The $\mathcal{L}_2$ (l_2) bumpless transfer problem for linear plants: its definition and solution

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Abstract

A novel characterization of bumpless transfer among alternative controllers, both in continuous and discrete time, is introduced. The bumpless transfer problem is to design a compensation scheme guaranteeing an $\mathcal{L}_2$ (respectively l_2) bound on the mismatch, after the switching time between the actual plant output and a particular target, ideal response. Minimization of the gain from initial plant state mismatch to $\mathcal{L}_2$ (respectively, l_2) plant output mismatch provides improved transients. A solution to the bumpless transfer problem is given, both for plants without input saturation and for plants with input saturation. The solution guarantees anti-windup features in the latter case.

Key words: Bumpless transfer, Anti-Windup, bounded control

1 Introduction

Bumpless authority transfer among different controllers is a goal that was formulated in the early stages of applied automatic control research. It arose because the application of classical control techniques often led to undesired transients and even instability after power-on and after switching control authority among controllers that induce different closed-loop performance properties (see, e.g., Morse (1995)). The first results on systematic bumpless transfer designs were given in Hanus et al. (1987), where an anti-windup scheme was modified in order to pre-condition the controller before the switch-on time. Since then, bumpless transfer schemes have been addressed mainly in conjunction with some anti-windup schemes. Many bumpless transfer schemes, combined with underlying anti-windup solutions, are described in the survey papers Edwards and Postlethwaite (1998); Kothare et al. (1994); Graebe and Ahlén (1996). The bumpless transfer problem has been addressed more recently and directly in Turner and Walker (2000) and its application reported in Turner et al. (2002). Other schemes that address the bumpless transfer goal, not in conjunction with anti-windup, are given in Lee et al. (2002) and also in Arehart and Wolovich (1996) where the bumpless idea is seen from a hybrid systems perspective. In the above mentioned recent work, "bumpless transfer" is generally achieved by first requiring that the plant input is continuous (or, at least, it exhibits small jumps) at the switching time. This specification, suggested in Morse (1995); Graebe and Ahlén (1996), is obtained by forcing the controller output to coincide with the plant input before the switch. Next the bumpless transfer scheme must be endowed with additional structure aimed at inducing a desirable output response. Indeed, not much plant output information can be obtained only from the property that its input is (almost) continuous. In recent years, bumpless transfer has also been addressed in the context of hybrid control systems with switching control laws. In this framework, the typical results provide conditions for asymptotic stability of the switched system under certain conditions on the switching signal (see, e.g., the references in Zaccarian and Teel (2002)). While establishing stability of the switched closed-loop is an interesting research goal, it does not fully capture the intuitive idea behind "bumpless transfer" which, at least from an industrialist's viewpoint, is mainly concerned with optimizing the transient after one single switch. This fact was discussed in Zaccarian and Teel (2002).

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[1] Work supported by AFOSR grant F49620-03-1-0203, NSF grant ECS-0324679, ENEA-Euratom, ASI and MIUR through PRIN and FIRB projects.

In this paper we propose a formal definition of the bumpless transfer problem, based on the introduction of a *target response*, generated by a target closed-loop system, characterizing the ideal closed-loop behavior after the switch. The bumpless transfer design goal can then be stated formally as that of recovering the ideal response in an $\mathcal{L}_2$ (l_2) sense. This characterization is more appealing than the sole requirement that the controller output matches the plant input at the switching time because: 1) it captures the intuitive goal of bumpless transfer which is concerned with the plant output rather than its input; 2) it allows for cases where (due to the presence of disturbances, or simply due to unpredictability of the switching time) it is not possible to impose the controller output value. On the other hand, the method requires a plant model. Other methods, such as Hanus et al. (1987); Lee et al. (2002); Turner and Walker (2000) do not require a plant model and are employed to aid the switching among several linear controllers designed for a nonlinear plant in different operating conditions. The extension of our scheme to that case would require the implementation of several linearized models in the scheme and might be uneasy to pursue, in general. In Zaccarian et al. (2004), the methods herein proposed have been successfully applied to a relevant simulation study. The paper is structured as follows: in Section 2 we define the problem and in Section 3 we give the solution. In Section 4, we show the effectiveness of the approach on a simulation example.

Notation. Given a function of continuous (respectively, discrete) time $x : \mathbb{R} \to \mathbb{R}^n$ (respectively, $x : \mathbb{Z} \to \mathbb{R}^n$), the notation $\Delta x(t)$, often abbreviated Δx, denotes the differentiation with respect to time $\dot{x}(t) = \frac{dx(t)}{dt}$, often abbreviated $\dot{x}$ (respectively, it denotes the push-forward map with respect to time $x^+(t) = x(t+1)$, often abbreviated x^+). The Euclidean norm is denoted $|\cdot|$. Given a signal $s : \mathbb{R} \to \mathbb{R}^m$ (respectively, a signal $s : \mathbb{Z} \to \mathbb{R}^m$), and two times $t_1 \leq t_2$ (possibly equal to infinity), $\|s(\cdot)[t_1, t_2]\|_2$ denotes the $\mathcal{L}_2$ (respectively, l_2) norm of $s(\cdot)$ restricted to the time interval $[t_1, t_2]$, namely $\|s(\cdot)[t_1, t_2]\|_2 = \left(\int_{t_1}^{t_2} |s(t)|^2 dt\right)^{\frac{1}{2}}$ (respectively, $\|s(\cdot)[t_1, t_2]\|_2 = \left(\sum_{t=t_1}^{t_2} |s(t)|^2\right)^{\frac{1}{2}}$). Moreover, $\|s(\cdot)\|_\infty$ denotes the $\mathcal{L}_\infty$ (respectively, l_∞) norm of $s(\cdot)$, namely $\|s(\cdot)\|_\infty = \sup_{t\in\mathbb{R}} |s(t)|$ (respectively, $\|s(\cdot)\|_\infty = \sup_{t\in\mathbb{Z}} |s(t)|$).

2 Problem definition

Consider a linear time-invariant plant of the form

$$\begin{aligned} \Delta x &= Ax + Bu + B_d d \\ y &= Cx + Du + D_d d\ , \quad z = C_z x + D_z u + D_{dz} d, \end{aligned} \tag{1}$$

where x is the plant state, u is the control input, d is a disturbance input, y is the measured output and z is the performance output. Assume the following (possibly nonlinear) controller has been designed for the plant (1) to ensure certain closed-loop performance and robustness properties:

$$\Delta x_c = f(x_c, u_c, r), \quad y_c = g(x_c, u_c, r), \tag{2}$$

where x_c is the controller state, u_c is the feedback measurement input and r is the reference input. For the closed-loop between the controller (2) and the plant (1) we make the following assumption, which can be verified easily in many cases of interest (see Remark 3).

Assumption 1 *The closed-loop between the plant (1) and the controller (2) via the interconnection equations*

$$u = y_c + w_1, \quad u_c = y + w_2 \tag{3}$$

is such that solutions exist and are unique (i.e., the interconnection is well-posed) and satisfies the following Bounded Input Bounded Difference (BIBD) condition from the input (w_1, w_2): given any inputs $r(\cdot)$, $d(\cdot)$, and initial conditions $x(0)$, $x_c(0)$ such that the closed-loop solution $(x_0(\cdot), x_{c0}(\cdot))$ with $(w_1(\cdot), w_2(\cdot)) \equiv (0, 0)$ is uniformly bounded, then for any uniformly bounded inputs $(w_1(\cdot), w_2(\cdot))$, there exists a constant M such that the arising response $(x(\cdot), x_c(\cdot))$ satisfies the following bound

$$\left\| \left(x(\cdot) - x_0(\cdot)\ ,\ x_c(\cdot) - x_{c0}(\cdot) \right) \right\|_\infty \leq M\ . \tag{4}$$

Remark 1 The BIBD property introduced in Assumption 1 is, in general, a weakened version of the classical Bounded Input Bounded State (BIBS) property. Indeed, if the closed-loop system (1), (2), (3) is BIBS from the input (w_1, w_2), then the inequality (4) follows by breaking in two the difference at the left hand side, thereby obtaining $\|(x(\cdot), x_c(\cdot)) - (x_0(\cdot), x_{c0}(\cdot))\|_\infty \leq \|(x(\cdot), x_c(\cdot))\|_\infty + \|(x_0(\cdot), x_{c0}(\cdot))\|_\infty$, and noting that the first term is uniformly bounded by the BIBS assumption and that the second term is uniformly bounded by assumption. ◦

Given the plant (1), the bumpless transfer problem here defined addresses the situation where the plant input u is interconnected to the controller output y_c at some switching instant t_s (where $t_s \in \mathbb{R}$ for the continuous-time case and $t_s \in \mathbb{Z}$ for the discrete-time case). To characterize the properties of the bumpless transfer system before and after the switching time, we will need to assume that the signals r, d driving the controller-plant pair are defined for all (positive and negative) times. Moreover, for the sake of generality, we will assume that before the switching time (namely, for all $t < t_s$) the plant control input $u(\cdot)$ is fixed as a generic (measurable) function $\hat{u}(\cdot) : (-\infty, t_s] \to \mathbb{R}^m$. This function may arise from a manual control input as well as from the output of

an alternative controller from which authority needs to be transferred to the controller (2). To be more specific, the closed-loop without bumpless transfer compensation (otherwise called "bumpy closed-loop" in the following) corresponds to the interconnection between (1) and (2) through the following equations and initial conditions:

$$u(t) = \begin{cases} \hat{u}(t), & t < t_s, \\ y_c(t), & t \geq t_s, \end{cases} \qquad (5)$$
$$u_c(t) = y(t),\ t \geq t_s, \quad x_c(t_s) = 0.$$

To suitably define the $\mathcal{L}_2$ (respectively, l_2) bumpless transfer problem, we need to introduce the concept of *target* plant response, as detailed in the next definition.

Definition 1 Given the plant (1) and the controller (2) consider choices of the external signals $r(t),\ d(t),\ t \in (-\infty, +\infty)$, such that the solution of the closed-loop system (1), (2) arising from taking the limit $t_s \to -\infty$ exists and is unique. Denote this closed-loop system by *target closed-loop system*, the corresponding (unique) solution by *target state response* $(x_T(t), x_{c,T}(t))$, $t \in (-\infty, +\infty)$ and the corresponding performance output response by *target output response* $z_T(t)$, $t \in (-\infty, +\infty)$. ◦

Remark 2 The intuitive idea behind the target response of Definition 1 is that we want to characterize the best possible response achievable on the plant (1) after the switching time t_s from the viewpoint of the controller (2). Since the undesired "bumpy" behavior arises from the presence of the switch between the external input signal $\hat{u}(\cdot)$ and the controller output $y_c(\cdot)$, the best that we can hope for when addressing a bumpless transfer problem is to completely recover an ideal plant response characterized by the absence of the switch: namely, characterized by the behavior of the controller permanently interconnected to the plant. To this aim, it makes sense to require that the ideal target behavior only depends on the structure of the plant-controller pair (1), (2) and not on the switching instant t_s. This is actually the case for our Definition 1. ◦

Remark 3 In many cases of practical interest, the controller (2) is linear and such that the interconnection (3) is well-posed and internally stable. Then, Assumption 1 follows in a straightforward way from the linearity of the closed-loop. Moreover, the target state response $x_T(\cdot)$ of Definition 1 can be written explicitly for all $t \in \mathbb{R}$ as

$$x_T(t) := \int_{-\infty}^{t} e^{A_{cl}(t-\tau)} \left(B_{cl,r} r(\tau) + B_{cl,d} d(\tau)\right) d\tau, \qquad (6)$$

where A_{cl} is the state matrix of the linear closed-loop and $B_{cl,r}$ and $B_{cl,d}$ are the input matrices for r and d, respectively. If the closed-loop system is stabilizable from (w_1, w_2) (it is usually made asymptotically stable, and hence stabilizable, by design) then the BIBD property of Assumption 1 implies that A_{cl} is Hurwitz. Note the similarity between equation (6) and the standard definition of "steady-state" response (see, e.g., (Gentili et al., 1998, §6.7)), typically known in the linear systems setting. In particular, observe that when A_{cl} is Hurwitz, a sufficient condition for the target solution to exist and to be unique is that $r(\cdot)$ and $d(\cdot)$ do not grow exponentially in reverse time, so that the integral in (6) converges to a unique finite value for all times. ◦

Based on the above definition of target response, we have characterized what a controller would have done (in light of certain reference and disturbance choices) if it had been always interconnected to the plant. The natural goal of the bumpless-transfer strategy is to reproduce this response at the plant output z after the switching time t_s, or in other words, to make the signal $z(t) - z_T(t)$ small (as small as possible) for all $t \geq t_s$. Notice however that a structural limitation to this goal arises from the fact that before the switching time t_s, the external input $\hat{u}(\cdot)$ may drive the plant state $x(\cdot)$ (and its output $z(\cdot)$) arbitrarily far from this ideal response. Therefore, the controller has no authority over $x(\cdot)$ for all $t \leq t_s$ and we need to assume at least (uniform) boundedness of this response in that time interval. Moreover, by the continuity of solutions to ODEs, the best that we can hope for will be limited by the plant state mismatch $x(t_s) - x_T(t_s)$ at the switching time.

Problem 1 *Given the plant-controller pair (1), (2), the* bumpless transfer (BT) problem *is to design a control scheme such that the following holds: for any choice of $r(\cdot), d(\cdot)$ such that the corresponding target response $x_T(\cdot)$ exists and is unique, for any switching time t_s and any uniformly bounded choice of the external signal $\hat{u}(\cdot)$ such that $x(\cdot)$ is uniformly bounded in $(-\infty, t_s)$,*
1) *if the target response is uniformly bounded, then the closed-loop solution is uniformly bounded;*
2) *there exists a nonlinear gain $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the plant's state and output $x(\cdot), z(\cdot)$ satisfy*

$$\|(z - z_T)(\cdot)[t_s, \infty)\|_2 \leq \gamma(|x(t_s) - x_T(t_s)|). \qquad (7)$$

Remark 4 We give interpretations of the two conditions in Problem 1. Item 1 ensures that the controller states are uniformly bounded before the switching time. This is not trivial to obtain in general as, for example, arbitrary small disturbances could make the output of a disconnected PID controller diverge. Item 2 guarantees properties of the switching transient: given that the output mismatch at the switching time cannot be compensated for (because the controller has no authority over $z(t)$ until time t_s), the energy of the remaining part of the output mismatch is bounded by a (possibly nonlinear) gain of the state mismatch at the switching time. Within this setting minimizing the gain $\gamma(\cdot)$ leads to improved

output responses so that this framework is suitable for the design of optimal bumpless transfer devices. ◦

We also will address bumpless transfer in conjunction with the anti-windup problem. When dealing with anti-windup and bumpless transfer problems, suitable implementations of the anti-windup solutions of Teel and Kapoor (1997) (for the continuous-time case) and Grimm et al. (2003) (for the discrete-time case) solve both the anti-windup problem therein defined and the bumpless transfer problem defined here. When formalizing the design goal of Anti-Windup Bumpless Transfer (AWBT) compensation, we need to account for the intrinsic limitations arising from the presence of the saturation nonlinearity. To this aim, we need to restrict the attention to the subset of target responses that can be tracked through the saturated input from any initial conditions. In the following definition we enforce the intuitive requirement that a minimum amount of input authority is available to the controller in all the input directions. This condition is necessary, in general for the problem to be solvable (see, (Teel and Kapoor, 1997, Remark 2.2) for a counterexample).

Definition 2 A target response characterized by a plant input $u_T(\cdot)$ is *trackable* if there exists a (small enough) constant δ such that for all (measurable in continuous-time) $q(\cdot)$ satisfying $\|q(\cdot)\|_\infty < \delta$, the following holds: $\|\mathrm{sat}(u_T(\cdot) + q(\cdot)) - (u_T(\cdot) + q(\cdot))\|_2 < \infty$, where $\mathrm{sat}(\cdot)$ denotes the saturation nonlinearity. [2] ◦

Problem 2 *Given the plant-controller pair (1), (2) and a saturation function at the plant input, the* anti-windup bumpless transfer (AWBT) problem *is to design a control scheme such that the corresponding closed-loop system satisfies the anti-windup requirements* [3] *defined in Teel and Kapoor (1997) ($t \in \mathbb{R}$) and Grimm et al. (2003) ($t \in \mathbb{Z}$). Moreover, for any choice of $r(\cdot), d(\cdot)$ such that the corresponding target response $x_T(\cdot)$ exists, is unique and trackable (according to Definition 2), the bumpless transfer properties of Problem 1 are satisfied.*

[2] For simplicity, $\mathrm{sat}(\cdot)$ may be considered here as a decentralized symmetric saturation. However, the results in this paper apply to a more general class of input nonlinearities (see, (Teel and Kapoor, 1997, Assumption 2) for details).

[3] To keep the focus of this paper on the bumpless transfer problem, we are not treating the anti-windup problem formally here. When dealing with a saturated plant and an original controller that performs desirably without input constraints, "anti-windup" is augmentation to the saturated closed-loop aimed at 1) preserving the local unconstrained response whenever the saturation remains inactive; 2) recovering as much as possible that ideal response in all other cases. Since the bumpless transfer solutions proposed here share the same structure as the anti-windup solutions in Teel and Kapoor (1997); Grimm et al. (2003), they can be used both for bumpless transfer and for anti-windup purposes.

3 Problem solution

To provide a solution to the bumpless transfer problem, we will use the following filter:

$$\begin{aligned} \Delta x_e &= A x_e + B(u - y_c) \\ v_1 &= f_e(x_e, u - y_c), \quad v_2 = -(C x_e + D(u - y_c)) \end{aligned} \tag{8}$$

where the function $f_e(\cdot,\cdot)$ is still to be defined. The proposed solution, [4] represented in Figure 1 corresponds to the interconnection of the plant (1), the controller (2) and the filter (8) through the following equations:

$$\begin{aligned} u(t) &= \begin{cases} \hat{u}(t), & t < t_s, \\ y_c(t) + v_1(t), & t \geq t_s, \end{cases} \\ u_c(t) &= y(t) + v_2(t), \; t \in (-\infty, +\infty), \end{aligned} \tag{9}$$

Note that, differently from the bumpy interconnection (5), we are defining inputs and solutions of the controller (2) (and also of the filter (8)) at all times $t \in (-\infty, +\infty)$. Therefore, we do not need to specify an initial condition for the controller and filter state variables because the corresponding transient response is disregarded (see the following Section 3.1 for comments on how this translates into a practical implementation framework).

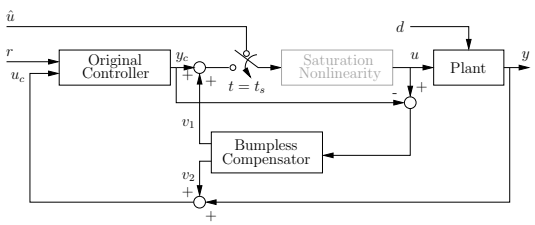

Fig. 1. The proposed bumpless transfer scheme.

The degrees of freedom available in the choice of the signal v_1 (through the selection of the function f_e) permits achieving the bumpless transfer goals. In particular, when only addressing the bumpless transfer problem (namely, Problem 1), the following choice can be made:

$$f_e(x_e, u - y_c) := K x_e, \tag{10}$$

where the matrix K is a stabilizing gain. A suitable choice is $K := XQ^{-1}$, where the matrices X and Q sat-

[4] In Figure 1 the saturation nonlinearity is depicted in gray because input saturation is not considered in the BT problem, whereas it is considered in the AWBT problem.

isfy the following LMI constraints for continuous time:

$$\begin{bmatrix} QA^T + AQ + BX + X^TB^T & \star \\ C_zQ + D_zX & -I \end{bmatrix} < 0 \tag{11a}$$

$$\begin{bmatrix} \alpha I & I \\ I & Q \end{bmatrix} > 0, \tag{11b}$$

and for discrete time:

$$\begin{bmatrix} -Q & \star & \star \\ AQ + BX & -Q & \star \\ C_zQ + D_zX & 0 & -I \end{bmatrix} < 0 \tag{12a}$$

$$\begin{bmatrix} \alpha I & I \\ I & Q \end{bmatrix} > 0\ . \tag{12b}$$

Theorem 1 *Given any plant-controller pair (1), (2) that satisfies Assumption 1, a solution to Problem 1 always exist and is given by the closed-loop system arising from the interconnection (9) with the filter (8) and the choice (10), where $K := XQ^{-1}$ and X, Q satisfy (11) for the continuous-time case (respectively, (12) for the discrete-time case). Moreover, the corresponding gain $\gamma(\cdot)$ is a linear gain $\gamma(s) = \overline{\gamma}s$ where $\overline{\gamma} = \sqrt{\alpha}$.*

Remark 5 Note that the value of the linear gain $\overline{\gamma}$ established in Theorem 1 can be minimized (thus providing a compensation matrix K inducing the minimum finite gain achievable by the architecture (10)) by solving the following LMI eigenvalue problem for the continuous-time case $\min_{X,Q,\alpha} \alpha$, subject to (11a), (11b), and the following one for the discrete-time case $\min_{X,Q,\alpha} \alpha$, subject to (12a), (12b). The corresponding solution should lead to an optimized output behavior. Note however that, based on the following proof of Theorem 1, the choice of the gain K in (10) could be carried out as any stabilizing state feedback gain for the plant (1) through the input u, thus arbitrarily imposing a linear behavior to the authority transfer. In general, however, the arising bound on the performance measure $\gamma(\cdot)$ would not be as small. ∘

Proof. To prove the theorem, we separate the time in two intervals: the first one before the switching time t_s and the second one after the switching time. Consider the first time interval (namely, $t < t_s$) and define $\bar{x} = -x_e$. The closed-loop system can be seen as the cascaded interconnection between the following two subsystems:

$$\begin{cases} \Delta x = Ax + B\hat{u} + B_d d \\ y = Cx + D\hat{u} + D_d d \end{cases} \tag{13a}$$

$$\begin{cases} \Delta\bar{x} = A\bar{x} + B(y_c - \hat{u}) \\ v_2 = C\bar{x} + D(y_c - \hat{u}) \\ \Delta x_c = f(x_c, v_2 + y, r) \\ y_c = g(x_c, v_2 + y, r). \end{cases} \tag{13b}$$

The second subsystem (13b) corresponds to the target closed-loop system with the interconnection (3) perturbed by the two external inputs $w_1 = -\hat{u}$ and $w_2 = y$, both generated by the first subsystem (13a). Since $x(\cdot)$ and $\hat{u}$ are uniformly bounded by assumption in the interval $(-\infty, t_s)$, then the inputs w_1, w_2 to the perturbed target closed-loop are uniformly bounded too and the uniform boundedness of the target trajectory together with the BIBD property of Assumption 1 guarantee that the solution $(x(\cdot), \bar{x}(\cdot), x_c(\cdot)) := (x(\cdot), -x_e(\cdot), x_c(\cdot))$ is uniformly bounded in the interval $(-\infty, t_s)$, thus proving item 1 in this time interval.

To prove item 1 on the remaining time interval and to prove item 2, consider the change of coordinates $(x_T, x_c, x_e) := (x - x_e, x_c, x_e)$ and notice that for all times, the closed-loop corresponds to the following cascaded interconnection (where we use $y_T := y + v_2$ and we introduce the new output z_T):

$$\begin{cases} \Delta x_T = Ax_T + By_c + B_d d \\ y_T = Cx_T + Dy_c + D_d d \\ z_T := C_z x_T + D_z y_c + D_{dz} d \\ \Delta x_c = f(x_c, y_T, r) \\ y_c = g(x_c, y_T, r) \end{cases} \tag{14a}$$

$$\begin{cases} \Delta x_e = Ax_e + B(u - y_c) \\ v_1 = Kx_e, \end{cases} \tag{14b}$$

Since the first subsystem (14a) coincides with the target closed-loop system, then the corresponding solution $x_T(t) = x(t) - x_e(t)$ coincides with the (unique) target response at all times (regardless of what the switching time is). Consider now the second subsystem and notice that by definition, at the switching time $t = t_s$, we have $x_e(t_s) = x(t_s) - x_T(t_s)$. Moreover, by (9), for all $t \geq t_s$, equation (14b) corresponds to the simplified dynamics

$$\Delta x_e(t) = (A + BK)x_e(t) \tag{15a}$$

which can be augmented with an output equation arising from substituting z and z_T into $z_e(t) := z(t) - z_T(t)$:

$$z_e(t) = (C_z + D_zK)x_e(t) \tag{15b}$$

The uniform boundedness proof at item 1 on the time interval $[t_s, +\infty)$ follows from the cascade structure (14) together with the uniform boundedness of the target response and the Hurwitz (respectively, discrete-time Hurwitz) property of the matrix $A+BK$ (which directly follows from the LMI (11a) (respectively, (12a) for $t \in \mathbb{Z}$).

Finally, the proof of item 2 is completed by the following claim, which is a classical result of linear LMI-based control design (see, e.g., (Boyd et al., 1994, §7.4.1) for a statement of the continuous-time result). •

Claim 1 *Given any solution X, Q, α to the LMI constraints (11) (respectively, (12)) and the continuous-time system (respectively, discrete-time system) (15), where $K = XQ^{-1}$, starting from the initial condition $x_e(t_s) = x_{e0}$, then $\|z_e(\cdot)[t_s, \infty)\|_2 \leq \sqrt{\alpha}|x_{e0}|$.*

For the AWBT problem stated in Problem 2 special attention needs to be devoted to the fact that the presence of saturation at the plant's input imposes severe structural limitations to the controller's authority. As it is well-known, stabilization of linear plants with input limitations can be achieved globally only if the plant does not have any exponentially unstable modes. In the opposite case, the best that can be done is restricted to operate locally within a subset of the state space strictly contained in a bounded null controllability region. A further complication for the discussion of anti-windup bumpless transfer schemes resides in the fact that plants containing poles on the stability boundary (i.e., on the imaginary axis for $t \in \mathbb{R}$ and on the unit circle for $t \in \mathbb{Z}$) cannot be exponentially stabilized by bounded inputs. As far as our problem is concerned, one important limitation of these plants is that global external stabilization can only be achieved by obtaining nonlinear gains and often nonlinear stabilizing laws are needed to achieve global stability properties. When dealing with anti-windup bumpless transfer schemes, the interconnection (5) should be replaced by the following one, incorporating the saturation effects:

$$\begin{aligned} u(t) &= \begin{cases} \mathrm{sat}(\hat{u}(t)), & t < t_s, \\ \mathrm{sat}(y_c(t) + v_1(t)), & t \geq t_s, \end{cases} \\ u_c(t) &= y(t) + v_2(t), \; t \in (-\infty, +\infty), \end{aligned} \tag{16}$$

where $\mathrm{sat}(\cdot)$ is the input saturation. However, in the AWBT case, the choice (10) for the function $f_e(\cdot,\cdot)$ no longer guarantees the global properties established in Theorem 1 (it actually only guarantees them in a local way) and alternative choices of $f_e(\cdot,\cdot)$ should be sought for to solve Problem 2. These choices should be carried out, e.g., following the guidelines in Teel and Kapoor (1997); Zaccarian and Teel (2002) for the continuous-time case and the guidelines in Grimm et al. (2003) for the discrete-time case. Overall, the amount of notation and tools required to recall these choices of $f_e(\cdot,\cdot)$ is too large to allow for a thorough discussion of the topic. Therefore, we refer the reader to Teel and Kapoor (1997); Grimm et al. (2003) for the details of the corresponding constructions and concentrate here on the bumpless transfer properties of the arising schemes, giving a sketch of the generalization of the proofs in Teel and Kapoor (1997); Grimm et al. (2003) to the AWBT case.

Theorem 2 *Given any plant-controller pair (1), (2) that satisfy Assumption 1,*
1) *if the plant is exponentially stable (namely its poles are strictly within the stability region), then a solution to the Problem 2 exists and guarantees a linear gain $\gamma(s) = \overline{\gamma}s$.*
2) *if the plant is not exponentially unstable (namely, its poles are within the stability region including its boundary), then a solution to the Problem 2 exists and guarantees a nonlinear gain $\gamma(s)$.*
3) *if the plant is exponentially unstable, then a (global) solution to the problem does not exist.*

Sketch of the proof. We only provide here a sketch of the proof which is constructive. All the solutions proposed correspond to the interconnection (16) between the filter (8) and the plant-controller pair (1), (2) using different selection strategies for the compensation function $f_e(\cdot,\cdot)$ in (8). Consider first item 3. The insolubility of the problem for cases where the plant is exponentially unstable directly follows from the fact that the null controllability region (namely, the subset of the state space from which there exists an input selection driving the state to zero) is bounded. Therefore, the signal $\hat{u}(\cdot)$ can drive the plant state outside the null controllability region before the switching time $t = t_s$, causing exponentially diverging responses (see also the following Remark 6). As for item 1, linear constructions for the function $f_e(\cdot,\cdot)$ are given for the continuous-time case in Teel and Kapoor (1997); Zaccarian and Teel (2002) (respectively, for the discrete-time case in Grimm et al. (2003)) that guarantee a finite $\mathcal{L}_2$ (respectively, l_2) bound on the state x_e based on the excess of saturation. Items 1 and 2 of Problem 1 can then be proven following similar steps to those carried out in the proof of Theorem 1 (in particular, Claim 1 should be replaced by an equivalent statement obtained from the finite gain bounds written with the initial conditions). Similarly, item 2 of the theorem follows by employing nonlinear constructions for the functions $f_e(\cdot,\cdot)$. In the continuous-time case these constructions are commented in Teel and Kapoor (1997). In the discrete-time case they are commented in Grimm et al. (2003). Note that these constructions allow to obtain nonlinear gains (linear gains cannot be achieved in this case). •

Remark 6 Theorem 2 does not given any constructive approach to address the AWBT problem when the plant contains exponentially unstable modes. It is however possible also in this case to rely on recent developments on anti-windup design for exponentially unstable plants Galeani et al. (2004) to guarantee useful bumpless transfer properties of the closed loop. However, in this exponentially unstable case, Theorem 2 establishes that the result can only be local, therefore the plant state should be constrained to operate within a suitable subset $\mathcal{U}$ of the (bounded) null-controllability region. To this aim, it will be necessary to impose a further constraint on the allowable manual inputs $\hat{u}(\cdot)$ before the switching time requiring that the corresponding state response guar-

antees $x(t_s) \in \mathcal{U}$, thus indirectly guaranteeing that the plant state can be controlled to zero after the switching time. Moreover, the performance bound (7) will need to be generalized to account for the requirement that the plant state remains within the region $\mathcal{U}$ for all times greater than the switching time t_s. $\circ$

3.1 Implementation issues

When implementing the proposed, ideal bumpless transfer solution, the controller-filter pair (2), (8) starts running at some time $t_\circ \in (-\infty, t_s]$. When $t_0 \ll t_s$, it is a reasonable approximation to assume that the controller-filter pair has been active for all times $t \leq t_s$. When the switching time t_s is relatively close to the power-on time t_0 of the control system, the following two strategies can be useful to speed-up the transient controller response thereby making it appear to the plant that $t_0 \ll t_s$:
1. the controller state at the initial time t_0 may be initialized with a value which matches (in an averaged sense) the current plant state. Note that no requirement is imposed on this initial state, although a good guess may reduce the transient response;
2. in cases where the plant is operating around the steady state and the disturbances and the references have a sufficiently uniform frequency content (or are constant), the controller and filter dynamics may be run at a faster rate during the interval before the switching time t_s. This strategy will accordingly speed up the controller transient and prepare the controller for the switch in a faster way. The correct timing of both controller and filter could then be reestablished at the switching time t_s.

Once the controller (2) has been activated with its filter (8) and the transient has expired, the control scheme may be switched on and off multiple times. After each switch on time, bumpless transfer will occur, based on the information stored into the state x_e of the filter. This feature may be useful in a multicontroller scheme where certain task oriented controllers are required to operate during some control phases. The bumpless transfer scheme would guarantee a smooth authority transfer among the different controllers. (in this setting, the filter (8) should be replicated for each controller involved in the multicontroller scheme). This approach could not guarantee stability for arbitrary switching, unless the selection of v_1 is carried out so to guarantee closed-loop properties, such as absolute stability (using, e.g., the techniques in Zaccarian and Teel (2002)). Another setting where this scheme might be useful is when certain controllers need to be tested in the feedback loop. In that case, a security transfer to a previously tested controller could be commanded at any time guaranteeing the bumpless transfer feature (see also Zaccarian and Teel (2002) for some developments in this direction).

When there is some flexibility in the switching time t_s, note that the term on the right hand side of (7), which determines performance, is known during the control operation. Indeed, due to the equality $x_e(t) = x(t) - x_T(t)$ which holds for all times, the size of the state of the filter (8) determines the size of the transient to be expected at the plant performance output after the switch has happened. It is then reasonable to monitor the norm of $x_e(\cdot)$ to select the switching instant as an instant when this size is small, subject also to the constraint that $t_0 \ll t_s$. The corresponding performance output will almost instantaneously follow the target output response. Ideally, if $t_0 = -\infty$ and one could find a time where $x_e(T) = 0$, then the switching transient would be completely removed by the bumpless transfer scheme. See the bold solid curve in the simulation results of the next section. It is worth noticing that, if this strategy was possible for all switches in a multicontroller scheme, then the plant response would be uniformly bounded regardless of the switching sequence, as it would always coincide with the target response associated with the active controller, which is uniformly bounded by assumption.

4 Simulation example

We now illustrate the effectiveness of the proposed bumpless transfer strategy. The reader is also referred to Zaccarian et al. (2004) where a more sophisticated case study consisting in the dynamics of an open water channel has been studied, both in discrete and continuous time. Those results show excellent performance of the bumpless transfer strategy. We select a simple marginally stable SISO system characterized by an input saturation level of 0.25 and by the state space matrices: $A = \left[\begin{smallmatrix} -500 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \end{smallmatrix}\right]$, and design two controllers. The first controller is an LQG controller which enforces a slow closed-loop behavior and is used at startup. The second controller is more aggressive and is designed with internal models aimed at guaranteeing zero steady-state error for step references and rejecting sinusoidal disturbances with unitary frequency. To suitably stabilize the closed-loop system the second controller is completed by designing an LQG stabilizer for the plant augmented with the internal models. To guarantee bumpless transfer, we augment the closed-loop with the filter (8) and select the function $f_e(\cdot,\cdot)$ according to the design procedure suggested in Teel and Kapoor (1997) for marginally stable plants, namely $v_1 = Kx_e = -B^T P x_e$, where $P = P^T > 0$ solves $A^T P + PA \leq 0$. The resulting gain is $K = \left[\begin{smallmatrix} 0.002 & -100 \end{smallmatrix}\right]$. Note that this scheme will work for any (even nonlinear) controller designed for this plant.

We simulate plant startup where the first controller is used and then authority is switched to the second controller to guarantee asymptotic reference tracking and disturbance rejection. The reference is a unit step and the disturbance is a sine wave of amplitude 0.2 affecting the plant input. Figure 2 gives three responses. The dotted line in the upper plot represents the reference

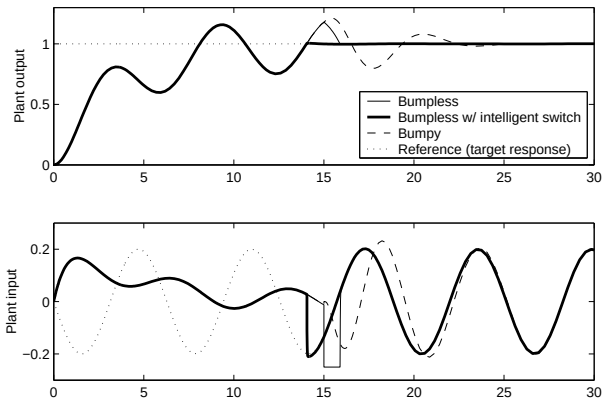

Fig. 2. Responses of the authority transfer for the example.

signal, which in this case is coincident with the target output response. The target input response corresponds to the dotted line in the lower plot. For the thin solid and dashed curves the switching time is fixed at $t_s = 15$ and it is evident from the upper plot that the bumpless scheme rapidly drives the plant response to the target one (the input authority is fully exploited by the scheme because in the lower plot, the plant input is kept into saturation until the transfer error becomes small enough). The bold solid curve represents the results arising from applying the idea in the last paragraph of the previous section. Namely, the switching time is commanded upon detection of a very small norm of the state x_e. In particular slightly after time $t = 14$, this state is almost zero and the bumpless switch happens without any transient.

5 Conclusions

We have proposed a novel definition of bumpless transfer that directly focuses on the behavior of the plant output after the switching time. Unlike previous bumpless transfer definitions, this definition is a mathematically rigorous one which allows for a characterization of the bumpless transfer performance. Based on the anti-windup scheme of Teel and Kapoor (1997); Grimm et al. (2003), a constructive compensation scheme is also proposed for the case where the plant input is unconstrained and for the case where it is subject to input saturation. In the latter case, the construction also guarantees the anti-windup goal. The results are given both in the continuous and discrete time. Simulation results confirm the effectiveness of the proposed approach.

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