable anti-windup design goals.

Over two decades ago, it had been proven in [92] that a linear system could only be globally exponentially stabilized by a bounded input if it was exponentially stable itself (see also [65]). The other important results established in [65], [92] were that 1) if the plant is not exponentially unstable (namely it only has poles in the closed left half plane) then it can be globally asymptotically (not necessarily exponentially) stabilized by a bounded input); 2) if the plant is exponentially unstable, then the null controllability region is bounded and global results cannot be achieved. Moreover, the null controllability region is unbounded along the eigenspaces corresponding to eigenvalues with nonpositive

<sup>1</sup>It should be emphasized that some anti-windup schemes do not fall into the general structure of Figure 1. In particular, the MPC-based discrete-time schemes named “reference (or command, or error) governors” (see [1], [29] and references therein) follow a different design paradigm that we don’t cover in this paper.

real part and is bounded along the remaining ones. Another relevant result was that of [19] where it was shown that no saturated linear controller can achieve global asymptotic stability for the triple integrator: this reveals the need for nonlinear control solutions when dealing with saturated plants (especially global properties are sought).

The general results on intrinsic limitations for saturated systems correspond to the mathematical formalization of reasonable intuition arising when looking at bounded stabilization: saturation can be well thought of as  $\text{sat}(y_c) = g_E(y_c)y_c$ , where  $g_E(\cdot) \in (0, 1]$  is an equivalent gain that becomes smaller and smaller as  $y_c$  grows large. Then the large signal behavior of a bounded control system can be associated with an equivalent gain which is arbitrarily close to zero, namely the best that one can obtain globally corresponds to the properties of the system with zero control input. This explains why global exponential stability cannot be achieved unless the plant is already exponentially stable. The key to successful saturated control then becomes the use of one or both of the following two approaches:

- 1) Seek nonglobal results that will not need to comply with the intrinsic limitations listed above so that good performance can be achieved for reasonably sized signals; we will denote by “regional” any approach that can guarantee stability and/or performance in a guaranteed nontrivial region of the closed-loop signals (keep in mind that local results are always achieved, even by no anti-windup, as noted in Section II-B)
- 2) Adopt nonlinear solutions where the control gains are larger for small signals and smaller for large signals. The arising control systems are far from being linear and are tailored on the specific peculiarities of saturation (small equivalent gain for large signals and large equivalent gain for small signals).

The observations above clarify how crucial certain aspects are when seeking anti-windup solutions. In particular, it should be recognized that not all solutions will work for all types of plants because the above limitations provide the following constraints:

- 1) global exponential stability will only be achievable with exponentially stable plants;
- 2) global asymptotic stability will only be achievable with non-exponentially unstable plants;
- 3) regional asymptotic stability is achievable with any type of plant but then large operating regions become most desirable.

The three items represent a very natural classification among the several constructive anti-windup recipes illustrated in this paper. This classification, which is in terms of the achievable closed-loop properties, could be similarly stated in terms of the type of plant under consideration, namely:

- 1) with exponentially stable plants all the provided algorithms will be applicable;
- 2) with marginally stable/unstable plants, only the algorithms providing global asymptotic stability or regional exponential stability will be applicable; here it will be in general necessary that the algorithm be nonlinear to achieve global results;
- 3) with exponentially unstable plants only algorithms yielding regional guarantees will be applicable.

The above classification of the proposed algorithms will be used in the overview that we will present in the next Section II-E.

#### D. Two approaches to modern anti-windup

While Figure 1 represents the direct approach to anti-windup design, a further classification should be made between two families of anti-windup compensation which most of the modern constructive anti-windup techniques fall into. It will be useful to introduce some notation for the block diagrams in Figure 1 in order to introduce these two families of solutions. Consider the following state-space representation for the continuous-time linear plant in the figure (for simplicity, the time dependence in the vector will be omitted):

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_{pu} u + B_{pw} w \\ y_p &= C_p x_p + D_{pu} u + D_{pw} w \\ z &= C_z x_p + D_{zu} u + D_{zw} w \end{aligned} \tag{1}$$

#	Applic	Arch	Guarant	Proofs	Order	Method	Notes
1	Any	DLAW	RES	Thms 3, 4	$n_p + n_c/n_p/0$	LMI	$\mathcal{L}_2$ -gain optimization
2	ES	DLAW	GAS	Thms 1, 2	$n_p + n_c/n_p/0$	LMI	$\mathcal{L}_2$ -gain optimization
3	Any	DLAW	RES	Thms 3, 4	$n_p + n_c/n_p/0$	LMI	Region of stability maximization
4	Any	DLAW	RES	Thms 3, 4	$n_1 + 2n_2 < n_p + n_c$	LMI	A priori fixed AW dynamics
5	ES	MRAW	GES	Refs [106], [121]	$n_p$	N/A	Simplest possible MRAW scheme
6	ES	MRAW	GES	Refs [106]	$n_p$	Lyap	Solve a Lyapunov equation
7	ES	MRAW	GES	Refs [118]	$n_p$	LMI	Constrained LQ
8	MS	MRAW	GAS	Refs [106]	$n_p$	Lyap	Solve a Lyapunov equation
9	Any	MRAW	None	N/A	$n_p$	LQR	No guarantees
10	Any	MRAW	RES	Refs [24]	$n_p$	LMI	Uses generalized sector
11	Any	MRAW	RES	Refs [27]	$n_p$	LMI	Fully nonlinear

TABLE I  
OVERVIEW OF THE ANTI-WINDUP ALGORITHMS.

where  $x_p \in \mathbb{R}^{n_p}$ ,  $u \in \mathbb{R}^m$ ,  $w \in \mathbb{R}^q$  and  $y_p \in \mathbb{R}^p$  are the state, the input, the exogenous input and the measured output vectors of the plant, respectively.  $z \in \mathbb{R}^l$  is the regulated output vector used for performance purposes. Matrices  $A_p$ ,  $B_{pu}$ ,  $B_{pw}$ ,  $C_p$ ,  $C_z$ ,  $D_{pu}$ ,  $D_{pw}$ ,  $D_{zu}$  and  $D_{zw}$  are real constant matrices of appropriate dimensions. Pairs  $(A_p, B_{pu})$  and  $(C_p, A_p)$  are assumed to be controllable and observable, respectively.

Considering the plant (1), we assume that an  $n_c$ 'th-order dynamic output stabilizing compensator

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c + B_{cw} w + v_x \\ y_c &= C_c x_c + D_c u_c + D_{cw} w + v_y \end{aligned} \quad (2)$$

(where  $x_c \in \mathbb{R}^{n_c}$  is the controller state,  $u_c \in \mathbb{R}^p$  is the controller input,  $y_c \in \mathbb{R}^m$  is the controller output and  $v_x, v_y$  are extra inputs used for anti-windup purposes, specified later) has been designed in order to guarantee some performance requirements and the stability of the closed-loop system in the absence of control saturation, that is when the following unconstrained interconnection is used:

$$u = y_c, u_c = y_p, v_x = 0, v_y = 0. \quad (3)$$

*Remark 1:* The closed-loop (1), (2), (3) is assumed to be well-posed. Therefore, it is evidently necessary that the matrix  $\Delta := I_m - D_c D_{pu}$  and the matrix  $I_p - D_{pu} D_c$  are both nonsingular. Moreover, the unconstrained closed-loop dynamic matrix

$$\mathbb{A} = \begin{bmatrix} A_p + B_{pu} \Delta^{-1} D_c C_p & B_{pu} \Delta^{-1} C_c \\ B_c (I_p + D_{pu} \Delta^{-1} D_c) C_p & A_c + B_c D_{pu} \Delta^{-1} C_c \end{bmatrix} \quad (4)$$

is necessarily Hurwitz, i.e., in the absence of control bounds, the closed-loop system would be globally exponentially stable.

Suppose now that the input vector  $u$  is subject to symmetric magnitude limitations as follows:<sup>2</sup>

$$-u_{0(i)} \leq u_{(i)} \leq u_{0(i)}, \quad u_{0(i)} > 0, \quad i = 1, \dots, m \quad (5)$$

As a consequence of the control bounds, the actual control signal to be injected into the system is a saturated one, that is, there is an input nonlinearity before the input variable  $u$  which is defined as  $u = \text{sat}_{u_0}(y_c)$ , where each component of the saturation function  $\text{sat}_{u_0}(\cdot)$  is classically defined  $\forall i = 1, \dots, m$  by:

$$\text{sat}_{u_0}(y_{c(i)}) = \text{sign}(y_{c(i)}) \min(|y_{c(i)}|, u_{0(i)}). \quad (6)$$

Within this setting, it is now possible to introduce the two main architectures for anti-windup design. The first one, somewhat natural, is called **direct linear anti-windup** (DLAW) and will be addressed in

<sup>2</sup>To keep the discussion simple we will use here symmetric saturation levels, but all the approaches illustrated in this tutorial (conservatively) work as well with nonsymmetric saturations as long as  $u_0$  is a lower bound on the two saturation levels.

detail in Section III. The second one, called **model recovery anti-windup** (MRAW) will be addressed in detail in Section IV.

*Direct linear anti-windup* corresponds to selecting the anti-windup filter in Figure 1 as a linear filter which produces the signals  $v_x$  and  $v_y$  as an output.

$$\begin{aligned}\dot{x}_{aw} &= A_{aw}x_{aw} + B_{aw}(\text{sat}_{u_0}(y_c) - y_c) \\ \begin{bmatrix} v_x \\ v_y \end{bmatrix} &= C_{aw}x_{aw} + D_{aw}(\text{sat}_{u_0}(y_c) - y_c) \\ u &= \text{sat}_{u_0}(y_c), \quad u_c = y_p,\end{aligned}\tag{7}$$

where  $x_{aw} \in \mathbb{R}^{n_{aw}}$  is the anti-windup state,  $u_{aw} = \text{sat}_{u_0}(y_c) - y_c =: dz(y_c)$  is the anti-windup input and  $(v_x, v_y) \in \mathbb{R}^{n_c+m}$  is the anti-windup output. The goal of DLAW design consists of selecting suitable matrices  $A_{aw}, B_{aw}, C_{aw}, D_{aw}$  in (7) so that the so-called anti-windup closed-loop system (1), (2), (7) satisfies desirable stability and performance properties. Representative algorithms solving this problem with different guaranteed properties will be presented in Section III and are summarized in the next section.

Historically, DLAW belongs to the large family of anti-windup controllers summarized in the early survey [62], but the most recent literature, addressing the corresponding design problem from the point of view of guaranteed stability and performance properties is strongly coupled with the use of linear matrix inequalities (see, e.g., [9]). Perhaps the first important paper in this direction is [76], which addresses static anti-windup design, namely only  $D_{aw}$  is to be designed in (7) and the remaining matrices are empty. At that time, the design procedure was based on global sector conditions on the saturation. Later works using global sector conditions and providing globally stabilizing results were given in [21], [39], [41], [43], [44], [108]. Since the use of LMIs, typically, can only show exponential stability of the arising closed-loop, by the intrinsic limitations highlighted in Section II-C, these results were only applicable to control systems with exponentially stable plants. Later work, using generalized sector conditions for the saturation (this condition allows for regional stability estimates) led to algorithms applicable also with non exponentially stable plants [8], [33], [56], [86], [95], [100], [112]. Discrete-time counterparts have been reported in [34], [51], [70], [93].

*Model recovery anti-windup* follows a completely different paradigm to anti-windup design and is based on selecting the anti-windup compensator in Figure 1 as a dynamical system incorporating a model of the plant transfer function from  $u$  to  $y_p$ . In particular, the compensator is selected as follows:

$$\begin{aligned}\dot{x}_{aw} &= A_p x_{aw} + B_{pu}(y_c - v_y - \text{sat}_{u_0}(y_c)) \\ v_x &= B_c (C_p x_{aw} + D_{pu}(y_c - v_y - \text{sat}_{u_0}(y_c))) \\ u &= \text{sat}_{u_0}(y_c), \quad u_c = y_p,\end{aligned}\tag{8}$$

where the signal  $v_y$  is purposely left unspecified because it corresponds to a degree of freedom to be exploited in the anti-windup design. The advantage of the model recovery anti-windup architecture stands in the fact that the anti-windup filter (8) keeps track (via  $x_{aw}$ ) of the amount of plant state response that is missing in the saturated closed-loop due to the undesired effects of saturation. A different way of saying this is that regardless of what the selection of  $v_y$  is, the closed-loop (1), (2), (8) is such that  $x_p + x_{aw}$  always coincides with the unconstrained plant state response (see Section IV-A for details) so that driving  $x_{aw}$  to zero will force the plant state  $x_p$  to recover the unconstrained response. This fact allows one to prove closed-loop stability and performance by way of a nice cascaded systems analysis and provides two useful advantages: 1) the controller (2) does not need to be linear (the MRAW scheme is controller independent) and robust closed-loop properties hold under the assumption that it satisfies mild Lipschitz conditions; 2) the stabilizing signal  $v_y$  could be selected nonlinearly, thereby allowing the design of nonlinear anti-windup laws, suitable for solving the trickiest anti-windup design problems within the intrinsic limitations of Section II-C (where nonlinear compensation is necessary to get global results).

The key ingredients of model recovery anti-windup designs were laid down in the companion papers [106], [107] (the approach has been called  $\mathcal{L}_2$  anti-windup for a long time). Later on, the architecture has

been further illustrated for exponentially stable linear plants in [5], [118], [119]. Nontrivial extensions to exponentially unstable linear plants have been given in [26], [27], [104] while approaches guaranteeing global asymptotic stability for non exponentially stable nor exponentially unstable plants (those globally asymptotically but not globally exponentially stabilizable by bounded inputs) are given in [24], [120].

Discrete-time counterparts can be found in [45]. A further property of the MRAW structure is that it can be applied (in a nontrivial way) to nonlinear plants too. An example where this has been done is [74].

### E. Anti-windup algorithms overview

The algorithms described in the following Sections III and IV correspond to different alternatives to address and solve suitable anti-windup problems. Table I contains a summary of all the algorithms reported next, emphasizing for each one of them the following aspects:

- **Applicability:** characterizes what type of windup problems can be addressed with that algorithm, based on the properties of the plant to be controlled: ES stands for exponentially stable (namely  $A_p$  Hurwitz), MS stands for marginally stable (namely all the eigenvalues of  $A_p$  in the closed left half plane and only single eigenvalues on the imaginary axis), Any stands for any plant.
- **Architecture:** Characterizes whether the anti-windup architecture is Direct Linear (Section III) or Model Recovery (Section IV).
- **Guarantees:** What type of guarantees are given on the closed-loop: these come either from theorems reported in this paper or from results in other references. The guarantees can be GES for global exponential stability, GAS for global asymptotic stability and RES for regional exponential stability, meaning that there is a guaranteed nontrivial basin of attraction. For Algorithm 9, there is no guarantee on the basin of attraction so, even though local exponential stability is trivially guaranteed, we report “None” in the table.
- **Proofs:** Indicates where to find the proofs of the properties of the previous column.
- **Order:** Indicates the order of the anti-windup compensator. Note that by construction all the MRAW schemes are of order  $n_p$ . This may make the static/low-order DLAW a better candidate for simple anti-windup solutions.
- **Method:** Characterizes the synthesis method. This can be Lyapunov based, LMI-based or LQR based.
- **Notes:** Brief notes on the peculiarities of the specific algorithm as compared to the other ones.

## III. DIRECT LINEAR ANTI-WINDUP DESIGN ALGORITHMS

In this section, system defined through (1), (2) and (7) can be also written as follows:

$$\begin{aligned}
 \dot{x}_p &= A_p x_p + B_{pu} \text{sat}_{u_0}(y_c) + B_{pw} w \\
 y_p &= C_p x_p + D_{pu} \text{sat}_{u_0}(y_c) + D_{pw} w \\
 z &= C_z x_p + D_{zu} \text{sat}_{u_0}(y_c) + D_{zw} w \\
 \dot{x}_c &= A_c x_c + B_c y_p + B_{cw} w + v_x \\
 y_c &= C_c x_c + D_c y_p + D_{cw} w + v_y \\
 \dot{x}_{aw} &= A_{aw} x_{aw} + B_{aw} (\text{sat}_{u_0}(y_c) - y_c) \\
 v_x &= \begin{bmatrix} I_{n_c} & 0 \end{bmatrix} (C_{aw} x_{aw} + D_{aw} (\text{sat}_{u_0}(y_c) - y_c)) \\
 v_y &= \begin{bmatrix} 0 & I_m \end{bmatrix} (C_{aw} x_{aw} + D_{aw} (\text{sat}_{u_0}(y_c) - y_c))
 \end{aligned} \tag{9}$$

*Remark 2:* The presence of the implicit loop in the closed-loop system due to  $v_y$  can be removed by considering a simplified anti-windup controller: the anti-windup output is only injected in the dynamics of  $x_c$ . In this case, the anti-windup output is such that  $v_x \in \mathbb{R}^{n_c}$ . Another way consists in filtering the signal  $v_y$  and therefore to inject  $\bar{v}_y = F(s)v_y$ . Thus, a low-pass filter  $F(s)$  can be used to avoid some chattering effects on the control signals as well as algebraic or implicit loops [7].

Two issues of interest with respect to system (9) are the stability ( $w = 0$ ) and performance problems ( $w \neq 0$ ). Hence, when  $w = 0$ , it is of interest to estimate the basin of attraction of system (9), denoted  $\mathcal{B}_a$  which is defined as the set of all  $(x_p, x_c, x_{aw}) \in \mathcal{R}^{n_p} \times \mathcal{R}^{n_c} \times \mathcal{R}^{n_{aw}}$  such that for any  $(x_p(0), x_c(0), x_{aw}(0))$  belonging to  $\mathcal{B}_a$ , the corresponding trajectory converges asymptotically to the origin. In particular, when global stability of the system is ensured the basin of attraction corresponds to the whole state space. However, more generally, the exact characterization, via systematic approaches, of the basin of attraction is not possible. In this case, it is important to be able to provide estimates of the basin of attraction. In this sense, regions of asymptotic stability can be used to estimate the basin of attraction [60]. On the other hand, in some practical applications one can be interested in ensuring the stability for a given set of admissible initial conditions. This set can be seen as a practical operating region for the system, or a region where the states of the system may be driven to by the action of temporary disturbances.

In the case where  $w = 0$ , one of the problems of interest with respect to the closed-loop system (9), modified by the addition of the two static anti-windup loops  $v_x$  and  $v_y$ , consists of computing the anti-windup gains in order to enlarge the basin of attraction of the resulting closed-loop system. In the case where  $w \neq 0$ , the problem of interest is to ensure a certain level of performance which can be measured, for example, by the finite  $\mathcal{L}_2$  gain from the exogenous input  $w$  to the performance output  $z$ . For this, the disturbance vector  $w$  is assumed to be limited in energy, that is,  $w \in \mathcal{L}_2$  and for some scalar  $\delta$ ,  $0 \leq \frac{1}{\delta} < \infty$ , it follows that:

$$\|w\|_2^2 = \int_0^\infty w'(t)w(t)dt \leq \frac{1}{\delta} \quad (10)$$

In this case, the problem can then be summarized as follows.

*Problem 1:* Determine the anti-windup matrices  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$  and a region of asymptotic stability, denoted  $\mathcal{E}_0$ , as large as possible, such that

- 1) The closed-loop system (9) with  $w = 0$  is asymptotically stable for any initial condition belonging to the set  $\mathcal{E}_0$ .
- 2) The map from  $w$  to  $z$  is finite  $\mathcal{L}_2$  gain stable with gain  $\gamma > 0$ .

The implicit objective in the first item of Problem 1 is to optimize the size of the basin of attraction for the closed-loop system (9) (with  $w = 0$ ) over the choice of matrices  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$ . This can be accomplished indirectly by searching for an anti-windup compensator defined from  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$  that leads to a region of stability for the closed-loop system as large as possible. Considering quadratic Lyapunov functions and ellipsoidal regions of stability, the maximization of the region of stability can be accomplished by using some well-known size optimization criteria for ellipsoidal sets, such as: minor axis maximization, volume maximization, or even the maximization of the ellipsoid in certain given directions. On the other hand, when the open-loop system is asymptotically stable, it can be possible to search for the controller matrices in order to guarantee the global asymptotic stability of the origin of the closed-loop system.

*Remark 3:* Problem 1 can be studied in the context of a static anti-windup gain by considering  $n_{aw} = 0$ ,  $A_{aw} = 0$ ,  $B_{aw} = 0$ ,  $C_{aw} = 0$  and by computing the gain  $D_{aw}$ .

In next subsections III-B and III-C, some results to address Problem 1 will be presented in the case of full-order or low (or reduced) order anti-windup controller. At this stage, it is very important to underline that the notion of full-order anti-windup has different meanings depending on the authors in the literature. For example, in [38], [107], [113], the authors use full order to mean plant order (i.e.,  $n_{aw} = n_p$ ). In contrast, in [6], [96], the authors use full order to mean  $n_{aw} = n_p + n_c$ . Both cases will be discussed.

### A. Preliminary elements

All the results developed in the DLAW context are based upon the use of dead-zone nonlinearities and associated modified sector conditions. Indeed, it is important to underline that every system,

which involves saturation-type nonlinearities, may be easily rewritten with dead-zone nonlinearities. Considering the saturation function  $\text{sat}_{u_0}(y_c)$ , the resulting dead-zone nonlinearity  $\phi(y_c)$  is obtained from  $\phi(y_c) = \text{sat}_{u_0}(y_c) - y_c$ . Thus, by considering the extended state vector  $x$  defined by

$$x = \begin{bmatrix} x_p \\ x_c \\ x_{aw} \end{bmatrix} \in \mathfrak{R}^{n_p+n_c+n_{aw}} \quad (11)$$

the closed-loop system is described as follows:

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}_1\phi(y_c) + \mathcal{B}_2w \\ y_c &= \mathcal{C}_1x + \mathcal{D}_{11}\phi(y_c) + \mathcal{D}_{12}w \\ z &= \mathcal{C}_2x + \mathcal{D}_{21}\phi(y_c) + \mathcal{D}_{22}w \end{aligned} \quad (12)$$

where

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} \mathbb{A} & B_v C_{aw} \\ 0 & A_{aw} \end{bmatrix}; \mathcal{B}_1 = \begin{bmatrix} B_\phi + B_v D_{aw} \\ B_{aw} \end{bmatrix} \\ \mathcal{B}_2 &= \begin{bmatrix} B_2 \\ 0 \end{bmatrix}; \mathcal{C}_1 = [ C_1 \quad C_{v1} C_{aw} ] \\ \mathcal{D}_{11} &= D_1 + C_{v1} D_{aw} \\ \mathcal{C}_2 &= [ C_2 \quad C_{v2} C_{aw} ]; \mathcal{D}_{21} = D_2 + C_{v2} D_{aw} \end{aligned} \quad (13)$$

with  $\mathbb{A}$  defined in Remark 1 and the following state-space matrices:

$$\begin{aligned} B_v &= \begin{bmatrix} B_{pu}\Delta^{-1} [ 0 \quad I_m ] \\ B_c D_{pu} \Delta^{-1} [ 0 \quad I_m ] + [ I_{n_c} \quad 0 ] \end{bmatrix} \\ B_\phi &= \begin{bmatrix} B_{pu}(I_m + \Delta^{-1} D_c D_{pu}) \\ B_c D_{pu}(I_m + \Delta^{-1} D_c D_{pu}) \end{bmatrix} \\ C_1 &= [ \Delta^{-1} D_c C_p \quad \Delta^{-1} C_c ] \\ C_{v1} &= \Delta^{-1} [ 0 \quad I_m ]; D_1 = \Delta^{-1} D_c D_{pu} \\ C_2 &= [ C_z + D_{zu} \Delta^{-1} D_c C_p \quad D_{zu} \Delta^{-1} C_c ] \\ C_{v2} &= D_{zu} \Delta^{-1} [ 0 \quad I_m ] \\ D_2 &= D_{zu}(I_m + \Delta^{-1} D_c D_{pu}) \\ B_2 &= \begin{bmatrix} B_{pu} \Delta^{-1} (D_{cw} + D_c D_{pw}) + B_{pw} \\ B_c D_{pu} \Delta^{-1} (D_{cw} + D_c D_{pw}) + B_{cw} + B_c D_{pw} \end{bmatrix} \\ \mathcal{D}_{12} &= \Delta^{-1} (D_{cw} + D_c D_{pw}) \\ \mathcal{D}_{22} &= D_{zw} + D_{zu} \Delta^{-1} (D_{cw} + D_c D_{pw}) \end{aligned} \quad (14)$$

The results developed in the sequel rest on a recent characterization of the dead-zone nonlinearity using some local sector conditions. Hence, in this context, let us define the following polyhedral set:

$$S(u_0) = \{y_c \in \mathfrak{R}^m, \omega \in \mathfrak{R}^m; -u_0 \preceq y_c - \omega \preceq u_0\} \quad (15)$$

*Lemma 1:* [99] If  $y_c$  and  $\omega$  are elements of  $S(u_0)$  then the nonlinearity  $\phi(y_c)$  satisfies the following inequality:

$$\phi(y_c)' S^{-1}(\phi(y_c) + \omega) \leq 0 \quad (16)$$

for any diagonal positive definite matrix  $S \in \mathfrak{R}^{m \times m}$ .

*Remark 4:* Particular formulations of Lemma 1 can be found in [33] (concerning the case of systems with a single saturation function) and in [83], [100] (concerning systems presenting both magnitude and dynamics restricted actuators). Moreover, it should be pointed out that  $\omega = \Lambda y_c$ , where  $\Lambda$  is a diagonal matrix such that  $0 < \Lambda \leq I_m$ , (see for instance [52], [60]), is a particular case of the generic formulation (16). A key advantage of condition (16) is that, contrary to the classical case with  $\omega = \Lambda y_c$ , it allows the formulation of conditions directly in LMI form. Moreover, Lemma 1 caters easily for nested saturations.

*Remark 5:* Particular formulations of Lemma 1 can be stated by considering

$$S_\omega(u_0) = \{\omega \in \mathfrak{R}^m; -u_0 \preceq \omega \preceq u_0\} \quad (17)$$

$$-\phi(y_c)' S^{-1}(\text{sat}_{u_0}(y_c) + \omega) \geq 0 \quad (18)$$

instead of (15) and (16), respectively. Such a formulation is used in [55] and [56] and gives simplified conditions in the case when  $y_c$  depends on  $\phi(y_c)$  leading to implicit function and nested conditions.

Lemma 1, as written, is rather dedicated to the regional case, but it can be considered in a global context. For this, it suffices to consider  $\omega = y_c$  and therefore the set  $S(u_0)$  is the state space. Hence, the sector condition (16) is globally satisfied. Note, however, that the global stability of the closed-loop system subject to such a nonlinearity will be obtained only if some assumptions on the stability of the open-loop system are verified.

### B. DLAW schemes with global guarantees

Solutions to the anti-windup problem in the global case are typically of somewhat lower complexity than the local anti-windup problem because global solutions do not require some sort of description of the saturated system's region of attraction. For stable linear plants (i.e.  $\Re\lambda_i(A_p) < 0, \forall i$ ) it is possible to develop anti-windup compensators which go beyond local guarantees and provide stability for all  $x \in \mathfrak{R}^{n_p+n_c+n_{aw}}$ . This simplifies the anti-windup problem somewhat as now a description of the region of attraction is not required, as it is the whole state-space.

Before presenting some conditions to design DLAW schemes with global guarantees, let us provide a quick overview regarding this global context. Although in principle the design of anti-windup compensators for stable linear systems subject to input saturation could be achieved using absolute stability tools such as the Circle and Popov criteria [60], these were most useful for single-loop systems and analysis. Design was somewhat harder until LMI's began to emerge, although useful classical tools are reported in [58], [114]. One of the first applications of LMI's to the anti-windup synthesis problem was given by [69] which considered the anti-windup synthesis problem as an application of absolute stability theory involving common Lyapunov functions. [69] followed [62] by considering only static anti-windup compensators, viz  $A_{aw}, B_{aw}, C_{aw}$  were all matrices of zero row and/or column dimension and only  $D_{aw}$  was sought. Similar ideas to the above were also exploited in [85] where the observer-based structure of anti-windup compensators was used (i.e. again the anti-windup compensator was static). Even if the papers [69] and [85] were important steps in anti-windup design and they both used the LMI framework as part of the anti-windup synthesis procedure, the design methods were not wholly LMI-based as the inequalities were really bilinear matrix inequalities which were linearized by fixing one of the free variables. The late part of the 20th and early part of the 21st century saw the development of two anti-windup synthesis methods which were wholly LMI-based. In [76], a method continues the static anti-windup theme but effectively uses the Circle Criterion with an  $\mathcal{L}_2$  gain constraint to devise a purely LMI based synthesis method. The work in [76] is similar to that in [90] where a small gain approach is used to synthesize an anti-windup compensator. The results in [90] are generally stated in terms of bilinear matrix inequalities which transpire to be linear in the special cases of single-input-single-output systems and also when the static multiplier is fixed. However the advantage of the results of [90] is that they can be applied to unstable systems, although no estimate of the region of attraction is provided. A similar result was reported in [108] except that the results were improved in two ways. Firstly provision for low order anti-windup synthesis was made, which significantly enlarges the class of compensators which can be designed and also allows compensators with superior performance (in terms of their  $\mathcal{L}_2$  gains) to be obtained. Secondly, [108] proposed a performance map which allowed deviation from linear performance to be minimized explicitly via LMI's. Moreover, in [40] and [38] conditions were given which allowed a general  $n_{aw}$ 'th order anti-windup compensator to be constructed using "almost" LMI conditions. In the general case these were non-convex but under certain conditions could be relaxed to be linear.

Let us first suppose that the anti-windup compensator is given (that is matrices  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$ ,  $D_{aw}$  are known) and let us denote  $n = n_p + n_c + n_{aw}$ . The following general result can be stated by using the framework developed in [7] (see also [38]).

*Theorem 1:* If there exist a symmetric positive definite matrix  $Q \in \mathfrak{R}^{n \times n}$ , a diagonal matrix  $S \in \mathfrak{R}^{m \times m}$ , a positive positive scalar  $\gamma$  such that the following conditions hold:

$$\begin{bmatrix} \text{He}[AQ] & B_1 S - QC'_1 & B_2 & QC'_2 \\ \star & \text{He}[-S - \mathcal{D}_{11}S] & -\mathcal{D}_{12} & S\mathcal{D}'_{21} \\ \star & \star & -I & \mathcal{D}'_{22} \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (19)$$

then,

- 1) when  $w = 0$ ,  $w(t) = 0, \forall t > t_1 \geq 0$ , the nonlinear closed-loop system (12) remains stable for all initial conditions  $x(0) \in \mathfrak{R}^n$ .
- 2) When  $w \neq 0$ ,
  - the closed-loop trajectories remain bounded for any  $w(t) \in \mathcal{L}_2$  and any initial conditions;
  - the map from  $w$  to  $z$  is finite  $\mathcal{L}_2$  gain stable with:

$$\int_0^T z(t)'z(t) dt \leq \gamma \int_0^T w(t)'w(t) dt + \gamma x(0)'Q^{-1}x(0), \forall T \geq 0. \quad (20)$$

Let us now focus on the synthesis issue as stated in Problem 1. Then, the analysis variable  $Q$  which is introduced in Theorem 1 and the matrices  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$ ,  $D_{aw}$  of the anti-windup compensator have to be optimized simultaneously. As a result, the inequality (19) which is a priori a BMI, is no longer convex. However, in the full-order case (i.e.  $n_{aw} = n_p + n_c$ ), some particular structures can be exploited to derive a convex characterization.

*Theorem 2:* There exists an anti-windup controller ( $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$ ,  $D_{aw}$ ) such that the conditions of Theorem 1 are satisfied if there exist two symmetric positive definite matrices  $X \in \mathfrak{R}^{(n_p+n_c) \times (n_p+n_c)}$ ,  $Y \in \mathfrak{R}^{(n_p+n_c) \times (n_p+n_c)}$ , a positive scalar  $\gamma$  such that the following conditions hold:

$$\begin{bmatrix} A'X + XA & XB_2 & C'_2 \\ \star & -I & \mathcal{D}'_{22} \\ \star & \star & -\gamma I \end{bmatrix} < 0 \quad (21)$$

$$\begin{bmatrix} Y_1 A'_p + A_p Y_1 & B_{pw} & Y_1 C'_z \\ \star & -I & \mathcal{D}'_{zw} \\ \star & \star & -\gamma I \end{bmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} X & \star \\ I & Y \end{bmatrix} > 0 \quad (23)$$

where  $Y_1 \in \mathfrak{R}^{n_p \times n_p}$  is the block (1, 1) of  $Y$ .

Let us underline that the notion of full-order anti-windup has different meaning depending on the authors in the literature. For example, in [42], [113] [55], the authors use full order to mean plant order (i.e.,  $n_{aw} = n_p$ ). To the contrary, in [6] or in the current paper, the full order means  $n_{aw} = n_p + n_c$ . In [38], a theorem involving two relations similar to (21) and (22) together with a rank condition is provided. This result is an existence condition, which is in general difficult to satisfy due to the nonconvex rank constraint. Such a result is indeed similar to the LMI  $\mathcal{H}_\infty$  synthesis problem which is generally a number of LMI's coupled with a rank constraint [20]. However, it transpires that, for the special cases of  $n_{aw} = 0$  and  $n_{aw} \geq n_p$ , that the rank constraint vanishes leaving simply a set of linear matrix inequalities. Although the above theorem is an existence condition, an anti-windup compensator yielding the  $\mathcal{L}_2$  gain  $\gamma > 0$  can then be constructed by solving a further LMI [38]. Furthermore, mirroring techniques used in low order robust controller design, a trace minimization can be performed to "remove" the rank constraint as advocated in [21]. Although this procedure is not guaranteed to work, experience in other areas of the control field has shown this to sometimes be successful.

### C. DLAW schemes with regional guarantees

Relatively few papers were dedicated to anti-windup strategy for exponentially unstable systems, that is in a regional or local context, until around 2000. The majority of those papers presented algorithms for computing anti-windup compensators, but without any guarantees about stability: see, for example [115] and [106]. However, a key element in the local case is the ability to guarantee the stability and therefore to characterize the region of stability for the closed-loop system (9). One of the first paper addressing clearly the local case with a guarantee of stability was Teel's paper [104]. In [104], an algorithm, which requires measurement of the exponentially unstable modes, was proposed. The results provided an anti-windup compensator extending those presented in [106] by removing some restrictions on the transient behavior of the unsaturated feedback loop. In [104], the conditions were not however in an LMI form. Indeed, one of the first applications of LMI's to the anti-windup synthesis problem in the local was given in [32], [33] by considering only static anti-windup loop ( $A_{aw}, B_{aw}, C_{aw}$  were all matrices of zero row and/or column dimension and only  $D_{aw}$  was sought). [33] followed in particular the papers [12], [37] in which the conditions proposed are not into LMI form but in BMI (bilinear matrix inequalities) form due mainly to the way chosen to model the saturation terms based on Linear Differential Inclusions or classical sector conditions.

Furthermore, in [112], an extension of [38] was proposed allowing the computation of dynamic anti-windup compensators. Nevertheless, contrary to [33], the region in which the stability of the closed-system is guaranteed is not clearly described. Recently, several papers dealing with performance, like  $\mathcal{L}_2$  performance, have been published mainly in the context of dynamic anti-windup compensator design: see, for example, [95] in which the first six chapters are dedicated to anti-windup strategies and their applications. See also [55], [56] and [6].

Regarding the regional context, the following general result can be stated, by using the same framework as in Theorem 1.

*Theorem 3:* If there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n \times n}$ , a matrix  $Z \in \mathbb{R}^{m \times n}$ , a positive diagonal matrix  $S \in \mathbb{R}^{m \times m}$ , a positive scalar  $\gamma$  such that the following conditions hold:

$$\begin{bmatrix} \text{He}[\mathcal{A}Q] & \mathcal{B}_1 S - QC'_1 - Z' & \mathcal{B}_2 & QC'_2 \\ \star & \text{He}[-S - \mathcal{D}_{11}S] & -\mathcal{D}_{12} & S\mathcal{D}'_{21} \\ \star & \star & -I & \mathcal{D}'_{22} \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (24)$$

$$\begin{bmatrix} Q & Z'_{(i)} \\ \star & \delta u_{0(i)}^2 \end{bmatrix} \geq 0, i = 1, \dots, m \quad (25)$$

then,

- 1) when  $w = 0$ ,  $w(t) = 0$ ,  $\forall t > t_1 \geq 0$ , the nonlinear closed-loop system (12) remains stable for all initial conditions  $x(0) \in \mathcal{E}(Q, \delta) = \{x \in \mathbb{R}^n; x'Q^{-1}x \leq \delta^{-1}\}$ .
- 2) When  $w \neq 0$ , for  $x(0) = 0$ ,
  - the closed-loop trajectories remain bounded in the set  $\mathcal{E}(Q, \delta)$ ;
  - the map from  $w$  to  $z$  is finite  $\mathcal{L}_2$  gain stable with:

$$\int_0^T z(t)'z(t) dt \leq \gamma \int_0^T w(t)'w(t) dt, \forall T \geq 0 \quad (26)$$

*Remark 6:* In the case of a non-null initial condition  $x(0)$ , a positive scalar  $\beta$  has to be considered in order to ensure that the closed-loop trajectories remain bounded in  $\mathcal{E}(Q, \beta^{-1} + \delta)$ ,  $\forall x(0) \in \mathcal{E}(Q, \beta^{-1})$ . Hence,  $\mathcal{E}(Q, \beta^{-1})$  will be seen as a set of admissible initial conditions. From this, there clearly appears a trade-off between the size of the set of admissible conditions (given basically by  $\beta$ ), the size of the region of stability (depending on  $\beta^{-1} + \delta$ ) and the bound on the admissible disturbance (given by  $\delta$ ). Furthermore, the finite  $\mathcal{L}_2$ -gain from  $w$  to  $z$  will present a bias term and will read:

$$\|z\|_2^2 \leq \gamma \|w\|_2^2 + \gamma x(0)'Q^{-1}x(0) \leq \gamma(\|w\|_2^2 + \beta).$$

A detailed discussion about this issue can be found in [13], where the stabilization via state feedback of systems presenting actuator saturation is considered.

Similarly to the global context, we can derive a convex characterization in order to design the anti-windup compensator satisfying Theorem 3.

*Theorem 4:* There exists an anti-windup controller  $(A_{aw}, B_{aw}, C_{aw}, D_{aw})$  such that the conditions of Theorem 3 are satisfied if there exist two symmetric positive definite matrices  $X \in \mathfrak{R}^{(n_p+n_c) \times (n_p+n_c)}$ ,  $Y \in \mathfrak{R}^{(n_p+n_c) \times (n_p+n_c)}$ , a matrix  $Z_1 \in \mathfrak{R}^{m \times (n_p+n_c)}$ , a matrix  $U \in \mathfrak{R}^{m \times (n_p+n_c)}$ , a positive scalar  $\gamma$  such that the following conditions hold:

$$\begin{bmatrix} \mathbb{A}'X + X\mathbb{A} & XB_2 & C_2' \\ * & -I & D_{22}' \\ * & * & -\gamma I \end{bmatrix} < 0 \quad (27)$$

$$\begin{bmatrix} \text{He} \left[ A_p Y_1 - B_{pu} Z_1 \begin{bmatrix} I_{n_p} \\ 0 \end{bmatrix} \right] & B_{pw} \\ * & -I \\ * & * \\ Y_1 C_z' - \begin{bmatrix} I_{n_p} & 0 \end{bmatrix} Z_1' D_{zu}' & * \\ & D_{zw}' \\ & -\gamma I \end{bmatrix} < 0 \quad (28)$$

$$\begin{bmatrix} X & * & * \\ I & Y & * \\ U_{(i)} & Z_{1(i)} & \delta u_{0(i)}^2 \end{bmatrix} \geq 0, i = 1, \dots, m \quad (29)$$

with  $Y_1 \in \mathfrak{R}^{n_p \times n_p}$  is the block  $(1, 1)$  of  $Y$ .

*Remark 7:* In order to optimize the size of the stability domain  $\mathcal{E}(Q, \delta)$ , the performance constraint can be relaxed by setting  $\gamma = \infty$ . In that case, inequality (24) is modified by removing the two last lines and columns, and therefore the LMI constraints (27) and (28) become respectively:

$$\mathbb{A}'X + X\mathbb{A} < 0 \quad (30)$$

$$\text{He} \left[ A_p Y_1 - B_{pu} Z_1 \begin{bmatrix} I_{n_p} \\ 0 \end{bmatrix} \right] < 0 \quad (31)$$

*Remark 8:* If the open-loop matrix  $A_p$  is Hurwitz, the global asymptotic stabilization problem can be addressed by considering  $U = 0$  and  $Z_1 = 0$  in Theorem 4 leading to Theorem 2.

*Remark 9:* Conditions of Theorem 3 (and Theorem 1) imply the satisfaction  $QA' + AQ < 0$ . This guarantees the asymptotic stability of the matrix  $A_{aw}$  of the anti-windup compensator, contrarily to the approach pursued in [28]. Moreover, in the spirit of [86], it would also be possible to modify conditions of Theorem 1, 2, 3 or 4 in order to constrain only the poles of the anti-windup controller and not the whole closed-loop plant dynamics.

#### D. Algorithms

Based on Theorems 2 and 4, several algorithms are now developed in order to compute full and fixed-order anti-windup compensators. Moreover, several LMI-based optimization problems can be proposed to compute the anti-windup controller in order to optimize one of the following criteria: maximization of the  $\mathcal{L}_2$  bound on the admissible disturbances (disturbance tolerance maximization which corresponds to minimize  $\delta$ ); the minimization of the induced  $\mathcal{L}_2$ -gain between the disturbance  $w$  and the regulated output  $z$  (disturbance rejection maximization which corresponds to minimize  $\gamma$  for a given  $\delta$ ); or the maximization of the region where the asymptotic stability of the closed-loop system is ensured (maximization of the region of attraction).

Using Theorem 4, the existence of a full-order anti-windup compensator is easily checked by solving a finite set of LMIs. Then the matrix  $Q$  (from Theorem 3) is obtained as:

$$Q = \begin{bmatrix} Y & I \\ N & 0 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & M \end{bmatrix}^{-1} \text{ with } M'N = I - XY \quad (32)$$

The synthesis variables  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$  can finally be calculated as the solution of (24) which becomes convex as soon as  $Q$  is fixed. Moreover, using a change of variables  $\bar{B}_{aw} = B_{aw}S$ ,  $\bar{D}_{aw} = D_{aw}S$ , it can be observed that the matrices  $S$  and  $Z$  do not have to be fixed. This offers some additional degrees-of-freedom that can be used for example to add constraints on the controller matrix  $A_{aw}$  [86].

*Algorithm 1:* Full-order synthesis (Regional case)

**Step 1.** Given  $\delta$ , minimize  $\gamma$  under the LMI constraints (27), (28), (29) with respect to the variables  $\gamma$ ,  $X$ ,  $Y$ ,  $U$  and  $Z_1$ .

**Step 2.** Compute  $Q$  as the solution of (32).

**Step 3.** Fix  $Q$  in inequality (24) and solve the convex feasibility problem with respect to the variables  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$ .

★

In the global context (Theorem 2), the previous algorithm is simplified as follows.

*Algorithm 2:* Full-order synthesis (Global case)

**Step 1.** Minimize  $\gamma$  under the LMI constraints (21), (22), (23) with respect to the variables  $\gamma$ ,  $X$ ,  $Y$ .

**Step 2.** Compute  $Q$  as the solution of (32).

**Step 3.** Fix  $Q$  in inequality (19) and solve the convex feasibility problem with respect to the variables  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$ .

★

As a preliminary step to Algorithm 1, it is interesting to compute the largest admissible stability region without a performance constraint. Note that, when the open-loop matrix  $A_p$  is asymptotically stable, if the conditions shown in Theorem 1 or Theorem 2 are feasible then the region of stability is the whole state space. Otherwise, the problem of maximizing the region of stability consists of maximizing the size of  $\mathcal{E}(Q, \delta)$ . Different linear optimization criteria  $J(\cdot)$ , associated to the size of  $\mathcal{E}(Q, \delta)$ , can be considered, like the volume:  $J = -\log(\det(\delta^{-1}Q))$ , or the size of the minor axis:  $J = -\lambda$ , with  $Q \geq \lambda I_n$ . A given *shape set*  $\Xi_0 \in \mathfrak{R}^n$  and a scaling factor  $\beta$ , where  $\Xi_0 = Co\{v_r \in \mathfrak{R}^n ; r = 1, \dots, n_r\}$  can also be considered and the associated criterion may then be to maximize the scaling factor  $\beta$  such that  $\beta\Xi_0 \subset \mathcal{E}(Q, \delta)$  [31], [54]. In particular it is interesting to address this problem in the plant space. For the sake of simplicity, in this case, we set  $\delta = 1$ . Using Remark 7, this can be done by changing Algorithm 1 as follows.

*Algorithm 3:* Region of stability

**Step 1.** Choose a set of interesting directions  $v_i \in \mathfrak{R}^{n_p}$ ,  $i = 1, \dots, q$ . Define  $\bar{v}_i = [v_i' \ 0] \in \mathfrak{R}^{n_p+n_c}$ ,  $i = 1, \dots, q$ .

**Step 2.** Minimize  $\mu$  under the LMI constraints (30), (31), (29) and  $\mu - \bar{v}_i' X \bar{v}_i \geq 0$ ,  $i = 1, \dots, q$ , with respect to the variables  $\mu$ ,  $X$ ,  $Y$ ,  $U$  and  $Z_1$ .

**Step 3.** Fix  $Q$  in the modified inequality (24) and solve the convex feasibility problem with respect to the variables  $A_{aw}$ ,  $B_{aw}$ ,  $C_{aw}$  and  $D_{aw}$ .

★

A special case of interest resides in the case where the matrices  $A_{aw}$  and  $C_{aw}$  of the anti-windup compensator are a priori fixed. Indeed, the BMI constraint (24) of Theorem 3 is convex as soon as the matrices  $A_{aw}$  and  $C_{aw}$  of the anti-windup controller are fixed. This allows the order of anti-windup compensator to be reduced, and also simplifies the computational effort. Based on this remark, a simple algorithm can be derived.

*Algorithm 4:* Fixed-dynamics synthesis

**Step 1.** Choose appropriate  $A_{aw}$  and  $C_{aw}$ .

**Step 2.** Minimize  $\gamma$  under the LMI problem constraints (24) and (25) with respect to the variables  $Q$ ,  $S$ ,  $Z$ ,  $\bar{B}_{aw}$ ,  $\bar{D}_{aw}$ .

**Step 3.** Compute  $B_{aw} = \bar{B}_{aw}S^{-1}$  and  $D_{aw} = \bar{D}_{aw}S^{-1}$ .

★

The main difficulty in the above algorithm resides in the first step, i.e., in choosing the matrices  $A_{aw}$  and  $C_{aw}$  adequately. However, according to the approach developed in [7] (see also [50], [108]) this choice may be carried out by considering the poles of the anti-windup controller. These poles can be chosen by selecting a part of those obtained in the full order design case. Typically, the slow and fast dynamics are eliminated. Alternatively, an iterative procedure starting from the static case can be used. The list of poles is then progressively enriched until the gap between the full and reduced order cases becomes small enough. Note then that the order of the controller is now given by  $n_{aw} = n_1 + 2n_2$  ( $n_1$  and  $n_2$  corresponding respectively to real and complex poles). The two parameters  $n_1$  and  $n_2$  have to be chosen sufficiently small, so that  $n_{aw} < n_p + n_c$ . Exploiting a similar idea, the algorithm proposed in [59] is based on a different decomposition of the anti-windup controller using dyadic forms. Even if such forms are slightly more general, the associated algorithm requires the user to specify output pole directions which may not be trivial.

Static anti-windup compensators are easily computed as particular solutions of previous algorithms since relation (19) in the global case or relation (24) in the regional case are simplified and allow to obtain directly  $\bar{D}_{aw} = D_{aw}S$  [33], [101]

*Remark 10:* A nonlinear  $\mathcal{L}_2$ -induced performance level does not unfortunately provide a direct answer to the most standard anti-windup control problem which consists in minimizing the saturation effects for a restricted class of reference signals. The class of  $\mathcal{L}_2$ -bounded signals which is typically considered is indeed very large, while in practice, it is most often sufficient to consider step-like reference inputs with bounded magnitudes. To avoid the introduction of external inputs as  $w$  which often lead to more conservative synthesis conditions, both reference and perturbation signals ( $w_r$  and  $w_p$ ) can be generated by a stable autonomous linear system [7], [87].

### E. Extensions

Several extensions can be considered: the presence of delays in the systems; the presence of saturation in the output or still the presence of rate and magnitude saturation.

1) *Time-delay systems:* Systems presenting time-delays have received special attention in the control systems literature: see for example [14], [46], [78], [84]. This interest naturally comes from the fact that time-delays appear in many kinds of control systems (e.g. chemical, mechanical and communication systems) and their presence can be source of performance degradation and instability. In this sense, the literature offers many works giving conditions for ensuring stability as well as performance and robustness requirements, considering or not the delay dependence.

Considering that many practical systems present both time-delays and saturating inputs, from the considerations above, it becomes important to study the stability issues regarding this kind of systems. With this aim, different techniques can be investigated: in particular the characterization of admissible regions of stability is often based on the use of Razumikhin or Lyapunov-Krasovskii functionals. In parallel, another popular technique consists of approximating the delay through a Padé approximation, which implies an increase in the order of the closed-loop system. It may be used in order to prove some robustness properties with respect to the presence of delays. Techniques based on Lyapunov functionals or Padé approximations can be used to analyze the stability and the performance of saturated systems.

In the anti-windup approach context, let us cite [80] and [117], where dynamic anti-windup strategies are considered for systems with input and output delays. It should be highlighted that the results in [80] can be applied only to open-loop stable systems and that in [117], the main focus is the formal definition and characterization of the  $\mathcal{L}_2$  gain based anti-windup. Differently from [80], [117], in [36], [97], [98], the design of anti-windup gains was studied with the aim of enlarging the region of attraction of the

closed-loop system. In [97] a method for computing anti-windup gains for systems presenting only input delays is proposed leading to BMI conditions. In contrast to [80], the proposed techniques can be applied to both stable and unstable open-loop systems. Finally, in [28] the design of a non-rational dynamic anti-windup controller is done in order to guarantee both that the trajectories of the system are bounded and a certain  $\mathcal{L}_2$  performance level is achieved by the regulated outputs.

Some application oriented studies can be found in [116] in the context of open water channels and in [7] in the context of a fighter aircraft, where the delays are replaced by first-order Padé approximations.

2) *Sensor saturation*: The study of systems subject to sensor saturation is less developed, with only a few papers devoted to this topic [94], [67], [61], [63], [11], [57]. Sensor saturation is normally found in applications where cost prohibits the use of sensors with adequate range, leading to sensor saturation for large reference/disturbance inputs. Alternatively sensor saturation can model the situation where only the sign of the output is known. In this case, the sign function can be modelled by a saturation function with a steep gradient.

A naive appraisal of the sensor saturation problem suggests that it is similar to the actuator saturation problem with the plant and the controller interchanged. In fact this is not the case [67], [94]; one of the crucial differences between the two problems resides in the availability of the “un-saturated” signal. In the actuator saturation case, knowledge of both the output produced by the linear controller and the saturated version of this (i.e. the signals either side of the saturation block) is generally assumed. In the sensor saturation case, it is not realistic to assume that the actual plant output is known; only the saturated version of this is known (otherwise there would be no problem!). This is problematic for the anti-windup approach, and hence an observer may be used to overcome this difficulty. If the study of systems subject to sensor saturation is under-developed, the study of anti-windup compensation for this class of systems is less developed still. To the best of the authors’ knowledge, the only literature discussing this approach are the papers [94], [109], [110] that establish conditions for both local stability and global stability with  $\mathcal{L}_2$  gain respectively. These two papers are also related to the paper by Park [81] in which state-constrained systems are considered and an anti-windup type approach is proposed.

3) *Rate and magnitude saturation*: Rate-limits are of particular importance in mechanical systems where the inertia in various components of the actuator prevents it from moving very fast, thereby limiting the rate of the control signal which it can pass to the plant. It is important to note that modelling of rate saturation nonlinearities is not identical within the constrained control literature [66], [77], [89] and is even disparate within the anti-windup literature [4], [88], [91]. However, a useful model of the rate-limit [35], [88], [89], [91] appears to be obtained by cascading the standard saturation function with an integrator and gain and enclosing this within a feedback loop. Such a representation of a rate-limit is attractive, even if one note of caution is the presence of the integrator: see [91] for example.

The merits of different ways to model actuator position and rate limits are discussed in [89]. Actuator rate-limits have recently attracted a lot of interest due to their role in pilot-induced-oscillations (PIOs) and the subsequent untimely demise of several aircraft due to rate-limited actuators - see [2], [10], [16], [83] for further details. In [22] and [25], a different model of rate saturation is proposed. An anti-windup scheme is developed in a similar way to that used when dealing with only magnitude saturation.

Moreover, in the context of rate or dynamics and magnitude limitations, both conditions in local and global cases can be derived to design static or dynamic anti-windup compensators. Non-constructive conditions are proposed in [4] to characterize a plant-order anti-windup controller. Constructive conditions to exhibit anti-windup schemes are proposed in [95] for magnitude and rate saturation, in [100] for magnitude and dynamics saturation. Some applications of such studies are given, for example, in [10], [72], [83], [105].

#### IV. MODEL RECOVERY ANTI-WINDUP DESIGN ALGORITHMS

##### A. Overview of the architecture

As anticipated in Section II-D (see in particular equation (8)), model recovery anti-windup corresponds to selecting the anti-windup compensator in Figure 1 as a dynamical filter containing a (possibly

approximated) model of the plant dynamics. The aim of this filter is to keep track of what the closed-loop response would be in the absence of saturation. In particular, call  $y_{c,\ell}$  the controller output response that one would get without saturation and  $z_\ell$  the plant performance output obtained without saturation. Then, under the assumption of perfect knowledge of the plant model, inserting the filter (8) in the closed-loop corresponds to introducing in the closed-loop the following dynamics:

$$\begin{aligned} \dot{x}_{aw} &= A_p x_{aw} + B_{pu}(\text{sat}_{u_0}(y_{c,\ell} + v_y) - y_{c,\ell}) \\ z - z_\ell &= C_z x_{aw} + D_{zu}(\text{sat}_{u_0}(y_{c,\ell} + v_y) - y_{c,\ell}), \end{aligned} \quad (33)$$

which clarifies the relation between the anti-windup dynamics (whose state is  $x_{aw}$ ) and the performance output deviation from the desirable unconstrained response  $z_\ell$ .

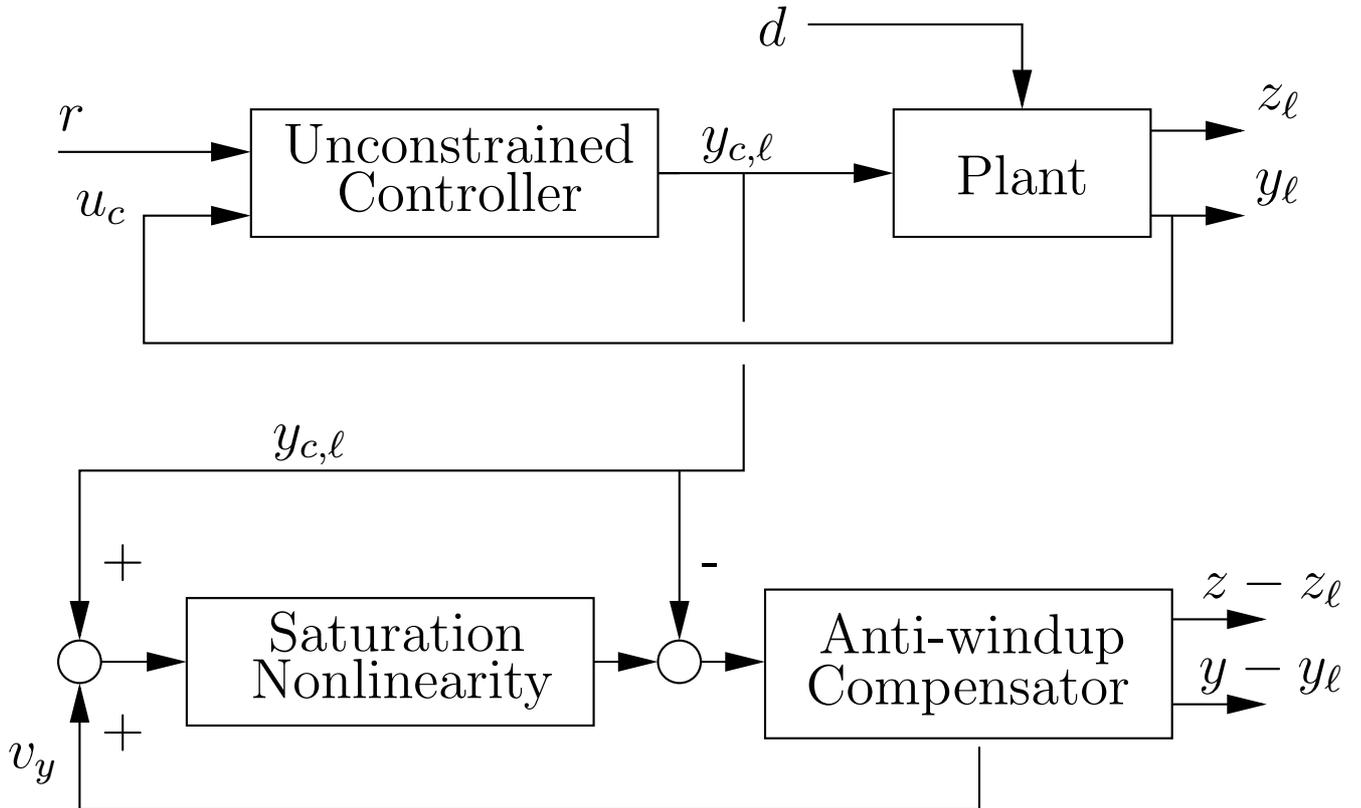


Fig. 2. The equivalent mismatch representation of the MRAW compensation scheme.

A schematic representation of equation (33) is illustrated in Figure 2 where the upper subsystem is given by the dynamics of the controller (2) (with  $v_x = 0$  and  $v_y = 0$ ) in closed-loop with the fictitious plant arising from studying the dynamics in the transformed state  $x_\ell := x - x_{aw}$ . A formal proof of the equivalence between the scheme of Figure 2 and the closed-loop MRAW dynamics of (1), (2), (8) can be found in [106], [118]. A parallel proof for nonlinear plants has been given in [74].<sup>3</sup> This formal proof also serves as a proof for the relation in equation (33) between the unconstrained signals  $y_{c,\ell}$ ,  $z_\ell$  and the performance output deviation  $z - z_\ell$ .

*Remark 11:* It should be emphasized that the notation adopted in this paper is inconsistent with some of the previous work on model recovery anti-windup (i.e.,  $\mathcal{L}_2$  anti-windup). We adopt this notation here for consistency with the DLA architecture, however for clarification purposes, we should point out that often in the MRAW literature the controller dynamics (2) is given without the signals  $v_x$  and

<sup>3</sup>See also [73], where extra intuition is given with respect to this property of unconstrained response recovery.

$v_y$ . Moreover, the MRAW compensator (8) is formulated as follows

$$\begin{aligned} \dot{x}_{aw} &= A_p x_{aw} + B_{pu}(\text{sat}_{u_0}(y_c + v_1) - y_c) \\ v_2 &= C_p x_{aw} + D_{pu}(\text{sat}_{u_0}(y_c + v_1) - y_c) \\ u &= \text{sat}_{u_0}(y_c + v_1), \quad u_c = y_p + v_2, \end{aligned} \quad (34)$$

where  $v_1$  coincides with the signal  $v_y$  that we use here and  $v_2$  is often also denoted as  $y_{aw}$  (for obvious reasons arising from its parallel definition to  $y$ ). All the results here reported are equivalent to those found in the literature, however stated with this different notation motivated by the fact that the discussion in this paper groups DLAW and MRAW schemes.

Based on the above discussion and on the result of equation (33), it becomes evident that the anti-windup goal of forcing as much as possible  $z$  to recovering the unconstrained response  $z_\ell$  amounts to the goal of driving to zero the anti-windup output  $z_{aw} := z - z_\ell$  in equation (33) as effectively as possible. In other words, the anti-windup goal is transformed into a bounded stabilization problem with an input matched disturbance  $y_{c,\ell}$ . Furthermore, the way the ‘‘disturbance’’  $y_{c,\ell}$  acts on this bounded stabilization problem is quite peculiar, indeed it shifts up and down the saturation levels of an equivalent time varying saturation nonlinearity affecting the stabilizing signal  $v_y$ . Due to this reason, the MRAW architecture branches out into many different solutions different from each other depending on how the signal  $v_y$  is designed to guarantee that, eventually, the anti-windup compensator state  $x_{aw}$  converges to zero thereby causing the output  $z_{aw} = z - z_\ell$  to converge to zero and, consequently, the plant output  $z$  to converge to the desirable unconstrained performance output response  $z_\ell$ .

Within this framework, the simplest anti-windup solutions that one can think of are solutions that focus on the nonlinear dynamics (33) in their linear regime. Namely, the saturation nonlinearity is disregarded and the anti-windup compensator (often called ‘‘mismatch’’) dynamics become linear:

$$\begin{aligned} \dot{x}_{aw} &= A_p x_{aw} + B_{pu} v_y \\ z - z_\ell &= C_z x_{aw} + D_{zu} v_y. \end{aligned} \quad (35)$$

This linear dynamics coincides locally (namely for small enough  $y_{c,\ell}$  and  $v_y$ ) with the nonlinear dynamics (33). Therefore, a first set of possible solutions to the MRAW problem are those given by the following Algorithm 9 where  $v_y$  is selected as a linear state feedback from  $x_{aw}$  designed completely disregarding the saturation effects. These solutions often lead to desirable responses even though they are not associated with any formal closed-loop stability guarantee.

When looking for alternative solutions that may give closed-loop stability (and performance) guarantees, we point to Algorithms 5, 6, 7, 8 and 10, which still provide linear selections of  $v_y$  even though these selections are carried out based on suitable characterizations of the saturation effects, thereby being better candidates for handling well large signal responses (which interest significantly the flat region of the saturation function). This typically corresponds to a worse small signal behavior, which is sacrificed to induce the increased large signal guarantees.

The third, most complicated, type of solutions that one can propose within the MRAW framework is that of selecting  $v_y$  as a nonlinear function of the anti-windup compensator state  $x_{aw}$  (and possibly of some extra available measurement from the closed-loop). This type of solution is certainly the most difficult to design and implement but is the most advanced scheme within this framework. Algorithm 11 and a number of techniques briefly commented in Section IV-E and not included here due to space constraints, follow this last paradigm.

Note that the MRAW architecture is independent of the controller dynamics. By this fact, any stabilizing controller can be used within the MRAW scheme and closed-loop stability will be guaranteed by the scheme. However, extra mild assumptions on the controller dynamics are highly desirable because they are necessary for robustness. These properties are some suitable incremental stability properties (see [106], [118] for details) which allow to carry out a small gain type argument on the closed-loop in the presence of uncertainties.

Perhaps the most important drawback of the MRAW scheme as compared to the DLAW solutions is that the compensation scheme is of the same order of the plant, which is often too complicated. However, the robustness property commented above becomes a key fact when implementing MRAW in real-life problems where one would like to use a rough model of the plant dynamics. This fact is certainly possible within the scheme and typically allows for a strong reduction of the order of the arising anti-windup compensator (see, e.g., [79] for an application study where several anti-windup order reduction techniques were implemented in the discrete-time MRAW setting).

### B. MRAW with exponentially stable plants

With exponentially stable plants, the simplest possible compensation scheme is given by the so-called IMC-based anti-windup, which corresponds to blindly use the controller as if no saturation was in place and delivering to the plant (after trimming it by way of saturation) the same signal that the controller would have produced without any saturation. This scheme is described in [75] (see also [121] and was also discussed independently in [47] (where it was attributed to Irving). When seen in the context of MRAW, the IMC scheme simply amounts to selecting  $v_y = 0$ .

*Algorithm 5:* IMC-based MRAW for exponentially stable plants

**Step 1.** Select  $v_y = 0$ .

★

A more sophisticated, although still simple, strategy for selecting the compensation signal  $v_y$  is given by the following Lyapunov-based procedure [106], where the degrees of freedom available in the design process can often lead to better performance, even though this should be tuned by means of trial and error approaches. We will consider here the case where the plant is asymptotically stable and then introduce in Section IV-C a generalization of the algorithm that also applies to the case when the plant is stable, but not asymptotically (namely, it has single poles on the imaginary axis).

The following algorithm requires two parameters to be chosen before being applied. One of them is the positive definite matrix  $Q$ , which has an impact on the way the different states of the plant are weighted within the control action. The second one is the positive scalar factor  $\rho$  which is proportional to the aggressiveness of the stabilizing action.

*Algorithm 6:* Lyapunov-based MRAW for exponentially stable plants

**Step 1.** Select a positive definite matrix  $Q$  and a positive number  $\rho$ .

**Step 2.** Solve the following Lyapunov equation:

$$A_p'P + PA_p = -Q$$

in the unknown  $P > 0$  (for example, use the MATLAB command  $P = \text{lyap}(A_p', Q)$ ).

**Step 3.** Select the compensation signal  $v_y$  as

$$v_y = K_v x_{aw} := -\rho B_{pu}' P x_{aw}. \quad (36)$$

★

Despite its simplicity, in many cases Algorithm 6 is sufficiently good to induce highly desirable responses on the arising closed-loop system. In the multi-input multi-output case, the constant  $\rho$  can be generalized to be a positive definite diagonal matrix, thus obtaining extra degrees of freedom in tuning the compensation signal  $v_y$ , where the different plant inputs would be weighted differently by the corresponding diagonal matrix gain.

We conclude this section by giving a further algorithm taken from [118], where the anti-windup gains are selected with the goal of guaranteeing closed-loop global exponential stability (by global sector conditions for the saturation function) while minimizing the LQ index:

$$J := \int_0^\infty (x_{aw}' Q_P x_{aw} + v_y' R_P v_y) dt$$

where  $Q_P$  and  $R_P$  are suitable positive definite design parameters.

*Algorithm 7:* LQ-based MRAW for exponentially stable plants

**Step 1.** Select positive definite matrices  $Q_P$  and  $R_P$ .

**Step 2.** Solve the following LMI:

$$\begin{aligned} & \min \gamma \quad \text{s.t.} \\ & \begin{bmatrix} QA'_p + A_pQ & B_{pu}U + X'_1 \\ UB'_{pu} + X_1 & X'_2 + X_2 - 2U \end{bmatrix} < 0, \\ & \begin{bmatrix} \gamma I & I \\ I & Q \end{bmatrix} > 0, \\ & \text{He} \begin{bmatrix} A_pQ + B_{pu}X_1 & 0 & 0 \\ Q & -Q_P^{-1}/2 & 0 \\ X_1 & 0 & -R_P^{-1}/2 \end{bmatrix} < 0, \end{aligned}$$

in the unknowns  $Q > 0$ ,  $X_1$ ,  $X_2$  and  $U > 0$  diagonal.

**Step 3.** Select the compensation signal  $v_y$  as

$$v_y = Kx_{aw} + L(\text{sat}_{u_0}(y_c + v_1) - y_c), \quad (37)$$

where  $K = X_1Q^{-1}$  and  $L = X_2U^{-1}$ .

★

### C. MRAW with non exponentially unstable plants

When dealing with marginally unstable plants (namely plants without poles with positive real part but having poles on the imaginary axis), it has been pointed out in Section II-C that the anti-windup problem, as well as the bounded stabilization problem, becomes hard. This is because global exponential stability cannot be achieved and only global asymptotic stability has to be sought for. Therefore, nonlinear algorithms become necessary, at least when wanting to induce global properties. For regional properties, the class of systems addressed in this section is no different from the exponentially unstable plants addressed in the next section and the reader should refer to that section, treating the non-exponentially stable modes as if they were exponentially unstable.

Due to space limitations and to avoid overloading the notation, we will only report here an algorithm addressing the simple situation of a marginally stable plant (namely a plant with only single poles on the imaginary axis). For this type of plant, an algorithm generalizing Algorithm 6 of the previous section has been suggested in [106] and later clarified and streamlined in [120] where it was applied to an experimental system. This algorithm is especially useful for practical situations with integrating plants which belong to the class addressed here.

The algorithm depends on the selection of suitable matrices and constants  $Q_s$ ,  $\rho_0$  and  $\rho$ , which parallel the ones in Algorithm 6 except for the new one  $\rho_0$ . This  $\rho_0$  is associated with the aggressiveness of the anti-windup action in the marginally stable direction with relation to its aggressiveness in the other directions.

*Algorithm 8:* Lyapunov-based MRAW for marginally stable plants

**Step 1.** Compute an invertible (real) matrix  $T$  such that  $T^{-1}A_pT = \begin{bmatrix} A_s & 0 \\ 0 & A_0 \end{bmatrix}$ , where the eigenvalues of the  $n_s \times n_s$  matrix  $A_s$  all lie in the open left-half plane (i.e.  $A_s$  is Hurwitz) and the eigenvalues of the  $n_0 \times n_0$  matrix  $A_0$  all lie on the imaginary axis (i.e.  $A_0$  is marginally stable).

**Step 2.** Select a positive definite matrix  $Q_s \in \mathfrak{R}^{n_s \times n_s}$  and define the positive definite matrix  $P_s$  to be the solution of the Lyapunov equation

$$A'_sP_s + P_sA_s = -Q_s.$$

**Step 3.** Select  $\rho_0 > 0$  and set  $P = (T^{-1})' \begin{bmatrix} P_s & 0 \\ 0 & \rho_0 I_{n_0} \end{bmatrix} T^{-1}$ , where  $I_{n_0}$  denotes the  $n_0 \times n_0$  identity matrix.

**Step 4.** Select  $\rho > 0$  and the compensation signal  $v_y$  as

$$v_y = K_v x_{aw} := -\rho B_{pu}' P x_{aw}. \quad (38)$$

★

Fully nonlinear anti-windup algorithms inducing global asymptotic stability for marginally unstable plants are also hinted in [106] where two alternative solutions are suggested based on the scheduled Riccati ideas of [71] and on the nested saturations ideas in [102] and references therein.

More recent algorithms inducing global asymptotic stability for certain classes of systems have been proposed in [23], [24], [26] but are not reported here due to space constraints.

#### D. MRAW with exponentially unstable plants

When dealing with exponentially unstable plants, the achievable region of attraction is necessarily bounded in the exponentially stable directions. Due to this fact, one can only rely on algorithms providing regional guarantees instead of global ones. Since saturation acts like an identity for small enough signals, the first trivial solution that we propose is based on simply designing  $v_y$  as a linear LQR gain disregarding saturation. The arising scheme will still be good in a region which might be very small for particularly difficult examples. As compared to Algorithm 7, this algorithm here does not provide any stability in the large guarantee (whereas Algorithm 7 provides globally stabilizing gains).

*Algorithm 9:* LQ-based MRAW for exponentially unstable plants

**Step 1.** Select positive definite matrices  $Q_P$  and  $R_P$ .

**Step 2.** Select the compensation signal  $v_y$  as

$$v_y = K_{lqr} x_{aw}, \quad (39)$$

where  $K_{lqr}$  is the optimal LQR gain for the plant (35) disregarding saturation (e.g., use the Matlab command `lqr(Ap, Bpu, Qp, Rp, 0)`).

★

Often the saturation blind solution of Algorithm 9 is associated with stability regions that are too small. For all those cases, the following algorithm might be a suitable alternative. The algorithm, taken from [24] is based on the use of the generalized sector condition for the design of  $v_y$ . Note that in this algorithm, first the stability region is maximized under a certain constraint on a desired performance level  $\bar{\gamma}$ . The larger  $\bar{\gamma}$ , the larger the stability region will be. Then, among all the compensator gains that induce that performance level in that region, the selected one is the one that maximizes the speed of convergence to zero.

*Algorithm 10:* LMI-based MRAW for exponentially unstable plants

**Step 1.** Choose a certain desired  $\mathcal{L}_2$  performance level  $\bar{\gamma}$ .

**Step 2.** Solve the following LMI optimization problem:

$$\begin{aligned} & \min \eta \quad \text{s.t.} \\ & \text{He} \begin{bmatrix} A_p Q + B_{pu} Y & B_{pu} & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_z Q - D_z Y & D_{zu} & -\bar{\gamma} \frac{I}{2} \end{bmatrix} < 0, \\ & \begin{bmatrix} \eta u_{0(k)}^2 & Y_{(k)} \\ Y_{(k)}' & Q \end{bmatrix}, \quad k = 1, \dots, m, \end{aligned}$$

(where  $Y_{(k)}$  denotes the  $k$ -th row of  $Y$ ) in the unknowns  $Q > 0$ ,  $Y$  and  $\eta$ . Minimizing  $\eta$  corresponds to maximizing the arising stability region.

**Step 3.** Based on the solution  $Q, Y$  from the previous step, solve the following generalized eigenvalue problem:

$$\begin{aligned} & \min \lambda \quad \text{s.t.} \\ & \text{He} [(A_p + B_{pu}\tilde{K})Q - \lambda Q] < 0, \\ & \text{He} \begin{bmatrix} A_p Q + B_{pu} Y & B_{pu} & 0 & B_{pu} \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z Q - D_{zu} Y & D_z & -\frac{\tilde{\gamma}}{2} I & D_{zu} \\ \tilde{K} Q - Y - \tilde{L} B' & -\tilde{L} & -\tilde{L} D'_{zu} & \tilde{L} - I \end{bmatrix} < 0, \end{aligned}$$

in the unknowns  $\tilde{K}$  and  $\tilde{L}$ .

**Step 4.** Select the compensation signal  $v_y$  as

$$v_y = K x_{aw} + L(\text{sat}_{u_0}(y_c + v_y) - y_c), \quad (40)$$

where  $L = -(I - \tilde{L})^{-1} \tilde{L}$  and  $K = (I - \tilde{L})^{-1} \tilde{K}$ .

Note that setting  $\tilde{L} = 0$  at step 3 of the previous algorithm, the anti-windup compensation is strictly proper and it is not necessary to implement and solve the nonlinear algebraic loop in (40). ★

We finally present an algorithm providing a fully nonlinear compensation scheme for the case when it is crucial to keep the exponentially unstable plant dynamics within a certain safety region (of course this is only possible if the disturbances do not push the plant state outside of the null controllability region). This algorithm is taken from [27] and is written here in a simplified form. For extensions and clarifications, the reader is referred to [27].

Note that this algorithm only works if one can directly measure the unstable states from the plant (denoted  $x_u$  below). Also, due to the extreme features required by the anti-windup compensation, the signal  $v_y$  has the authority of cancelling out completely the controller output  $y_c$  in the case when the plant trajectory is reaching the boundary of the (bounded) null controllability region.

*Algorithm 11:* Nonlinear MRAW for exponentially unstable plants

**Step 1.** Transform the anti-windup state space representation so that  $A_p = \begin{bmatrix} A_s & A_{12} \\ 0 & A_u \end{bmatrix}$  and  $B_{pu} = \begin{bmatrix} B_s \\ B_u \end{bmatrix}$ , where  $A_s$  is Hurwitz.

**Step 2.** Solve the following LMI optimization problem in the variables  $G, Q = Q' > 0$  and  $\gamma$  to obtain a state feedback stabilizer:

$$\begin{aligned} & \min \gamma \quad \text{s.t.} \\ & \begin{bmatrix} 1 & G^{(i)} \\ G'^{(i)} & Q \end{bmatrix} \geq 0, \quad i = 1, \dots, m, \\ & \begin{bmatrix} \gamma I & I \\ I & Q \end{bmatrix} \geq 0, \\ & Q A'_u + A_u Q + G' B'_u + B_u G < 0, \end{aligned}$$

and select  $P = Q^{-1}$ ,  $F_u := GP$ .

**Step 3.** Fix a small  $\epsilon \ll 1$  (e.g.,  $\epsilon = 0.05$ ) and define  $P_\epsilon := P/(1 - \epsilon)^2$  and the following functions:

$$\begin{aligned} \beta(x_u) &= \min \left\{ 1, \max \left\{ 0, \frac{1 - x'_u P x_u}{1 - (1 - \epsilon)^2} \right\} \right\} \\ \Psi(x_u, x_u^*) &= \frac{(x_u^*)' P v + \sqrt{((x_u^*)' P v)^2 + (v' P v) w}}{w}, \\ \Psi_\epsilon(x_u, x_u^*) &= \frac{(x_u^*)' P_\epsilon v + \sqrt{((x_u^*)' P_\epsilon v)^2 + (v' P_\epsilon v) w_\epsilon}}{w_\epsilon}, \end{aligned}$$

where  $v = x_u - x_u^*$ ,  $w = 1 - (x_u^*)' P x_u^*$  and  $w_\epsilon = 1 - (x_u^*)' P_\epsilon x_u^*$ .

**Step 4.** Select the following functions too:

$$\begin{aligned}\mathcal{P}(x_u) &= \frac{1}{\max\{1, \Psi_\epsilon(x_u, 0)\}} x_u, \\ \mathcal{P}_u(x_u) &= -(B'_u B_u)^{-1} B'_u A_u x_u \\ \gamma(x_u, x_u^*) &= \Psi(x_u, x_u^*) F_u \tilde{x}_u + (1 - \Psi(x_u, x_u^*)) \mathcal{P}_u(x_u^*), \\ \alpha(x_u, \xi_u, y_c) &= \gamma(x_u, \mathcal{P}(\xi_u)) + \beta(x_u)(y_c - \mathcal{P}_u(\xi_u)).\end{aligned}$$

**Step 5.** Select the compensation signal  $v_y$  as

$$v_y = \alpha(x_u, x_u - x_{aw,u}, \tilde{y}_c) - \tilde{y}_c, \quad (41)$$

where, according to the block separation at step 1,  $x_u$  and  $x_{aw,u}$  are the second block components of  $x$  and  $x_{aw}$ , respectively, and  $\tilde{y}_c = C_c x_c + D_c u_c + D_{cw} w$  corresponds to the controller output before the addition of the anti-windup correction.

★

### E. Extensions

1) *Boosting performance via switching and scheduling:* Already in [119], the idea of switching among a family of linear gains selected to induce desirable properties in nested neighborhoods of the origin has been applied for the design of  $v_y$ . This type of strategy is advantageous as a matter of fact it allows to increase the aggressiveness of the stabilizer as the state gets closer and closer to the origin (namely to the zone where saturation acts mainly like an identity). The need for these types of nonlinear controller had been already pointed out in [103] more than a decade ago.

While switching among families of different controllers is desirable and induces extreme performance, scheduling algorithms have been also recently proposed in [23], [24], [26] where the controller is selected by relying on a convex combination of the LMIs in Algorithm 10, thus resulting in a fully nonlinear control law scheduled by a parameter which indicates how far the anti-windup compensator state is from the origin. This is probably the most advanced control law currently available for MRAW designs. It is not described here due to its technical complication.

2) *Dead-time plants:* It has been proved in [117] that when the plant input is subject to a finite, known, delay, all the anti-windup solutions that do not require a direct measurement from the plant (therefore, all the algorithms given above except for Algorithm 11) can be still applied as long as the anti-windup compensator is driven by the excess of saturation before the plant delay. Indeed, by interconnecting the compensator in this way, the saturation compensation problem is fully decoupled from the dead time compensation problem which is assumed to be addressed directly by the unconstrained controller design.

3) *Rate and magnitude saturation:* There are several papers illustrating how rate and magnitude saturation can be addressed by slightly modifying the architecture of the anti-windup scheme. Quite natural but non constructive solutions to the problem have been given in [3], [105]. More recently, in [18], which will be part of the tutorial session corresponding to this paper, a novel way of looking at this problem via MRAW architectures is illustrated. The advantage with this novel look at the problem is that it leads to constructive techniques to design anti-windup compensator gains with stability and performance guarantees. These were not available with the previous techniques cited above.

4) *Reduced order design:* Perhaps the main negative aspect of the MRAW approach is that anti-windup compensation is always of the same order of the plant. Nevertheless, it is discussed in [106] (see also [5]) that the scheme is robust to variations of the plant parameters. Therefore it is reasonable to seek for simplified versions of the MRAW scheme where the anti-windup compensator dynamics are a reduced order model of the plant. This type of solution is possible to implement. Some discussions about reduced order implementation has been carried out in the discrete-time context in [79].

## V. CONCLUSIONS

This paper has introduced and summarised two approaches to anti-windup design. The DLAW approach, a category into which most modern AW compensators belong, has been described and sample algorithms with which one may synthesise this type of compensator have been given. The second half of the paper concentrates on the specific MRAW approach to anti-windup designs and several specific algorithms have been described. It is hoped that this paper will be of use to those just beginning their study of anti-windup, and will also act as a useful summary of some of the varying approaches to the design of anti-windup compensators.

## REFERENCES

- [1] D. Angeli and E. Mosca. Command governors for constrained nonlinear systems. *IEEE Trans. Aut. Cont.*, 44(4):816–820, April 1999.
- [2] anon. Why the grippen crashed. *Aerospace America*, page 11, 1994.
- [3] C. Barbu, S. Galeani, A.R. Teel, and L. Zaccarian. Nonlinear anti-windup for manual flight control. *International Journal of Control*, 78(14):1111–1129, September 2005.
- [4] C. Barbu, R. Reginatto, A. R. Teel, and L. Zaccarian. Anti-windup for exponentially unstable linear systems with inputs limited in magnitude and rate. In *American Control Conference*, Chicago, IL, June 2000.
- [5] A. Bemporad, A.R. Teel, and L. Zaccarian. Anti-windup synthesis via sampled-data piecewise affine optimal control. *Automatica*, 40(4):549–562, 2004.
- [6] J-M. Biannic, C. Roos, and S. Tarbouriech. A practical method for fixed-order anti-windup design. In *7<sup>th</sup> IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Pretoria, South Africa, 2007.
- [7] J-M. Biannic and S. Tarbouriech. Optimization and implementation of dynamic anti-windup compensators with multiple saturations in flight control systems. *Control Engineering Practice*, 17:703–713, 2009.
- [8] J-M. Biannic, S. Tarbouriech, and D. Farret. A practical approach to performance analysis of saturated systems with application to fighter aircraft flight controllers. In *5<sup>th</sup> IFAC Symposium ROCOND*, Toulouse, France, July 2006.
- [9] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, 1994.
- [10] O. Brieger, M. Kerr, D. Leissling, I. Postlethwaite, J. Sofrony, and M.C. Turner. Anti-windup compensation of rate saturation in an experimental aircraft. In *Proc. American Control Conference*, New York, 2007.
- [11] Y.Y. Cao, Z. Lin, and B.M. Chen. An output feedback  $\mathcal{H}_\infty$  controller for linear systems subject to sensor nonlinearities. *IEEE Transactions on Circuits and Systems-I*, 50(7):914–921, 2003.
- [12] Y.Y. Cao, Z. Lin, and D.G. Ward. An antiwindup approach to enlarging domain of attraction for linear systems subject to actuator saturation. *IEEE Trans. Aut. Cont.*, 47(1):140–145, 2002.
- [13] E.B. Castelan, S. Tarbouriech, J. M. Gomes da Silva Jr., and I. Queinnec.  $\mathcal{L}_2$ -stabilization of continuous-time systems with saturating actuators. *Int. J. Robust and Nonlinear Control*, 16:935–944, 2006.
- [14] J. Chiasson and J-J. Loiseau (Eds.). *Applications of time-delay systems*. LNCIS, vol.352, Springer Verlag, 2007.
- [15] J.C. Doyle, R.S. Smith, and D.F. Enns. Control of plants with input saturation nonlinearities. In *ACC*, pages 1034–39, Minneapolis (MN), USA, June 1987.
- [16] H. Duda. Prediction of pilot-int-the-loop oscillations due to rate saturation. *Journal of Guidance, Control and Dynamics*, 20(3):581–587, 1997.
- [17] H.A. Fertik and C.W.Ross. Direct digital control algorithm with anti-windup feature. *ISA Transactions*, 6(4):317–328, 1967.
- [18] F. Forni, S. Galeani, and L. Zaccarian. Model recovery anti-windup for plants with rate and magnitude saturation. In *European Control Conference*, Budapest (Hungary), August 2009.
- [19] A.T. Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *Int. J. Contr.*, 10:457–480, 1969.
- [20] P. Gahinet and P. Apkarian. A linear matrix inequality approach to  $\mathcal{H}_\infty$  control. *Int. J. Robust and Nonlinear Control*, 4:421–448, 1994.
- [21] S. Galeani, M. Massimetti, A.R. Teel, and L. Zaccarian. Reduced order linear anti-windup augmentation for stable linear systems. *International Journal of Systems Science*, 37(2):115–127, 2006.
- [22] S. Galeani, S. Onori, A.R. Teel, and L. Zaccarian. Further results on static linear anti-windup design for control systems subject to magnitude and rate saturation. In *IEEE Conference on Decision and Control*, San Diego, CA, USA, December 2006.
- [23] S. Galeani, S. Onori, A.R. Teel, and L. Zaccarian. Nonlinear  $\mathcal{L}_2$  anti-windup for enlarged stability regions and regional performance. In *Symposium on Nonlinear Control Systems (NOLCOS)*, pages 539–544, Pretoria (South Africa), August 2007.
- [24] S. Galeani, S. Onori, A.R. Teel, and L. Zaccarian. Regional, semiglobal, global nonlinear anti-windup via switching design. In *European Control Conference*, pages 5403–5410, Kos (Greece), July 2007.
- [25] S. Galeani, S. Onori, A.R. Teel, and L. Zaccarian. A magnitude and rate saturation model and its use in the solution of a static anti-windup problem. *Systems and Control Letters*, 57(1):1–9, 2008.
- [26] S. Galeani, S. Onori, and L. Zaccarian. Nonlinear scheduled control for linear systems subject to saturation with application to anti-windup control. In *Conference on Decision and Control*, pages 1168–1173, New Orleans (LA), USA, December 2007.
- [27] S. Galeani, A.R. Teel, and L. Zaccarian. Constructive nonlinear anti-windup design for exponentially unstable linear plants. *Systems and Control Letters*, 56(5):357–365, 2007.
- [28] I. Ghiggi, A. Bender, and J.M. Gomes da Silva Jr. Dynamic non-rational anti-windup for time-delay systems with saturating inputs. In *Proceedings of the World IFAC Congress*, Seoul, Korea, 2008.

- [29] E.G. Gilbert and I. Kolmanovsky. Fast reference governors for systems with state and control constraints and disturbance inputs. *Internat. J. Robust Nonlinear Control*, 9(15):1117–1141, 1999.
- [30] A.H. Glattfelder and W. Schaufelberger. *Control Systems with Input and Output Constraints*. Springer, 2003.
- [31] J. M. Gomes da Silva Jr. and S. Tarbouriech. Local stabilization of discrete-time systems with saturating control: an LMI-based approach. *IEEE Transactions on Automatic Control*, 46(1):119–121, 2001.
- [32] J.M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranteed regions of stability: an LMI-based approach. In *Proceedings of the 42th IEEE Conference on Decision and Control*, Hawaii, USA, 2003.
- [33] J.M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranteed region of stability: an LMI-based approach. *IEEE Trans. Aut. Cont.*, 50(1):106–111, 2005.
- [34] J.M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranteed regions of stability for discrete-time linear systems. *Systems and Control Letters*, 55(3):184–192, 2006.
- [35] J.M. Gomes da Silva Jr., S. Tarbouriech, and G. Garcia. Local stabilization of linear systems under amplitude and rate saturating actuators. *IEEE Trans. Aut. Cont.*, 48(5):842–847, 2003.
- [36] J.M. Gomes da Silva Jr., S. Tarbouriech, and G. Garcia. Anti-windup design for time-delay systems subject to input saturation: An LMI-based approach. *European Journal of Control*, 6:1–13, 2006.
- [37] J.M. Gomes da Silva Jr., S. Tarbouriech, and R. Reginatto. Analysis of regions of stability for linear systems with saturating inputs through an anti-windup scheme. In *IEEE-Conference on Control Applications 2002*, Glasgow, UK, 2002.
- [38] G. Grimm, J. Hatfield, I. Postlethwaite, A.R. Teel, M.C. Turner, and L. Zaccarian. Anti-windup for stable linear systems with input saturation: an LMI based synthesis. *IEEE Transactions on Automatic Control*, 48(9):1509–1525, 2003.
- [39] G. Grimm, J. Hatfield, I. Postlethwaite, A.R. Teel, M.C. Turner, and L. Zaccarian. Antiwindup for stable linear systems with input saturation: an LMI-based synthesis. *IEEE Trans. Aut. Cont. (A)*, 48(9):1509–1525, September 2003.
- [40] G. Grimm, I. Postlethwaite, A.R. Teel, M.C. Turner, and L. Zaccarian. Linear matrix inequalities for full and reduced order anti-windup synthesis. In *Proceedings of the American control conference*, pages 4134 – 4139, Arlington, VA, 2001.
- [41] G. Grimm, I. Postlethwaite, A.R. Teel, M.C. Turner, and L. Zaccarian. Case studies using LMIs in anti-windup synthesis for stable linear systems with input saturation. *European Journal of Control*, 9(5):459–469, 2003.
- [42] G. Grimm, A.R. Teel, and L. Zaccarian. The  $l_2$  anti-windup problem for discrete-time linear systems: definition and solutions. In *American Control Conference*, pages 5329–5334, Denver (CO), USA, June 2003.
- [43] G. Grimm, A.R. Teel, and L. Zaccarian. Linear LMI-based external anti-windup augmentation for stable linear systems. *Automatica (B)*, 40(11):1987–1996, 2004.
- [44] G. Grimm, A.R. Teel, and L. Zaccarian. Robust linear anti-windup synthesis for recovery of unconstrained performance. *Int. J. Robust and Nonlinear Control (A)*, 14(13-15):1133–1168, 2004.
- [45] G. Grimm, A.R. Teel, and L. Zaccarian. The  $l_2$  anti-windup problem for discrete-time linear systems: definition and solutions. *Systems and Control Letters*, 57(4):356–364, 2008.
- [46] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of time-delay systems*. Birkh<sup>u</sup>ser, Boston, 2003.
- [47] R. Hanus. Antiwindup and bumpless transfer: a survey. In *Proceedings of the 12th IMACS World Congress*, volume 2, pages 59–65, Paris, France, July 1988.
- [48] R. Hanus and M. Kinnaert. Control of constrained multivariable systems using conditioning technique. In *American Control Conference*, pages 1712–1718, Pittsburgh, 1989.
- [49] R. Hanus, M. Kinnaert, and J. L. Henrotte. Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica*, 23:729–739, 1987.
- [50] G. Herrmann, M.C. Turner, and I. Postlethwaite. Practical implementation of a novel anti-windup scheme in a hdd-dual-stage servo-system. *IEEE/ASME Transactions on Mechatronics*, 9(3):580–592, 2004.
- [51] G. Herrmann, M.C. Turner, and I. Postlethwaite. Discrete-time and sampled-data anti-windup synthesis: stability and performance. *International Journal of Systems Science*, 37(2):91–113, 2006.
- [52] H. Hindi and S. Boyd. Analysis of linear systems with saturation using convex optimization. In *Proc. of 37th Conference on Decision and Control (CDC'98)*, pages 903–908, Tampa, USA, 1998.
- [53] P. Hippe. *Windup in control. Its effects and their prevention*. AIC, Springer, Germany, 2006.
- [54] T. Hu and Z. Lin. *Control systems with actuator saturation: analysis and design*. Birkh<sup>u</sup>ser, Boston, 2001.
- [55] T. Hu, A.R. Teel, and L. Zaccarian. Regional anti-windup compensation for linear systems with input saturation. In *Proceedings of the American control conference*, pages 3397 – 3402, Portland, OR, 2005.
- [56] T. Hu, A.R. Teel, and L. Zaccarian. Anti-windup synthesis for linear control systems with input saturation: achieving regional, nonlinear performance. *Automatica (B)*, 64(2):512–519, 2008.
- [57] G. Kaliora and A. Astolfi. Nonlinear control of feedforward systems with bounded signals. *IEEE Transactions on Automatic Control*, 49(11):1975–1990, 2004.
- [58] M. Kerr, E. Villota, and S. Jayasuriya. Robust anti-windup design for input constrained SISO systems. *Proc. 8th Int. Symposium on QFT and Robust Frequency Domain Methods.*, 2007.
- [59] M.L. Kerr, M.C. Turner, and I. Postlethwaite. Practical approaches to low-order anti-windup compensator design: a flight control comparison. In *Proceedings of the World IFAC Congress*, Seoul, Korea, 2008.
- [60] H. K. Khalil. *Nonlinear Systems*. MacMillan, 1992.
- [61] R.B. Koplton, M.L.J. Hautus, and E.D. Sontag. Observability of linear systems with saturated outputs. *Linear Algebra and Applications*, 205-206:909–936, 1994.
- [62] M.V. Kothare, P.J. Campo, M. Morari, and N. Nett. A unified framework for the study of anti-windup designs. *Automatica*, 30(12):1869–1883, 1994.
- [63] G. Kreisselmeier. Stabilisation of linear systems in the presence of output measurement saturation. *Systems and Control Letters*, 29:27–30, 1996.

- [64] N. J. Krikelis. State feedback integral control with intelligent integrator. *Int. J. Control*, 32:465–473, 1980.
- [65] J.B. Lasserre. Reachable, controllable sets and stabilizing control of constrained linear systems. *Automatica*, 29(2):531–536, 1992.
- [66] Z. Lin. Semi-global stabilisation of linear systems with position and rate-limited actuators. *Systems and Control letters*, 30:1–11, 1997.
- [67] Z. Lin and T. Hu. Semi-global stabilisation of linear systems subject to output saturation. *Systems and Control Letters*, 43:211–217, 2001.
- [68] J.C. Lozier. A steady-state approach to the theory of saturable servo systems. *IRE Transactions on Automatic Control*, 1:19–39, May 1956.
- [69] V.R. Marcopoli and S.M. Phillips. Analysis and synthesis tools for a class of actuator-limited multivariable control systems: a linear matrix inequality approach. *International Journal of Robust and Nonlinear Control*, 6(9-10):1045–1063, 1996.
- [70] M. Massimetti, L. Zaccarian, T. Hu, and A.R. Teel. LMI-based linear anti-windup for discrete time linear control systems. In *Conference on Decision and Control*, pages 6173–6178, San Diego (CA), USA, December 2006.
- [71] A. Megretski.  $L_2$  BIBO output feedback stabilization with saturated control. In *13th Triennial IFAC World Congress*, pages 435–440, San Francisco, USA, 1996.
- [72] R. Miller and M. Pachter. Manual flight control with saturating actuators. *IEEE Control Systems*, 18(1):10–19, 1998.
- [73] F. Morabito, S. Nicosia, A.R. Teel, and L. Zaccarian. Measuring and improving performance in anti-windup laws for robot manipulators. In B. Siciliano, A. De Luca, C. Melchiorri, and G. Casalino, editors, *Advances in Control of Articulated and Mobile Robots*, chapter 3, pages 61–85. Springer Tracts in Advanced Robotics, 2004.
- [74] F. Morabito, A.R. Teel, and L. Zaccarian. Nonlinear anti-windup applied to Euler-Lagrange systems. *IEEE Trans. Rob. Aut. (A)*, 20(3):526–537, 2004.
- [75] M. Morari and E. Zafriou. *Robust process control*. Prentice Hall, Englewood Cliffs, NJ, 1989.
- [76] E.F. Mulder, M.V. Kothare, and M. Morari. Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, 37:1407–1416, 2001.
- [77] T. Nguyen and F. Jabbari. Output feedback controllers for disturbance attenuation with actuator amplitude and rate saturation. *Automatica*, 36(9):1339–1346, 2000.
- [78] S-I. Niculescu. *Delay Effects on Stability. A Robust Control Approach*. Springer-Verlag, Berlin, Germany, 2001.
- [79] L. Pagnotta, L. Zaccarian, A. Constantinescu, and S. Galeani. Anti-windup applied to adaptive rejection of unknown narrow band disturbances. In *European Control Conference*, pages 150–157, Kos (Greece), July 2007.
- [80] J-K. Park, C-H. Choi, and H. Choo. Dynamic anti-windup method for a class of time-delay control systems with input saturation. *Int. J. Robust and Nonlinear Control*, 10:457–488, 2000.
- [81] J-K. Park and H.Y. Youn. Dynamic anti-windup based control method for state constrained systems. *Automatica*, 39:1915–1922, 2003.
- [82] I. Postlethwaite, M.C. Turner, and G. Herrmann. Robust control applications. *Annual Reviews in Control*, 31(1):27–39, 2007.
- [83] I. Queinnec, S. Tarbouriech, and G. Garcia. Anti-windup design for aircraft control. In *Proceedings of the IEEE Conference on Control Applications (CCA)*, Munich, Germany, 2006.
- [84] J.P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1604, 2003.
- [85] B.G. Romanchuk. Some comments on anti-windup synthesis using LMI's. *International Journal of Robust and Nonlinear Control*, 9(10):717–734, 1999.
- [86] C. Roos and J-M. Biannic. A convex characterization of dynamically-constrained anti-windup controllers. *Automatica*, 44:2449–2452, 2008.
- [87] C. Roos, J-M. Biannic, S. Tarbouriech, and C. Prieur. *On-ground aircraft control design using an LPV anti-windup approach*. Chap. 7, LNCIS, vol. 365, Springer Verlag, 2007.
- [88] L. Rundquist and K. Stahl-Gunnarsson. Phase compensation of rate-limiters in unstable aircraft. *Proc. IEEE Conference on Control Applications*, 1996.
- [89] A. Saberi and A. Stoorvogel. Output regulation of linear plants with actuators subject to amplitude and rate constraints. *Int. J. Robust and Nonlinear Control*, 9(10):631–657, 1999.
- [90] M. Saeki and N. Wada. Synthesis of a static anti-windup compensator via linear matrix inequalities. *Int. J. Robust and Nonlinear Control*, 12:927–953, 2002.
- [91] J. Sofrony, M.C. Turner, and I. Postlethwaite. Anti-windup synthesis for systems with rate-limits: a Riccati equation approach. *Proc. SICE Annual Conference*, 2005.
- [92] E.D. Sontag. An algebraic approach to bounded controllability of linear systems. *Int. Journal of Control*, 39(1):181–188, 1984.
- [93] A. Syaichu-Rohman and R.H. Middleton. Anti-windup schemes for discrete time systems: an lmi-based design. In *Asian Control Conference*, pages 554–561, Melbourne (VIC), Australia, July 2004.
- [94] S. Tarbouriech and G. Garcia. Preliminary results about anti-windup strategy for systems subject to actuator and sensor saturations. In *Proc. of the IFAC World Congress*, Prague, 2005.
- [95] S. Tarbouriech, G. Garcia, and A.H. Glatfelter (Eds.). *Advanced strategies in control systems with input and output constraints*. LNCIS, vol.346, Springer Verlag., 2007.
- [96] S. Tarbouriech, J.M. Gomes da Silva Jr., and F.A. Bender. Dynamic anti-windup synthesis for discrete-time linear systems subject to input saturation and  $L_2$  disturbances. In *Proceedings of the 5th IFAC Symposium on Robust Control Design (ROCOND'2006)*, Toulouse, France, 2006.
- [97] S. Tarbouriech, J.M. Gomes da Silva Jr., and G. Garcia. Delay-dependent anti-windup loops for enlarging the stability region of time-delay systems with saturating inputs. *ASME - Journal of Dynamic Systems, Measurement and Control*, 125(1):265–267, June 2003.
- [98] S. Tarbouriech, J.M. Gomes da Silva Jr., and G. Garcia. Delay-dependent anti-windup strategy for linear systems with saturating inputs and delayed outputs. *International Journal of Robust and Nonlinear Control*, 14:665–682, 2004.

- [99] S. Tarbouriech, C. Prieur, and J.M. Gomes da Silva Jr. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Trans. Aut. Cont.*, 51(8):1364–1371, 2006.
- [100] S. Tarbouriech, I. Queinnec, and G. Garcia. Stability region enlargement through anti-windup strategy for linear systems with dynamics restricted actuator. *International Journal of Systems Science*, 37(2):79–90, 2006.
- [101] S. Tarbouriech and M.C. Turner. Anti-windup design: an overview of some recent advances and open problems. *IET Control Theory and Application*, 3(1):1–19, 2009.
- [102] A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. Aut. Cont.*, 41(9):1256–1270, September 1996.
- [103] A.R. Teel. A nonlinear control viewpoint on anti-windup and related problems. In *4th Nonlinear control systems design symposium (NOLCOS)*, pages 115–120, Enschede, The Netherlands, July 1998.
- [104] A.R. Teel. Anti-windup for exponentially unstable linear systems. *Int. J. Robust and Nonlinear Control*, 9:701–716, 1999.
- [105] A.R. Teel and J.B. Buffington. Anti-windup for an F-16’s daisy chain control allocator. In *Proc. of AIAA GNC Conference*, pages 748–754, New Orleans (LA), USA, August 1997.
- [106] A.R. Teel and N. Kapoor. The  $\mathcal{L}_2$  anti-windup problem: Its definition and solution. In *Proc. 4th ECC*, Brussels, Belgium, July 1997.
- [107] A.R. Teel and N. Kapoor. Uniting local and global controllers. In *Proc. 4th ECC*, Brussels, Belgium, July 1997.
- [108] M.C. Turner and I. Postlethwaite. A new perspective on static and low order anti-windup synthesis. *International Journal of Control*, 77(1):27–44, 2004.
- [109] M.C. Turner and S. Tarbouriech. Anti-windup for linear systems with sensor saturation: Sufficient conditions for global stability and  $\mathcal{L}_2$  gain. In *Proceedings of the IEEE Conference on Decision and Control (CDC)*, San Diego, USA, 2006.
- [110] M.C. Turner and S. Tarbouriech. Anti-windup compensation for systems with sensor saturation: a study of architecture and structure. *International Journal of Control*, 82(7):1253–1266, 2009.
- [111] K. S. Walgama, S. Ronnback, and J. Sternby. Generalization of conditioning technique for anti-windup compensators. *IEE Proc. D*, 139(2):109–118, 1992.
- [112] F. Wu and B. Lu. Anti-windup control design for exponentially unstable LTI systems with actuator saturation. *Systems and Control Letters*, 52:305–322, 2004.
- [113] F. Wu and M. Soto. Extended anti-windup control schemes for LTI and LFT systems with actuator saturations. *Int. J. Robust and Nonlinear Control*, 14:1255–1281, 2004.
- [114] W. Wu and S. Jayasuriya. A new QFT design methodology for feedback systems under input saturation. *J. Dyn. Sys., Meas. and Cont.*, 123:225–23, 2001.
- [115] C. Wurmthaler and P. Hippe. Closed-loop design for stable and unstable plants with input saturation. In *European Control Conference*, pages 1084–1088, Gröningen, The Netherlands, 1993.
- [116] L. Zaccarian, Y. Li, E. Weyer, M. Cantoni, and A.R. Teel. Anti-windup for marginally stable plants and its application to open water channel control systems. *Control Engineering Practice*, 15(2):261–272, 2007.
- [117] L. Zaccarian, D. Netic, and A.R. Teel.  $\mathcal{L}_2$  anti-windup for linear dead-time systems. *Systems and Control Letters*, 54(12):1205–1217, 2005.
- [118] L. Zaccarian and A.R. Teel. A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica (B)*, 38(10):1735–1744, 2002.
- [119] L. Zaccarian and A.R. Teel. Nonlinear scheduled anti-windup design for linear systems. *IEEE Trans. Aut. Cont. (B)*, 49(11):2055–2061, 2004.
- [120] L. Zaccarian, E. Weyer, A.R. Teel, Y. Li, and M. Cantoni. Anti-windup for marginally stable plants and its application to open water channel control systems. *Control Engineering Practice*, 15(2):261–272, 2007.
- [121] A. Zheng, M. V. Kothare, and M. Morari. Anti-windup design for internal model control. *Int. J. of Control*, 60(5):1015–1024, 1994.