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Dynamic allocation for input-redundant control systems

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Abstract

In this paper we address control systems with redundant actuators and characterize the concepts of weak and strong input redundancy. Based on this characterization, we propose a dynamic augmentation to a control scheme which performs the plant input allocation with the goal of employing each actuator in a suitable way, based on its magnitude and rate limits. The proposed theory is first developed for redundant plants without saturation and then extended to the case of magnitude saturation first and of magnitude and rate saturation next. Several simulation examples illustrate the proposed technique and show its advantages for practical application.

1 Introduction

Many modern control applications are characterized by the presence of multiple actuators inducing the same effects on the plant dynamics (state) or at least the same steady-state effect on the plant output. In any such situation, it is relevant to investigate the possibility of designing a control system which performs a suitable allocation of the actuators, for example with the goal in mind of minimizing the total energy required by the actuators, or maybe of promoting certain actuators over the others for safety, power consumption or other application specific reasons.

In the literature control allocation problems have been mostly addressed with reference to specific applications. The main application field where actuators redundancy is very much addressed and studied is that of reconfigurable flight control. In particular, in this field several techniques have been proposed (see [16] for a comprehensive survey). Much has been also published for control allocation of ships and underwater vehicles (see [5] for a survey). Finally, a relevant field where a simpler control allocation problem has been addressed in different ways is that of dual stage actuator control in hard disk drives (see [15] for a survey). Even though the literature in these fields is quite disconnected, there is an increasing effort in trying to unify the different approaches (see, e.g., the recent invited session [1]).

Emerging control applications are also in need of solutions to this relevant problem. For example, actuators redundancy with different rate and magnitude saturation levels and quite complex redundancy surfaces occur in modern Tokamak devices for nuclear fusion reactions (see [21] for a relevant case study where actuators redundancy handling is required). In the nuclear fusion context, the demand for fast and efficient input allocators will become more and more relevant as superconducting coils (associated with severe rate limitations) will be adopted in conjunction with conducting coils (characterized by severe magnitude limitations). This type of problem statement generalizes the typical situation experienced, e.g., in dual stage actuator control where a fast and small actuator is connected in parallel to a slow and large actuator. A further example where actuator redundancy handling is mandatory are the novel actuator devices recently proposed in [10].

Most of the actuation techniques adopted in practice correspond to static allocators which aim at optimizing certain criteria on the allocated input (see, for example [8] and references therein). For the

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aerospace applications, [3] recognizes the strong coupling between the control allocation problem and the presence of rate and magnitude saturation limits characterizing each actuator. Also [4] proposes allocation techniques for nonlinear actuators, while in [17] efficient algorithms are proposed for the on-line computation of the optimal static allocation. Finally, in [18] spacecraft thruster configuration is addressed based on efficient optimization methods.

While static techniques are more popular for input allocation, some dynamic approaches have also been proposed where the control allocation strategy is adjusted on-line based on the operating conditions of the plant and/or of the actuators. For example, in [11] a dynamic approach is proposed where the dynamical allocation asymptotically recovers the pseudo-inverse allocation for redundant linear systems. The approach is therein also generalized to nonlinear systems with application to over-actuated ship maneuvering. A more complex dynamic allocation scheme was proposed in [14] based on a sampled-data scheme using RHC ideas.

This paper gives a contribution in the context of dynamic input allocation. The core of the idea presented here is to recognize that when input redundancy is at hand, it is possible to inject an arbitrary signal in certain input directions without affecting the state-response of the plant (in the strongly redundant case) or the steady-state output response of the plant (in the weakly redundant case). Once these redundant directions are isolated, the injected signals are selected as the outputs of a suitable number of integrators whose state is adjusted on-line based on certain gains intuitively chosen to promote/penalize the different actuators based on their rate or magnitude saturation levels. The arising dynamic allocator simply corresponds to a linear filter which augments the existing controller thereby constituting a small computational overload for real-time implementation. The proposed ideas are only presented in the linear case, even though they can be generalized also to the nonlinear case. In particular, the generalization to nonlinear controllers is straightforward, while the generalization to nonlinear plants is more involved because closed-form solutions to the stabilization of rate and magnitude saturated plants are not available, and are of course required for a general constructive solution to the allocation problem.

The paper is structured as follows: in Section 2 we give some basic definitions. In Section 3 we describe the input allocation scheme for linear control systems. In Section 4 we generalize the approach to the case of systems with magnitude saturation and in Section 5 we further extend it to the case of magnitude and rate saturation. Several examples will be given throughout the paper to illustrate the effectiveness of the proposed techniques.

2 Input redundant systems

Consider a linear control system involving a linear plant

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu + B_d d \\ y & = & Cx + Du + D_d d, \end{array} \tag{1}$$

(where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the plant input, $y \in \mathbb{R}^{n_y}$ is the plant output and $d \in \mathbb{R}^{n_d}$ is a disturbance input) satisfying an input-redundancy assumption, as formalized in the next definition.

Definition 1 The plant (1) is input-redundant if one of the following two conditions is satisfied

- it is *strongly input-redundant* from u if it satisfies $\mathrm{Ker}\left(\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right) \neq \emptyset$; in this case it is useful to introduce the following full column-rank matrix:

$$B_\perp \text{ such that } \mathrm{Im}(B_\perp) = \mathrm{Ker}\left(\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]\right); \tag{2}$$

- it is *weakly input-redundant* from u to y if $P^\star := \lim_{s\to 0}(C(sI-A)^{-1}B+D)$ is a finite matrix which satisfies $\mathrm{Ker}(P^\star) \neq \emptyset$; in this case it is useful to introduce the following full column-rank matrix:

$$B_\perp \text{ such that } \mathrm{Im}(B_\perp) = \mathrm{Ker}(P^\star). \tag{3}$$

○

Remark 1 It is worth understanding what weak input-redundancy corresponds to when there's at least one pole at the origin. In this case, the DC gain of the corresponding input-output pair is infinity so that at the steady-state the corresponding input must be zero. This fact makes it impossible to re-allocate the input which must evidently remain at zero in any steady-state situation. Therefore, the requirement that $P^\star$ is finite in Definition 1 is not restrictive. Note also that one may focus on a restricted number of inputs to obtain $P^\star < \infty$. The corresponding allocator would then be of reduced order. ○

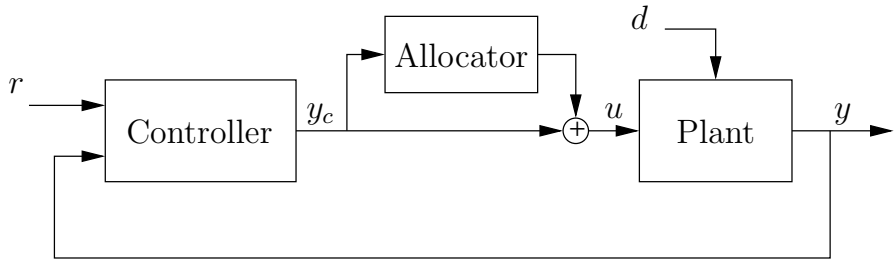

Figure 1: The input allocator in an input-redundant control system.

In this paper we address the problem of exploiting the input redundancy of the plant (1) to induce desirable input allocations during the evolution of a generic control system which can be thought of as the feedback interconnection of (1) with the following linear controller [1]

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c + B_r r \\ y_c &= C_c x_c + D_c u_c + D_r r, \end{aligned} \tag{4a}$$

$$u_c = y. \tag{4b}$$

The main intuition behind the approach proposed here is represented in Figure 1, where an input allocator block is inserted between the redundant input u of the plant and the controller output y_c. In particular, for the case of strong input-redundancy, the input allocator will be designed in such a way that it is completely invisible to the controller. Conversely, with weak input-redundancy, the allocator will cause transients at the plant output and will inevitably interact with the control system, so that it will be necessary to analyze its effects in conjunction with the controller itself.

3 Linear dynamic input allocation

3.1 Strong input redundancy

Assume that the plant (1) is strongly input redundant and consider the basis of the null space of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ introduced in (2). Note that the matrix $B_\perp$ satisfies $B_\perp^T B_\perp > 0$ and $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] B_\perp = 0$.

Consider the addition of the following dynamic input allocator at the input of the plant (1)

$$\begin{aligned} \dot{w} &= -K B_\perp^T \bar{W} u \\ u &= y_c + B_\perp w, \end{aligned} \tag{5}$$

where K and $\bar{W}$ are suitable matrices to be specified next. Then the following result holds.

[1] The controller (4) can be generalized to a generic locally Lipschitz controller. The attention here is restricted to the linear case to increase the clarity of the paper.

Theorem 1 *Assume that the plant (1) is strongly input redundant and consider the control system with input allocation namely (1), (4), (5), (2) and the control system without input allocation namely (1), (4), $u = y_c$.*

If K and $\bar{W}$ in (5) are symmetric and satisfy $K > 0$ and $B_\perp^T \bar{W} B_\perp > 0$, then

1. *the control system with input allocation is internally stable if and only if the control system without input allocation is internally stable*
2. *given any initial condition $(x(0), x_c(0), w(0))$ and any selection of the external signals $r(\cdot)$, $d(\cdot)$, the plant output responses of the two systems coincide for all times.*

Remark 2 The proof of Theorems 1 and 2 are carried out using Lyapunov-based arguments rather than linear techniques to allow an easy generalization to the nonlinear extensions of Sections 4 and 5. ◦

Proof. It is straightforward to verify that since $BB_\perp = 0$, the control system with input allocation can be written as the cascade interconnection of a first subsystem consisting in (4) and (1) with $u = y_c$, which drives via the signal y_c a second subsystem consisting in the dynamic allocator (5). Therefore the plant output y is independent of the dynamic allocator output w.

To show internal stability it is sufficient to show internal stability of the dynamic allocator (because of the cascade interconnection). To this aim consider the Lyapunov function $V(w) = \frac{1}{2} w^T K^{-1} w$ and note that

$$\dot{V} = -w^T B_\perp^T \bar{W} B_\perp w - w^T B_\perp^T \bar{W} y_c,$$

which implies internal stability because $B_\perp^T \bar{W} B_\perp > 0$ by assumption. •

Remark 3 (Speed of the dynamic allocator) Theorem 1 establishes that any (arbitrarily fast) dynamics in the dynamic allocator (5) can be implemented because the dynamics of w are completely invisible to the plant-controller dynamics. This fact can be interpreted by noting that given desirable matrices W_d and K_d, the speed of the allocator can be adjusted from extremely slow to extremely fast by selecting different values of a positive parameter ρ in the selection $K = \rho K_d$, $\bar{W} = W_d$. Indeed, by Theorem 1 any positive value of ρ will be allowed and given any controller output y_c^* the magnitude of ρ will regulate the speed of convergence to the steady-state input allocation, which is independent of K_d and is given by

$$u^\star = \left(I - B_\perp (B_\perp^T \bar{W} B_\perp)^{-1} B_\perp^T \bar{W}\right) y_c^\star. \tag{6}$$

This equation is easily derived from (5) by imposing the steady-state condition $\dot{w} = 0$ and corresponds to the solution to the following optimization problem:

$$\begin{aligned} \min_w \quad & J(u) := u^T \bar{W} u, \text{ subject to} \\ & u = y_c^\star + B_\perp w, \end{aligned} \tag{7}$$

as a matter of fact, imposing the stationary condition $\frac{\partial J(u(w))}{\partial w} = 0$, equation (6) is easily determined. ◦

Remark 4 (Shifting the allocator center) It is worth mentioning that an arbitrary (not necessarily constant) signal $u_0 \in \mathbb{R}^{n_u}$ can be subtracted from u at the right hand side of the input allocator (5), as well as its generalizations reported next. In this way, the distance of u from u_0 (rather than from the origin) is penalized by the allocator. This trick allows, for example, to account for non-symmetric saturations, as well as certain desirable operating conditions for some of the plant inputs. For example, if the first equation in (5) is replaced by $\dot{w} = -K B_\perp^T \bar{W}(u - \bar{u}_0)$, where $\bar{u}_0$ is constant, then the steady state input allocation (6) shifts to

$$u^\star = y_c^\star - B_\perp (B_\perp^T \bar{W} B_\perp)^{-1} B_\perp^T \bar{W}(y_c^\star - \bar{u}_0),$$

which is easily shown to be the solution to the following optimization problem, generalizing (7):

$$\begin{array}{ll} \min\limits_{w} & J_{\bar{u}_0}(u) := (u-\bar{u}_0)^T \bar{W}(u-\bar{u}_0), \text{ subject to} \\ & u = y_c^\star + B_\perp w. \end{array}$$

∘

Remark 5 (On the role of the parameters K and $\bar{W}$) Theorem 1 implies that any positive definite pair of matrices K and $\bar{W}$ in the dynamic allocator (5) will guarantee internal stability of the modified closed-loop. This degree of freedom can be exploited in different ways. As an example, using a diagonal $\bar{W}$, it is possible to promote or penalize the size of the different components of the plant input by suitably selecting, respectively, smaller or larger entries in the diagonal of the matrix $\bar{W}$. Indeed, the larger an entry in $\bar{W}$ will be, the larger will be the correction automatically enforced by the allocator when the corresponding input is nonzero.

As for K, it can be used to represent the control allocator with any basis B_0 of the null space of B, as a matter of fact, given an orthonormal basis $B_\perp$ and any basis B_0, there always exists a coordinate change T such that $B_0 T = B_\perp$ and selecting $\tilde{K} = TKT^T$ and defining $\tilde{w} = Tw$, one gets the equivalent representation for (5):

$$\begin{array}{rcl} \dot{\tilde{w}} & = & -\tilde{K} B_0^T \bar{W} u \\ u & = & y_c + B_0 \tilde{w}, \end{array} \tag{8}$$

which corresponds to selecting B_0 as any basis of the null space of B in (5). The general representation (8) is especially useful when wanting to penalize the use of certain redundant directions which can be isolated in the generic basis B_0. In that case, selecting the matrix $\tilde{K}$ as a diagonal positive definite matrix, some redundant input directions (corresponding to suitable columns of B_0 can be penalized by making the corresponding entries in $\tilde{K}$ small. ∘

Remark 6 (Robustness with respect to errors on $B_\perp$) Since the proof of Theorem 1 (as well as the following theorems) relies on Lyapunov-based arguments and all the right hand sides are Lipschitz, the scheme is robust with respect to arbitrarily small errors in $B_\perp$. Moreover, a formal robustness analysis with nontrivial estimates of the tolerable variations of $B_\perp$ can be carried out based on a small gain argument. ∘

Example 1 Consider the following randomly generated exponentially stable plant

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|ccc} -0.157 & -0.094 & 0.87 & 0.253 & 0.743 \\ -0.416 & -0.45 & 0.39 & 0.354 & 0.65 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

The plant is strongly input redundant with the orthonormal selection $B_\perp = [-0.27512\ -0.76512\ 0.58215]^T$. We design a controller guaranteeing asymptotic tracking of constant references via 1) negative error feedback interconnection; 2) inserting an integrator; 3) designing a stabilizing LQG controller which only uses the first two input channels. The LQG controller is designed using identity matrices both for the input and state weights in the LQ index and for the covariance matrix of the state and output equations noises. The arising matrices in (4) correspond to

$$\left[\begin{array}{c|c} A_c & B_r \\ \hline C_c & D_r \end{array}\right] = \left[\begin{array}{cccc|c} -1.57 & 0.5767 & 0.822 & -0.65 & 0 \\ -0.9 & -0.501 & -0.94 & 0.802 & 0 \\ 0 & 1 & -1.61 & 1.614 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1.81 & -1.2 & -0.46 & 0 & 0 \\ -0.62 & 1.47 & 0.89 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

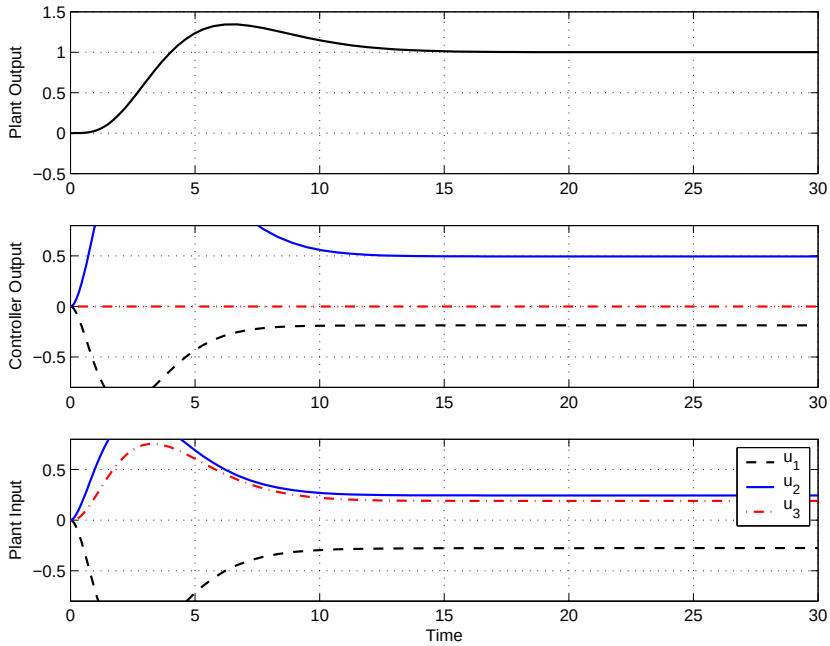

Figure 2: Example 1: responses to a unit step reference.

and to $B_c = -B_r$, $D_c = -D_r$.

By construction, the controller never uses the third input channel, so the input allocation for a given desired plant response can be improved in terms of size of each allocated input. We implement the input allocator (5) selecting $K = 10I$ and $\bar{W} = I$. The corresponding response to a unit step reference is reported in Figure 2 where the top plot represents the output response y (this is independent of the control allocation), the middle plot represents the controller output y_c (here, the third input channel is zero) and the bottom plot represents the plant input u (note that the control allocator allows to reduce the steady state size of the largest input to roughly one half as compared to the middle plot).

Following the comments in Remark 5 about the selection of W, we simulate three variants of the control allocator, selecting first $\bar{W} = \text{diag}([100\ 1\ 1])$, then $\bar{W} = \text{diag}([1\ 100\ 1])$ and finally $\bar{W} = \text{diag}([1\ 1\ 100])$. The resulting input allocations are shown in the top, middle and bottom plots of Figure 3, respectively (the plant output responses are not reported because they all coincide with the top plot of Figure 2). Figure 3 clearly illustrates the abilities of the control allocator to promote or penalize each control input.

Finally, based on the observations in Remark 3, we select $K = \rho K_d$, with $K_d = I$ and $\rho = 0.01$ and run the simulation up to time $t = 200$ to appreciate the slow input allocation performed by the dynamic allocator (this reconfiguration is, once again, invisible at the plant output). Figure 4 compares the response of Figure 2 ($K = 10I$) reported with thin lines to the response obtained using $K = 0.01I$ (thick lines). Note that the latter slowly converges to the former one (bottom plot) while the plant output is completely unaffected by this reconfiguration (top plot). $\star$

3.2 Weak input redundancy

The key difference between strong and weak input redundancy is that in the latter case, the dynamic allocation transients are visible at the plant output y, therefore there's a strong limit in the dynamic allocation speed commented in Remark 3. In particular, the main result for weakly input redundant plants establishes that there is a small enough speed such that the input-output allocation is effective (namely, closed-loop stability is retained). The corresponding dynamic allocator is given by (5) where $B_\perp$ now assumes the meaning in (3).

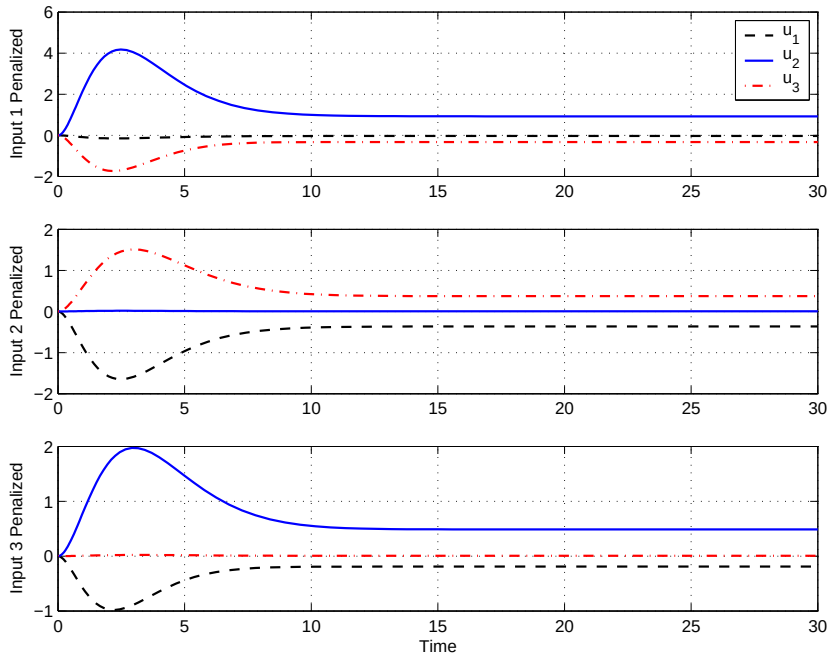

Figure 3: Example 1: different allocations based on the choice of $\bar{W}$.

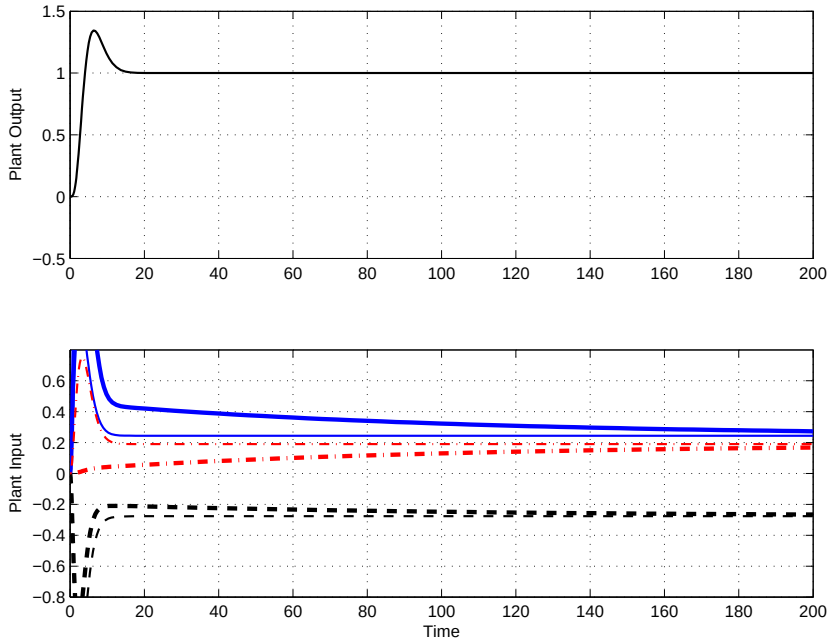

Figure 4: Example 1: different allocation speeds based on the choice of K.

Theorem 2 *Assume that the plant (1) is weakly input redundant and that the control system without input allocation (namely (1), (4), $u = y_c$) is internally stable.*

Consider the input allocator (5) with $B_\perp$ as in (3) and with $K = \rho\bar{K}$. If $\bar{K}$ and $\bar{W}$ are symmetric and satisfy $\bar{K} > 0$ and $B_\perp^T \bar{W} B_\perp > 0$, then there exists a small enough ρ such that

1. *the control system with input allocation (namely (1), (4), (5), (3), $K = \rho\bar{K}$) is internally stable and*

2. *Given any converging selection of the external signals $r(\cdot)$, $d(\cdot)$, the plant output responses of the*

two systems converge to the same values.

Proof. The closed-loop system with dynamic input allocation (1), (4), (5), (3), $K = \rho\bar{K}$ can be written in the following form

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= A_{cl} \begin{bmatrix} x \\ x_c \end{bmatrix} + B_{cl,d}d + B_{cl,r}r + B_{cl,w}w \\ y_c &= C_{cl} \begin{bmatrix} x \\ x_c \end{bmatrix} + D_{cl,d}d + D_{cl,r}r + D_{cl,w}w \\ \frac{\dot{w}}{\rho} &= -\bar{K}B_\perp^T\bar{W}(B_\perp w + y_c), \end{aligned}$$

where the matrices with subscripts "*cl*" can be uniquely determined from the plant and controller matrices in (1) and (4). Then, by the internal stability assumption for the closed-loop without input allocation, standard results on systems with two time scales (see, e.g., [13]) can be applied to conclude internal stability of the closed-loop with input allocation.

As for item 2), internal stability and linearity imply that any converging selection of external inputs will lead to converging state responses. Since the plant (1) is linear, denoting by $\star$ the steady-state values of the signals, the steady-state plant output will correspond to the sum of the contributions arising from each one of the plant inputs, namely $y^\star = y^\star_{y_c} + y^\star_d + y^\star_w = y^\star_{y_c} + y^\star_d$ because $y^\star_w = P^\star B_\perp(B_\perp^T\bar{W}B_\perp)^{-1}B_\perp^T\bar{W}y^\star_c = 0$ by the definition of $B_\perp$ in (3). Then by uniqueness of solutions, the steady-state response $y^\star$ will necessarily coincide with the steady-state response of the closed-loop without input allocation. •

Remark 7 The results in Theorem 2 imply that for a given control system where different control allocations can lead to the same steady state output, it is possible to implement a slow enough control allocator such that the corresponding dynamics is on a different time scale from that of the plant-controller closed-loop. This approach is especially useful when wanting the plant inputs to slowly drift toward a desired control allocation solution (see, e.g., the problems in [21]). In this case, the control allocator takes care of the slow corrections on the actuators, while the controller is requesting a certain output response regardless of what the input allocation selection is. ∘

Remark 8 In the presence of both strong and weak input redundancy, it is possible to select the matrix $B_\perp$ so that the first columns are in the strong redundancy directions. Then via suitable diagonal selections of the matrix K it is possible to combine the approaches of Theorems 1 and 2 by simultaneously using different speeds of allocation (see the end of the following Example 2 and Figure 7). ∘

Example 2 Consider the control system introduced in Example 1. The plant is evidently weakly input redundant and an orthonormal selection for $B_\perp$ is $B_\perp = \left[\begin{smallmatrix} -0.135 & 0.99 & -0.042 \\ -0.56 & -0.042 & 0.825 \end{smallmatrix}\right]^T$. Control allocation can be effected by picking $K = I$ and $\bar{W} = I$ and choosing a small enough ρ to ensure closed-loop stability.

Figures 5 and 6 illustrate the closed-loop responses with $\rho = 1$ and $\rho = 0.1$, respectively. The unstable behavior of the former case illustrates the limit on the speed of the dynamic control allocator in the case of weak input redundancy. Note that despite the very little use of u_3 in the bottom plot of Figure 6, the dynamic allocator allows to significantly reduce the magnitude of the inputs u_1 and u_2 (compare to the middle plot of the same figure). Also note that in the strong input redundancy case, as commented in Remark 3 this speed can be arbitrarily large.

According to Remark 8 it is possible to make use of the strong input redundancy direction by replacing the first column of $B_\perp$ by the vector $B_\perp$ determined in Example 1. Then it is possible to replace ρ by the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \rho_2 \end{smallmatrix}\right]$ and retain stability for a small enough ρ_2. In particular, selecting $\bar{K} = \left[\begin{smallmatrix} 100 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\rho_2 = 0.1$, the desirable responses of Figure 7 illustrate the effectiveness of this alternative approach. $\star$

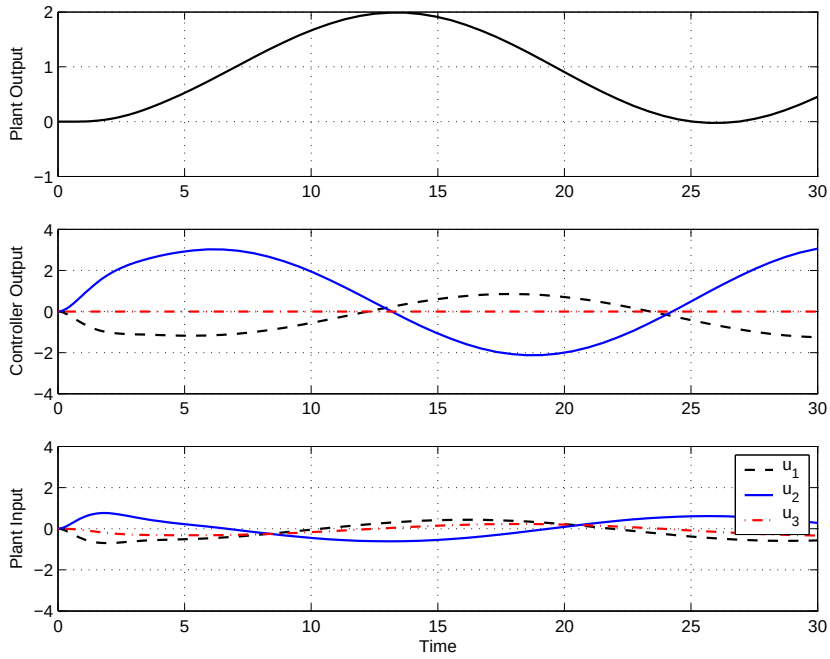

Figure 5: Example 2: unstable response caused by the selection $\rho = 1$.

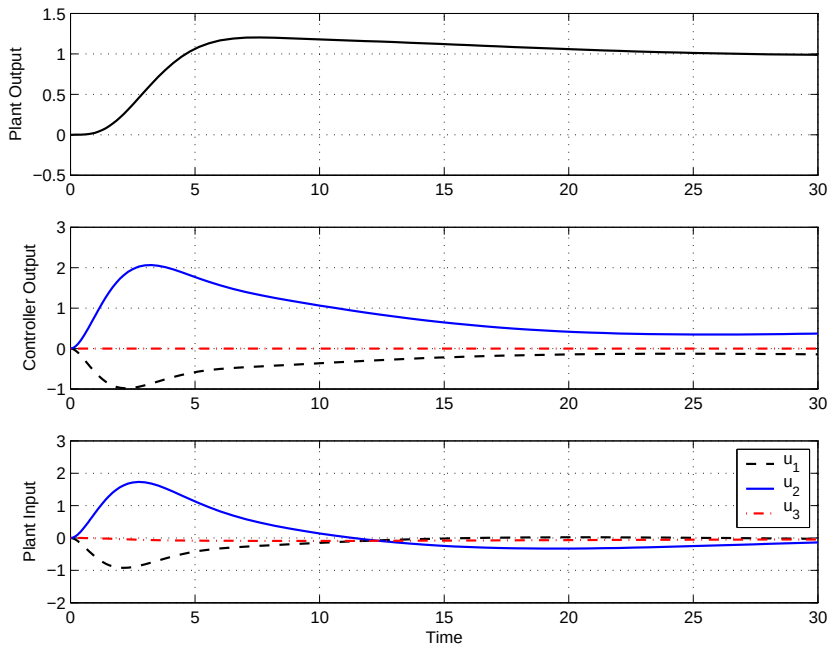

Figure 6: Example 2: stable response corresponding to $\rho = 0.1$.

4 Nonlinear input allocation for magnitude saturated plants

In this section we extend the input allocation approaches introduced in the previous section to the case where the plant input u is subject to magnitude saturation. This type of scenario is inevitably relevant when dealing with redundant actuators, as a matter of fact most plants with redundant actuators are equipped with such devices to (partially) overcome the limitations of the use of a single actuator by enhancing the control capability via the use of multiple hardware, possibly characterized by different features, such as different input ranges (which can be modeled as magnitude saturation effects) and/or

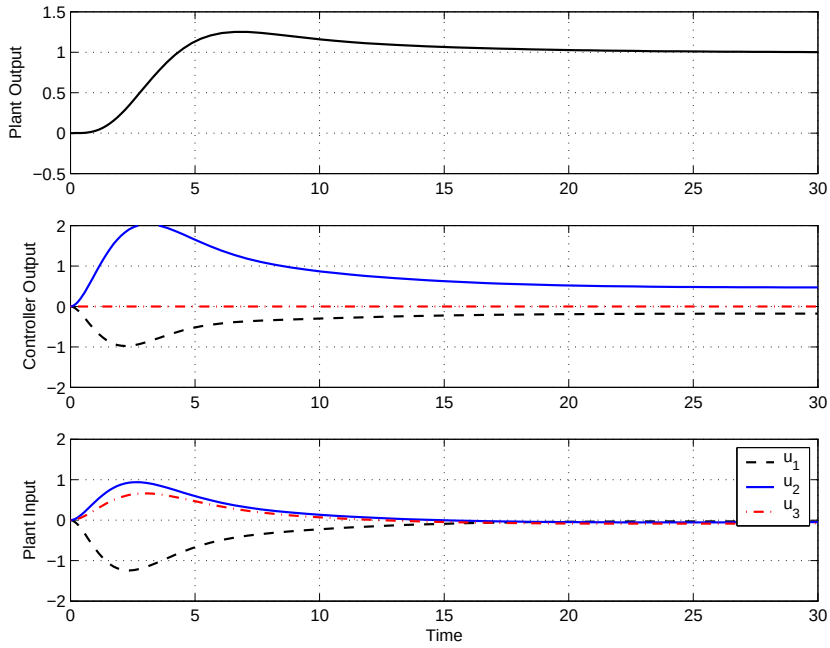

Figure 7: Example 2: response combining strong and weak redundancy.

different input variation capabilities (which can be modeled as magnitude saturation effects).

In the case when only magnitude saturation is present (see the next section for the case with rate saturation), the control allocation goal is strongly coupled with the actuators limits, as a matter of fact, the control allocation goal is mostly to promote the use of actuators capable of large excursions (namely characterized by large saturation limits) and penalize the use of actuators having small excursions (namely those with more restrictive saturation limits). To this aim, the linear allocation approach introduced in Section 3 can be suitably extended by generalizing the constant matrix $\bar{W}$ to a non-constant matrix function $W(\cdot)$ which allows to adjust on-line the selection of the actuators to be promoted or penalized. As compared to static allocation methods, the advantage of this solution is that the allocation goal is (slowly, if necessary) adjusted on-line during operation.

To account for plant input magnitude saturation, consider the following input selection for the plant (1):

$$u = \mathrm{sat}_M(y_u + v_1) = \mathrm{sat}_M(y_c + B_\perp w + v_1), \tag{9}$$

where $y_u = y_c + B_\perp w$ is the controller output after the control allocation, v_1 is a stabilizing signal defined next and $\mathrm{sat}_M(\cdot)$ is defined as the decentralized symmetric saturation function [2] having saturation limits, respectively, $m_1, \ldots, m_{n_u}$, and where $M = [m_1 \ \cdots \ m_{n_u}]^T$, so that for any $w \in \mathbb{R}^{n_u}$,

$$\mathrm{sat}_M(w) := \begin{bmatrix} m_1\sigma(w_1/m_1) \\ \vdots \\ m_{n_u}\sigma(w_{n_u}/m_{n_u}) \end{bmatrix},$$

with $\sigma(\cdot)$ being the unit saturation function defined for any $s \in \mathbb{R}$ as $\sigma(s) := \mathrm{sign}(s)\min\{|s|, 1\}$.

Given a small constant $\epsilon \in (0, 1)$ (for example, $\epsilon = 0.01$ has been often used in the simulation studies)

[2] To simplify the discussion, we only focus on symmetric saturations here. However, following the ideas in Remark 4, it is possible to extend the approach to non-symmetric saturations.

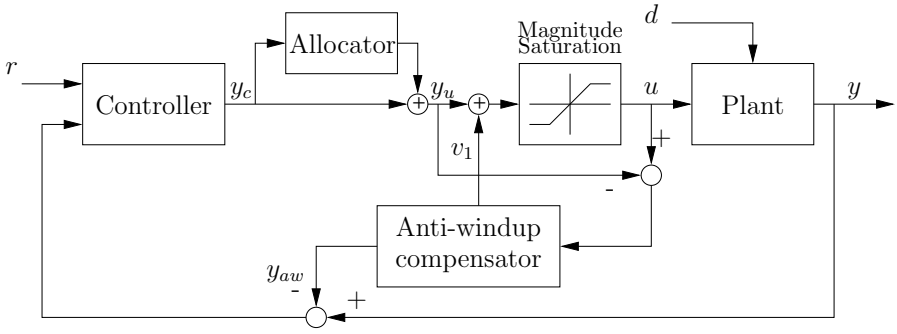

Figure 8: Dynamic input allocation with anti-windup for magnitude saturated plants.

the dynamic input allocator (5) is modified as follows:

$$\begin{aligned} \dot{w} &= -KB_\perp^T W(y_u) y_u \\ y_u &= y_c + B_\perp w \\ W(y_u) &= \left(\mathrm{diag}((1+\epsilon)M - \mathrm{abs}(\mathrm{sat}_M(y_u)))\right)^{-1}, \end{aligned} \tag{10}$$

where given a vector w, $\mathrm{abs}(w)$ denotes the vector whose components are respectively the absolute values of the components of w. The allocator (10) is interconnected via equation (9) where v_1 is a suitable correction signal generated to guarantee closed-loop asymptotic stability in light of saturation, by the following $\mathcal{L}_2$ anti-windup compensator, which is added to the closed-loop as shown in Figure 8.

$$\begin{aligned} \dot{x}_{aw} &= Ax_{aw} + B(u - y_u) \\ y_{aw} &= Cx_{aw} + D(u - y_u) \\ v_1 &= k(x_{aw}) \end{aligned} \tag{11}$$

and interconnected to the controller dynamics (4a) by replacing the interconnection equation (4b) via the following modified interconnection:

$$u_c = y - y_{aw}. \tag{12}$$

Following the anti-windup approach of [20], the function $k(\cdot)$ in equation (11) should be selected in such a way that the system

$$\dot{x}_{aw} = Ax_{aw} + B\left(\mathrm{sat}_M(y_u + k(x_{aw})) - y_u\right) \tag{13}$$

is $\mathcal{L}_2$ stable from $y_u - \mathrm{sat}_{M-\varepsilon}(y_u)$ to x_{aw} (ε is an arbitrarily small constant necessary to guarantee feasibility of the construction of $k(\cdot)$). This property can be either achieved globally (whenever the plant is not exponentially unstable) or regionally (if the plant is exponentially unstable) and the corresponding selections of the function $k(\cdot)$ are nonlinear and quite involved in general, [3] however in the special case where the plant is exponentially stable (namely, A is Hurwitz), the trivial selection $k(x_{aw}) \equiv 0$ is already sufficient for closed-loop exponential stability.

Since the focus of this paper is on the dynamic allocation aspects, rather than the anti-windup solutions, we will make use of the following definition of *feasible control signal* for (13) in the following, disregarding how the function $k(\cdot)$ (or its generalizations) should be selected to guarantee desirable anti-windup performance and large operating regions, but rather only using the necessary stability results induced by v_1 to formalize our dynamic allocation theorems. In the following we will denote as "converging signal" any function $t \mapsto s(t)$ such that $\lim_{t\to\infty} s(t)$ is well defined and finite.

[3] For more details on the selection of $k(\cdot)$, and possible alternative enhanced selections of v_1 aimed at performance improvement, see [20, Lemma 1] and the subsequent works in [22, 2, 23, 7].

Definition 2 A function $y_u(t), t \geq 0$ is a *feasible control signal* for (13) if $y_u - \text{sat}_{M-\varepsilon}(y_u) \in \mathcal{L}_2$ and the solution of (13) satisfies $x_{aw} \in \mathcal{L}_2$ ◦

Theorem 3 *Consider an input redundant plant (1) subject to input magnitude saturation of level M and the controller (4) designed in such a way that their interconnection without saturation and without allocation (namely, $u = y_c$) is internally stable.*

Then, in the strong [respectively, weak] input redundancy case, the control system with input allocation and without saturation (1), (4), (10), $u = y_c + B_\perp w$ is globally exponentially stable [respectively, it is globally exponentially stable for a small enough ρ in $K = \rho\bar{K}$].

Moreover, for any initial condition $(x(0), x_c(0), w(0))$ and input selections $r(\cdot), d(\cdot)$ such that this system generates a feasible control signal *$y_u = y_c + B_\perp w$, the following holds for the closed-loop with input allocation and with saturation (1), (4a), (10), (11), (12) with $B_\perp$ as in (2) [respectively, as in (3)]:*

1. *the plant input response u converges in the $\mathcal{L}_2$ sense to the plant input response of the control system with input allocation and without saturation;*

2) *[if $r(\cdot)$ and $d(\cdot)$ are converging signals], the plant output response y converges in the $\mathcal{L}_2$ sense to the plant output response of the control system without saturation and without allocation.*

Proof. Global exponential stability of the closed-loop without saturation (and with dynamic allocation) is easily proved by the same arguments used in Theorems 1 and 2, also noticing that for any diagonal positive definite $\bar{W}$ one gets $B_\perp^T \bar{W} B_\perp > 0$ and that $W(\cdot)$ in (10) satisfies $W(y_u) > \big(\text{diag}((1+\epsilon)M)\big)^{-1} =: W_0 > 0$, so that the Lyapunov arguments of Theorems 1 and 2 apply, despite the nonlinear nature of (10).

Consider now the whole closed-loop dynamics (1), (4a), (10), (11), (9), (12) and write them in the coordinates $(x_e, x_c, x_{aw}, w) := (x - x_{aw}, x_c, x_{aw}, w)$, which lead to a cascade structure where the first subsystem, corresponding to:

$$\begin{array}{rcl} \dot{x}_e &=& Ax_e + By_u + B_d d \\ \dot{x}_c &=& A_c x_c + B_c y_e + B_r r \\ \dot{w} &=& -KB_\perp^T W(y_u) y_u \\ y_e &=& Cx_e + Dy_u + D_d d \\ y_u &=& C_c x_c + D_c y_e + D_r r + B_\perp w \end{array} \tag{14}$$

drives, by way of y_u, a second subsystem corresponding to (13).

Since (14) reproduces the closed-loop without saturation and with dynamic allocation, then, by assumption, the signal y_u driving (13) is a feasible control signal. Hence, $x_{aw} = x - x_e \in \mathcal{L}_2$, namely x converges in the $\mathcal{L}_2$ sense to the response without saturation x_e. Therefore the response of the closed-loop with saturation converges in the $\mathcal{L}_2$ sense to the response without saturation. Based on this convergence property, the proof is completed disregarding saturation and following the same steps as in the proof of Theorem 2. •

Remark 9 Theorem 3 establishes that dynamic allocation can be employed for systems with magnitude saturation by way of an anti-windup compensator. The advantage in using an anti-windup approach is that one can design the control allocator to ensure stability in the absence of saturation (an easier task, in light of the actual saturation limits) and then address the saturation problem via the anti-windup action. Since effective anti-windup approaches are available, this trick allows to simplify the dynamic allocation problem for magnitude saturated systems. In particular, the theorem shows that the control allocator is effective for all the trajectories that can be handled by the anti-windup compensator. Finally observe that with nonlinear allocation it does not make sense to talk about "internal stability" as in Theorems 1 and 2, therefore what we prove for the allocated system before saturation is global exponential stability (which is implied by the internal stability property for the linear case). ◦

Example 3 We revisit the control system introduced in Examples 1 and 2 and assume now that all three inputs are subject to saturation, with saturation limits $M = [1\ 0.01\ 0.2]^T$. Note that these limits are randomly chosen because for any selection of M, the control allocator will automatically try to keep each control input far from its saturation limit. We consider the case where the first column of $B_\perp$ is the strongly redundant direction, while the second column completes the weakly redundant directions (similar to the case of Figure 7). However, in this case to achieve desirable responses in light of saturation, we select $K = \left[\begin{smallmatrix} 10 & 0 \\ 0 & 0.001 \end{smallmatrix}\right]$, which shows that the speed of convergence in the weakly redundant direction has to be chosen quite smaller than the case of Example 2 where saturation was not present.

For the anti-windup compensator function $k(\cdot)$ we make a naive selection ignoring input saturation and performing an LQR design with identity weights both on the state and the input. The corresponding selection is $k(x_{aw}) = K_{lqr} x_{aw}$ with $K_{lqr} = \left[\begin{smallmatrix} 0.944 & 0.0837 & 0.556 \\ -0.244 & 0.203 & 0.15 \end{smallmatrix}\right]$. This selection is sufficient to achieve regional stabilization (see, e.g., [19] for an explanation of why any stabilizing gain for (A, B) is already sufficient for regional anti-windup compensation).

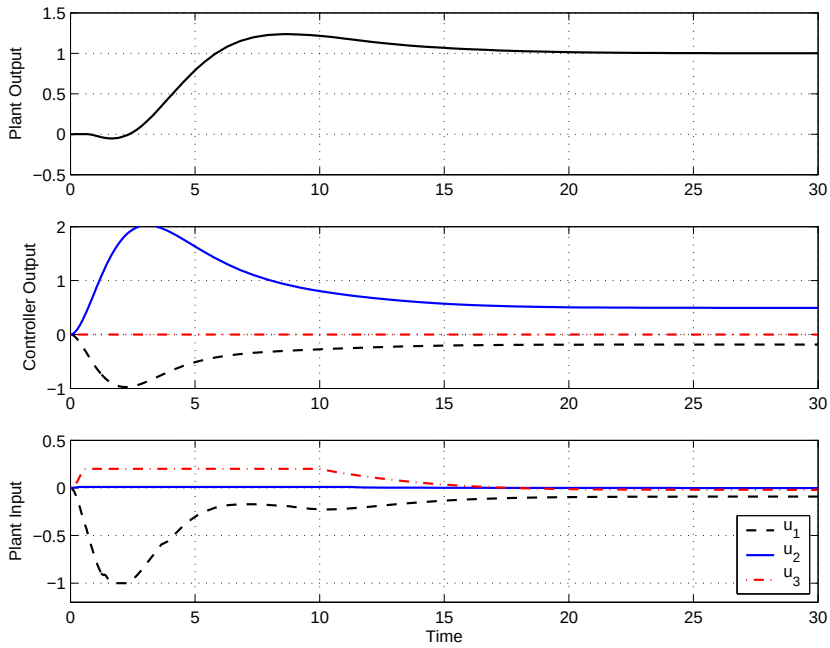

Figure 9: Example 3: response to the unit step reference.

The response to the same unit reference used in the previous examples is represented in Figure 9, where the top plot shows that with these strong saturation constraints there is some plant output performance deterioration. The middle and bottom plots represent the controller output y_c before input allocation and the plant input u after saturation. The plant input converges to the values $[-0.095\ \ -0.000337\ \ -0.014]$ which means that the percentage of use of the three inputs (as compared to the saturation levels) is roughly $[9.5\ 3.37\ 7]\%$. This useful allocation (each input is used by less than 10% of its maximal effort) is automatically enforced by the dynamic allocator. Note that the original allocation performed by the controller (middle plot) is not even within the saturation limits (for example $u_2^* \approx 0.5 \gg m_2 = 0.01$). $\star$

5 Nonlinear input allocation for magnitude and rate saturated plants

The most complex situation addressed in this paper is that when the control inputs corresponding to the different actuators are both subject to magnitude and rate saturation. To address this general case, it is necessary to suitably represent the control system in light of rate saturation: a phenomenon largely studied in the literature, which corresponded to different models ranging from the most accurate ones

leading to discontinuous dynamics representable using differential inclusions to less accurate ones, possibly consisting of linear approximations (see the references in [6] for an overview of different rate saturation models). In this paper we will use the same approach used in [6] (see also [12]), which relies on the strong assumption that the controller (4) is strictly proper (namely, $D_c = 0$ and $D_r = 0$). This assumption allows us to naturally extend the dynamic allocation approaches of the previous sections although the main ideas could possibly be usable also for non strictly proper controllers. Nevertheless, we regard this extension as future work.

Following a similar trick to that in [6], we equip the controller with an extra output corresponding to the derivative of its original output:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c + B_r r \\ y_u &= C_c x_c + B_\perp w, \\ y_{u,d} &= C_c A_c x_c + C_c B_c u_c + C_c B_r r + B_\perp \dot{w}, \end{aligned} \tag{15}$$

where $\dot{w}$ denotes the right hand side of the dynamic allocator equation, introduced next. With rate saturation, the dynamic allocator generalizes to the following extended version of (10): [4]

$$\begin{aligned} \dot{w} &= -K B_\perp^T W(y_u) y_u - K_r B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}) \\ y_u &= y_c + B_\perp w \\ W(y_u) &= \left(\mathrm{diag}((1+\epsilon)M - \mathrm{abs}(\mathrm{sat}_M(y_u)))\right)^{-1}, \end{aligned} \tag{16}$$

where $R = [r_1 \ \cdots \ r_{n_u}]^T$ represents the rate saturation limits of each input channel and $\mathrm{dz}_R(v) := v - \mathrm{sat}_R(v)$ for all $v \in \mathbb{R}^{n_u}$.

The new parameters W_r and K_r in the control allocator (16) play the same role as K and W for the rate saturation aspects. As an example, restricting the attention to diagonal selections of W_r and K_r, each diagonal term in W_r determines how penalized the variation of the r-th plant input is (namely how much the control allocator should try to distribute that control effort variation on the other redundant inputs), while each diagonal term in K_r determines how much use of the corresponding redundant direction (with reference to $B_\perp$) should be made by the control allocator. Since rate saturation allocation is only necessary during the transients, we will assume that only the diagonal terms in K_r corresponding to strongly redundant columns of $B_\perp$ are nonzero.

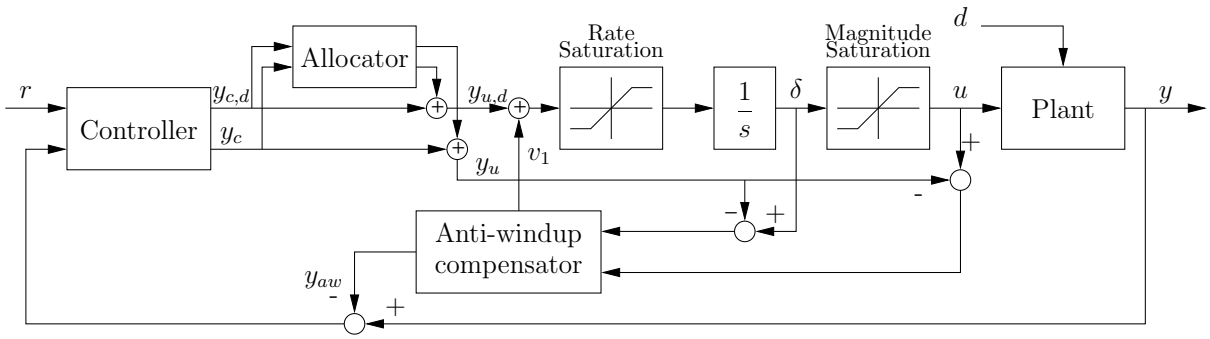

Figure 10: Dynamic input allocation with anti-windup for magnitude and rate saturated plants.

Also in the presence of rate saturation, the augmented control system requires anti-windup action. In particular, to suitably account for rate and magnitude saturation, as represented in the block diagram

[4] Note that in the absence of rate saturation, namely when $R = +\infty$, the dynamic allocator (16) reduces to (10).

of Figure 10, the following dynamics should be added to the control scheme:

$$\begin{array}{rcl} \dot{x}_{aw} & = & Ax_{aw} + B(u - y_u) \\ \dot{\delta} & = & \mathrm{sat}_R(y_{u,d} + v_1) \\ u & = & \mathrm{sat}_M(\delta) \\ y_{aw} & = & Cx_{aw} + D(u - y_u) \\ v_1 & = & k_r\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_u \end{smallmatrix}\right]\right) \end{array} \tag{17}$$

which should be once again interconnected to the controller input via equation (12) and where $k_r(\cdot)$ is a stabilizing function generalizing the function $k(\cdot)$ of (11), to be defined next.

Note that substituting the right hand side of the dynamics (16) in the third equation of (15), one obtains the following nonlinear algebraic loop, which is important to suitably adjust on-line the rate of the plant control inputs:

$$W_r y_{u,d} = -W_r B_\perp K_r B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}) + W_r v, \tag{18}$$

where $v = C_c A_c x_c + C_c B_c u_c + C_c B_r r - B_\perp K B_\perp^T W(y_u) y_u$ is independent of $y_{u,d}$ and does not affect the well-posedness of the algebraic loop. The following result is then necessary to ensure well-posedness.

Lemma 1 *If $K_r = K_r^T \geq 0$ and $W_r = W_r^T$, then the algebraic loop (18) is well-posed.*

Proof. By [9, Claim 2] and [22, Proposition 1], the algebraic loop is well-posed if there exists a diagonal and positive definite U such that $-2U - UW_r B_\perp K_r B_\perp^T W_r - W_r B_\perp K_r B_\perp^T W_r U < 0$. Using $U = I$ this inequality becomes $I + W_r B_\perp K_r B_\perp^T W_r > 0$ which is always satisfied because $K_r \geq 0$. •

Remark 10 From an implementation viewpoint, the algebraic loop (18) resulting from the combination of (16) and (15) may be hard to implement and solve on-line (its solution is well-known to be a piecewise affine map in the space of v). An effective alternative is to introduce a full column rank matrix $K_r^{1/2}$ such that $K_r^{1/2}(K_r^{1/2})^T = K_r \geq 0$ and insert a bank of low pass filters with state x_r in the input allocator (16) as follows:

$$\begin{array}{rcl} \dot{w} & = & -KB_\perp^T W(y_u) y_u - K_r^{1/2} x_r \\ \dfrac{\dot{x}_r}{\sigma} & = & -x_r + (K_r^{1/2})^T B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}), \end{array}$$

where σ is a large enough constant so that there's a time-scale separation between the filter dynamics and the remaining closed-loop dynamics. The arising fast dynamics of the filter would always be globally exponentially stable because they would correspond to:

$$\begin{array}{rcl} \dfrac{\dot{x}_r}{\sigma} & = & -x_r + (K_r^{1/2})^T B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}) \\ & = & -x_r - (K_r^{1/2})^T B_\perp^T W_r \mathrm{dz}_R(W_r B_\perp K_r^{1/2} x_r - v) \end{array}$$

which is globally exponentially stable by the sector properties of the deadzone function. ◦

Regarding the selection of the compensation function $k_r(\cdot)$ in (17), paralleling the case of magnitude saturation in Definition 2, the design goal is that given two functions $y_u(t), y_{u,d}(t)$, $t \geq 0$ such that $y_{u,d}(t) = \dot{y}_u(t)$ almost everywhere, the following system

$$\begin{array}{rcl} \dot{x}_{aw} & = & Ax_{aw} + B(\mathrm{sat}_M(\delta_{aw} + y_u) - y_u) \\ \dot{\delta}_{aw} & = & \mathrm{sat}_R\left(y_{u,d} + k_r\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right)\right) - y_{u,d} \end{array} \tag{19}$$

is $\mathcal{L}_2$ stable from $\left[\begin{smallmatrix} y_u - \mathrm{sat}_{M-\varepsilon}(y_u) \\ y_{u,d} - \mathrm{sat}_{R-\varepsilon}(y_{u,d}) \end{smallmatrix}\right]$ to (x_{aw}, δ_{aw}). Paralleling the discussion in Section 4 we can again make the following definition.

Definition 3 A function $y_u(t), t \geq 0$ for which there exists $y_{u,d}(t), t \geq 0$ such that $y_{u,d}(t) = \dot{y}_u(t)$ almost everywhere, is a *feasible control signal* for (19) if $\begin{bmatrix} y_u - \mathrm{sat}_{M-\varepsilon}(y_u) \\ y_{u,d} - \mathrm{sat}_{R-\varepsilon}(y_{u,d}) \end{bmatrix} \in \mathcal{L}_2$ and the solution of (19) satisfies $(x_{aw}, \delta_{aw}) \in \mathcal{L}_2$ ◦

Theorem 4 *Consider an input redundant plant (1) subject to input magnitude saturation of level M, rate saturation of level R and the controller (15) designed in such a way that their interconnection without rate nor magnitude saturation and without allocation (namely, $u = C_c x_c$, $u_c = y$) is internally stable.*

Assume that $B_\perp = [B_{s\perp} B_{o\perp}]$ where the first columns generate the strongly redundant directions and the remaining ones complete the basis of the weakly redundant directions and pick $K_r = K_r^T = \begin{bmatrix} K_{rs} & 0 \\ 0 & 0 \end{bmatrix} \geq 0$, $W_r \geq 0$ diagonal and $K = K^T = \rho \bar{K} > 0$ where $\rho \in \mathbb{R}$ is a positive scalar. Then the origin of the control system with input allocation and without saturation (1), (15), (16), $u = y_u$, $u_c = y$ is semiglobally exponentially stable with ρ (namely, for each arbitrarily large ball, there is a small enough ρ such that the ball is inside the basin of attraction of the exponentially stable origin).

Moreover, for any initial condition $(x(0), x_c(0), w(0))$ in the domain of attraction and any input selections $r(\cdot), d(\cdot)$ such that this system generates a feasible control signal *y_u, the following holds for the closed-loop with input allocation and with saturation (1), (15), (16), (17), (12):*

1. *the plant input response u converges in the $\mathcal{L}_2$ sense to the plant input response of the control system with input allocation and without saturation;*

2) *if $r(\cdot)$ and $d(\cdot)$ are converging signals, the plant output response y converges in the $\mathcal{L}_2$ sense to the plant output response of the control system without saturation and without allocation.*

Remark 11 The semiglobal stability result in Theorem 4 conveys the fact that two allocators act at the same time: the magnitude one and the rate one. For these allocators to be arbitrary and generic it is necessary that they act on different time scales: the rate one is fast and acts during the transients, while the magnitude one is slow enough - as imposed by ρ - and imposes the steady state values of the plant input. Note that it could be possible to allow the two allocators to act at the same time but to ensure closed-loop stability this would require imposing extra compatibility conditions among the parameters K, $W(\cdot)$, K_r and W_r. Instead, having time scale separation between the two allocators allows to leave the magnitude parameters (K and $W(\cdot)$) and the rate parameters (K_r and W_r) to be generic and independent of each other. ◦

Remark 12 Paralleling the results in Theorem 3, if the plant only has strong input redundant directions, then the second block components of K and K_r disappear and it is not necessary to require that $r(\cdot)$ and $d(\cdot)$ are converging signals at item 1 of the theorem. ◦

Proof. Semiglobal exponential stability of the closed-loop without saturation (and with dynamic allocation) follows by the same argument of the proof of Theorem 3 except for the new term appearing on the right hand side of the dynamic allocator (16). This term does not interact with the plant-controller dynamic because K_r is only nonzero in the state-redundant directions. However, it does interact with the w dynamics which become

$$\begin{aligned} \dot{w} &= -KB_\perp^T W(y_u) B_\perp w - KB_\perp^T W(y_u) y_c \\ &\quad -K_r B_\perp^T W_r \mathrm{dz}_R(W_r B_\perp \dot{w} + W_r C_c \dot{x}_c) \end{aligned} \tag{20}$$

where, by definition of $W(\cdot)$, there exist diagonal positive definite matrices W_0 and W_M such that $0 < W_0 < W(y_u(t)) < W_M$. Due to the cascaded structure of the closed-loop without saturation (also repeating the two-time scale argument of the proof of Theorem 2), since (20) is a globally Lipschitz system under the action of two bounded signals (y_c and $\cdot x_c$), it is sufficient to establish its exponential stability to conclude exponential stability of the interconnected system. To this aim, we will use the following lemma, which follows from using a generalized sector condition for the deadzone and studying the regional exponential stability of (20) with the quadratic Lyapunov function $w^T P w$ (see [9] for details).

Lemma 2 *If there exist matrices $P = P^T > 0$, $U > 0$ diagonal and $t \mapsto H(t)$ such that (where we use the shorthand notation $W_t = W(y_u(t))$ and $H_t = H(t)$)*

$$\mathrm{He}\begin{bmatrix} -PKB_\perp^T W_t B_\perp & (H_t^T - B_\perp^T W_t B_\perp K B_\perp^T W_r)U \\ -W_r B_\perp K_r P & -U - UW_r B_\perp K_r B_\perp^T W_r \end{bmatrix} < 0 \tag{21a}$$

$$\begin{bmatrix} r_i^2 & H_i(t)^T \\ H_i(t) & P \end{bmatrix} > 0, \quad i = 1, \ldots, n_u \tag{21b}$$

for all t (where r_i denotes the i-th component of R and $H_i(t)$ denotes the i-th row of $H(t)$), then the origin of (20) is exponentially stable with domain of attraction including the set $\mathcal{E}(P) := \{x : x^T P x \leq 1\}$.

To show semiglobal stability with arbitrarily small ρ, given any ball of arbitrary size, first select P such that $\mathrm{He}(-P\bar{K}B_\perp^T W_0 B_\perp) < 0$ and such that $\mathcal{E}(P)$ contains the desired exponential stability set (this P always exists because $B_\perp^T W_0 B_\perp > 0$ and any candidate P can be scaled arbitrarily). Then pick $H^T(t) = B_\perp^T W(t) B_\perp K B_\perp^T W_r = B_\perp^T W(t) B_\perp K B_\perp^T W_r$ and pick ρ small enough so that this $H(\cdot)$ satisfies equations (21b) for all t (note that this is always possible because $W_0 < W(t) < W_M$ for all t). Finally, since this selection of $H(\cdot)$ zeroes out the top right entry of (21a) it is sufficient to choose U large enough in (21a) to guarantee that (21) is satisfied. Then semiglobal stability follows from the lemma.

Consider now the whole closed-loop dynamics (1), (15), (16), (17), (12) and write them in the coordinates $(x_e, x_c, w, x_{aw}, \delta_{aw}) := (x - x_{aw}, x_c, w, x_{aw}, \delta - y_u)$, where $y_u = C_c x_c + B_\perp w$. This leads to a cascade structure where the first subsystem, corresponding to:

$$\begin{array}{rcl} \dot{x}_e &=& Ax_e + By_u + B_d d \\ \dot{x}_c &=& A_c x_c + B_c y_e + B_r r \\ \dot{w} &=& -KB_\perp^T W(y_u) y_u - K_r B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}) \\ y_e &=& Cx_e \\ y_u &=& C_c x_c + B_\perp w \end{array} \tag{22}$$

drives, by way of y_u and $y_{u,d} = C_c(A_c x_c + B_c y_e + B_r r) - B_\perp(KB_\perp^T W(y_u) y_u + K_r B_\perp^T W_r \mathrm{dz}_R(W_r y_{u,d}))$ a second subsystem corresponding to (19). Then, the $\mathcal{L}_2$ convergence property follows from the assumption that y_u is a feasible control signal.

As for the two items of the theorem, since by definition $x = x_e + x_{aw}$, the actual plant state will converge to the response of the fictitious plant in (22) as long as (x_{aw}, δ_{aw}) converge to zero. Once again, the result follows from the properties of system (22) together with the assumption that y_u is a feasible control signal. •

Example 4 Consider the same system used in Example 3 and introduce a rate saturation level at all inputs given by $R = [0.3\ 10\ 1]^T$, so that, quite reasonably, input 1 which has weak magnitude saturation is characterized by severe rate saturation. For this example we select the same parameters as in Example 3 and use $K_r = \left[\begin{smallmatrix} 1000 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $W_r = I$, so that only the state redundant directions are used for input rate allocation. Similar to the magnitude saturated case, the anti-windup gain is selected as $k_r\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right) = K_{lqr}\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]$, where K_{lqr} is an LQR gain for system (19) without saturations selected using identity penalty matrices. The closed-loop response to the same unit step reference used in the other examples is represented in Figure 11. Note that in spite of the restrictive rate and magnitude saturation limits, the control allocator with anti-windup compensation allows to guarantee a desirable transient and converge to a situation with a desirable input allocation (this is actually the same one reached in the simulation of Example 3. ⋆

Example 5 As a last example, we consider a simple plant with $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$ $D = \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]$ and where the controller is an LQG controller with an extra integral action to guarantee asymptotic regulation of constant references, which only acts on one of the two identical input channels.

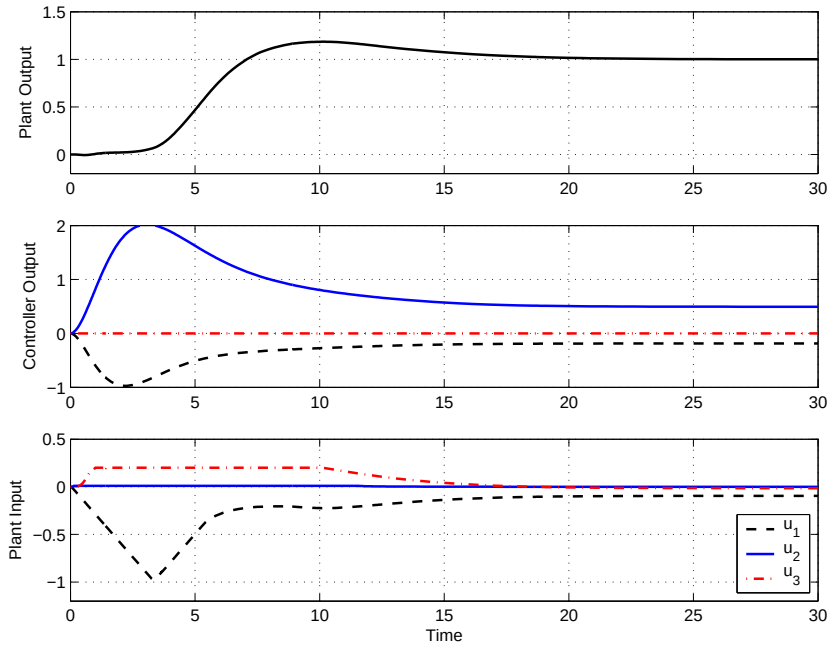

Figure 11: Example 5: response to the unit step reference.

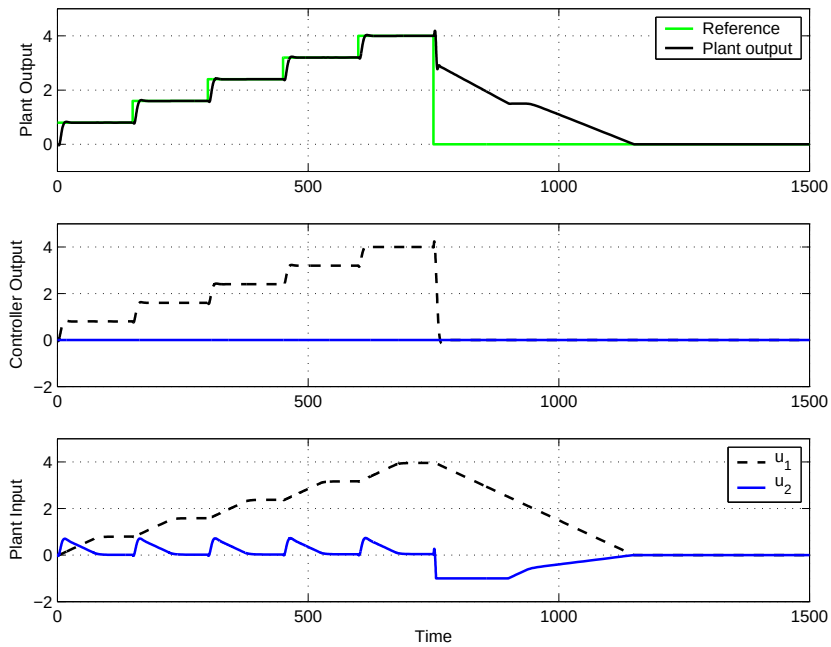

Figure 12: Example 5: response to small and large step references.

To reproduce a typical practical situation we select $M = [100\ 1]^T$ and $R = [.01\ 100]^T$ namely the first actuator is very powerful and slow while the second one is fast but has small excursions.

The plant is only strongly redundant and we pick $B_\perp = [-1\ 1]^T$. Moreover, we select $\bar{W} = I$, $\bar{K} = 0.1$ and $K_r = 1000$, $W_r = 1$, so that the rate allocation is faster than the magnitude allocation (this typically leads to improved responses) and the anti-windup gain K_{lqr} is selected as in the previous example. Figure 12 represents the response to a reference consisting of small steps first and then a large step at the end, which is shown by a dashed curve in the top plot.

From the middle plot, the automatic allocation performed by the dynamic allocator is quite intuitive:

for small steps the fast actuator is fully employed but also discharged as fast as possible to come back to the middle of its small operating range as fast as possible. The slow actuator, in turns, grows at its maximal rate to compensate for the centering of the fast actuator. For each one of the small reference steps, there is an automatic reconfiguration of the two actuators which is invisible at the plant output (namely, while the slow actuator grows, the fast actuator comes back to its central position). When the large step reference comes into place, the fast actuator immediately saturates and the slow actuator ramps down at maximal speed. The output response is inevitably deteriorated due to the intrinsic limits of the actuators. A desirable behavior during the response to the large step is that the fast actuator starts coming back to the central position (moves away from saturation) before the end of the transient. This behavior is useful so that the fast actuator is ready to perform fine corrections at the end of the response, if necessary and it is automatically performed by the fact that the magnitude allocator (performed by K) starts becoming dominant as compared to the rate allocator (performed by K_r). $\star$

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6 Conclusions

In this paper we presented a novel technique to distribute control authority among redundant actuators. The proposed dynamic allocation technique is intuitive and computationally efficient and has been successfully illustrated on several examples. To effectively characterize situations where re-allocation can be arbitrarily fast or should be sufficiently slow, the concepts of strong and weak redundancy have been introduced. Moreover, the theory has been illustrated in conjunction with magnitude and rate saturation phenomena which cannot be neglected whenever redundant actuators are in place. Future work includes the extension of this techniques to the case of nonlinear plants, at least for some relevant special cases.

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