Nonlinear static state feedback for saturated linear plants via a polynomial approach

G. Valmorbida, L. Zaccarian, S. Tarbouriech, I. Queinnec and A. Papachristodoulou

***Abstract*—The paper revisits the local exponential stabilization and global asymptotic stabilization problems of saturated linear systems using nonlinear control laws. The proposed nonlinear control law has rational dependence on a parameter σ, which is computed by solving an implicit equation depending on the state. Constructive solutions are obtained, based on a sum-of-squares formulation of the proposed conditions.**

I. Introduction

Saturation mechanisms are present in almost all interconnected practical systems. They remain a major challenge for the control designer as they may cause instability or loss in performance. Usually local design methods are considered rather than global approaches as they allow larger control gains and reduced convergence time. Even in this case, performance in terms of response time remains limited when constant control gains are computed. It is then preferable to compute solutions where control gains are smaller when the trajectory is far from the origin and become larger when the trajectory approaches it. Key results in the 80's had also proven that global asymptotic (exponential) stability is not achievable by saturated static feedback for plants with exponentially unstable dynamics [18], [17], [10]. The even former work of Fuller had established that no saturated linear controller could globally asymptotically stabilize the triple integrator [3]. All these aspects explain the importance of proposing nonlinear control strategies when dealing with saturated linear plants.

High performance, static, state feedback stabilization of saturated systems, that is, a selection of nonlinear control gains that induce a desirable high convergence rate, have been thoroughly investigated since the early '90s, when the nested saturations [20], [22] and scheduled Riccati [13] techniques were introduced. In particular, much attention has been devoted to the problem of stabilizing a chain of integrators with bounded feedback. This particular class belongs to the very peculiar set of plants called ANCBI (asymptotically null-controllable with bounded inputs), which possess poles on the imaginary axis but no poles in the right half plane. In particular, the double integrator with input saturation continues to attract a strong interest for research due to the fact that it characterizes the main dynamics of several practical systems. This interest especially arises in the global context [2], [26] where convergence rate performance is a key aspect.

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The aim of this paper is to propose a nonlinear, static, state feedback design for input-saturated linear plants with a guaranteed closed-loop stability region and a guaranteed convergence rate that may possibly deteriorate as the norm of the initial condition becomes larger. This degradation of the guaranteed convergence rate is reasonable since input saturation limits uniformly the globally achievable rate. It is necessary for ANCBI plants whose origin can be (uniformly) globally asymptotically stabilized (GAS) but cannot be (uniformly) globally exponentially stabilized (GES). Then different exponential bounds must be considered because no uniform global exponential bound can be obtained. In the sequel we will talk about GAS and GES without specifically emphasizing uniformity in our acronym, even though it should be kept in mind that this is a central aspect behind the standard definition of GAS and GES.

Our approach is inspired by some of the results presented in [4] and [5], where similar scheduled state feedback laws were proposed. The nonlinear controllers designed in this paper arise from rational polynomial matrices in a parameter σ, associated to an implicit equation. Such a polynomial approach to design a nonlinear controller had already been exploited in [27] for the problem of semi global stabilization of exponentially unstable linear systems subject to actuator saturation, in which the low gain parameter represented the convergence rate of the closed-loop system. A parameterized Algebraic Ricatti Equation (ARE) approach had also been used in [6] and extended to Linear Matrix Inequalities (LMI) conditions in [8], to allow one to take into account uncertainty in the model. A valid alternative to our construction is the recent work [12], however only global solutions for non-exponentially unstable plants are given there, whereas here we also address local solution for any type of linear plant. Moreover, our method is radically different from the one of [12] since we rely on polynomial constraints solvable by way of convex sum of squares (SOS) techniques, while [12] relies on the solution of a partial differential matrix inequality. Another strategy, originally formulated for the regulation problem [1], [11], parametrizes the gain on the state with the use of a scalar, bounded nonlinear function.

The paper is organized as follows. Section II gives our main local and global design solutions. Section III contains a theoretical example, while Section IV presents SOS numerical implementations and their illustrations on examples. All the proofs and necessary lemmas are presented is the Appendix.

Notation. For $x \in \mathbb{R}^m$ the ring of polynomials in x is denoted $\mathcal{R}[x]$, the ring of polynomial matrices of dimensions $n \times m$ is denoted $\mathcal{R}^{n\times m}[x]$, the ring of diagonal polynomial matrices of dimension n, $\mathcal{R}^{n\times n}_{diag}[x]$. The set of sum-of-squares polynomials on variable x, $\{p(x) \in \mathcal{R}[x] | p(x) = \sum_{i=1}^{k} g_i^2(x), g_i(x) \in \mathcal{R}[x]\}$ is denoted $\Sigma[x]$. The set of sum-of-squares matrices of dimension n is denoted $\Sigma^{n\times n}[x]$ and the set of diagonal sum-of-squares matrices of dimension n, $\Sigma^{n\times n}_{diag}[x]$. The sublevel set α^{-1} of a quadratic function $x^T M x$, where M is positive definite, $\{x | x^T M x \leq \alpha^{-1}\}$, is denoted $\mathcal{E}(M, \alpha^{-1})$. The identity matrix is denoted by I. The Hermitian operator on square matrices $\mathrm{He}(\cdot)$ is defined as $\mathrm{He}(X) := X + X^T$.

II. Saturated state feedback design

In this paper we address the problem of designing a nonlinear state feedback for the following general linear plant with input saturation

$$\dot{x} = Ax + B\mathrm{sat}(u) \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\mathrm{sat}(u)$ is the standard decentralized saturation function with unitary saturation limits[1].

Given plant (1), our aim is to design a high performance nonlinear feedback stabilizer

$$u = \kappa(x), \tag{2}$$

with a guaranteed stability region and convergence rate.

The following two theorems establish the desirable properties of the proposed construction. The first one addresses the local stabilization problem with guaranteed region of attraction and guaranteed convergence rate. The second one extends the first result to the global case. Notice that the stability conclusion is different for the two cases. Indeed, while the first statement deals with bounded stability regions (or estimates of the basin of attraction) from which uniform exponential stability is easily established, the second theorem establishes global properties that require to relax uniform exponential convergence to uniform asymptotic convergence. The proofs of both theorems are given in the Appendix.

Theorem 1: Given plant (1) and two scalars $\underline{\sigma}$, $\overline{\sigma}$, $0 < \underline{\sigma} < \overline{\sigma}$, assume that there exist a symmetric positive definite matrix $W(\sigma) \in \mathbb{R}^{n\times n}$, a diagonal positive definite matrix $T(\sigma) \in \mathbb{R}^{m\times m}$, two matrices $R(\sigma) \in \mathbb{R}^{m\times n}$ and $G(\sigma) \in \mathbb{R}^{m\times n}$, and a positive scalar α such that the following inequalities hold for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$:

$$N(\sigma) = \begin{bmatrix} \sigma AW(\sigma) + \sigma BR(\sigma) + W(\sigma) & -\sigma BT(\sigma) \\ \sigma R(\sigma) - \sigma G(\sigma) & -\sigma T(\sigma) \end{bmatrix}$$

$$-\mathrm{He}(N(\sigma)) \geq 0 \tag{3a}$$

$$W(\sigma) \geq I \tag{3b}$$

$$0 < \frac{dW(\sigma)}{d\sigma} \leq \alpha W(\sigma) \tag{3c}$$

$$\begin{bmatrix} W(\sigma) & G_i^T(\sigma) \\ G_i(\sigma) & \alpha \end{bmatrix} \geq 0,\ i = 1, \ldots, m, \tag{3d}$$

where G_i denotes the ith row of matrix G. Then, denoting $\overline{P} := W^{-1}(\overline{\sigma})$ and $\underline{P} := W^{-1}(\underline{\sigma})$, the following holds:

1. Well posedness. For each $x \in \mathcal{E}(\overline{P}, \alpha^{-1}) \setminus \mathcal{E}(\underline{P}, \alpha^{-1})$, the implicit equation:

$$\varphi(x, \sigma) := x^T W^{-1}(\sigma) x - \alpha^{-1} = 0 \tag{4}$$

has a unique solution $\sigma = \sigma_\varphi(x)$, and the function

$$\sigma^*(x) := \begin{cases} \sigma_\varphi(x), & \text{if } x^T \underline{P} x \geq \alpha^{-1}, \\ \underline{\sigma}, & \text{if } x^T \underline{P} x \leq \alpha^{-1} \end{cases} \tag{5}$$

is Lipschitz for all $x \in \mathcal{E}(\overline{P}, \alpha^{-1})$.

2. Exponential Stability. The static nonlinear state feedback

$$\kappa(x) = K(x)x := R(\sigma^*(x))W^{-1}(\sigma^*(x))x \tag{6}$$

exponentially stabilizes the origin of (1), (2) with basin of attraction containing the set $\mathcal{E}(\overline{P}, \alpha^{-1})$.

3. Exponential performance. There exists $M > 0$ such that for each $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, all solutions to the nonlinear closed loop (1)-(6) satisfy

$$x(t_0) \in \mathcal{E}(W^{-1}(\sigma), \alpha^{-1}) \Rightarrow |x(t)| \leq Me^{-\frac{t-t_0}{\sigma}}|x(t_0)|. \tag{7}$$

Theorem 1 establishes local results with guaranteed region of attraction and convergence rate, once a suitable bounded set $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ of feasibility of the (infinite dimensional) inequalities (3) is found. The following theorem shows that it is possible to make these local results global by simply ensuring feasibility of these inequalities in an unbounded set $\sigma \in [\underline{\sigma}, +\infty)$.

Theorem 2: Given plant (1) and a scalar $\underline{\sigma} > 0$, assume that inequalities (3) hold for all $\sigma > \underline{\sigma}$. Then the implicit equation (4) has a unique solution $\sigma = \sigma_\varphi(x)$ for all $x \notin \mathcal{E}(\underline{P}, \alpha^{-1})$ and $\sigma^*(x)$ in (5) is locally Lipschitz. Moreover, the feedback law (6) induces GAS and LES of the origin of (1)-(2), (5)-(6). Finally, for each $\sigma \geq \underline{\sigma}$, there exists M_σ such that all solutions to the nonlinear closed loop (1)-(2), (5)-(6) satisfy (7) with $M = M_\sigma$.

Remark 1: Note that due to the well-posedness result provided in Theorems 1 and 2, the established (local or global) asymptotic stability is robust to sufficiently small perturbations, either due to noise or to uncertainty affecting the system model. This fact follows from the intrinsic robustness established in [7, Thm 7.21] for compact attractors (which in our case is only one point, namely the origin). ⋆

III. An illustrative example

The inequalities in (3) are difficult to check in general. In this section we illustrate how these can be solved for a scalar open-loop unstable plant with input saturation, highlighting the potential of Theorem 1. Consider the scalar system

$$\dot{x} = \eta x + \mathrm{sat}(u), \tag{8}$$

where $\eta > 0$. System (8) corresponds to (1) with $A = \eta$, $B = 1$, and it is an example of system that can be only locally stabilized from u since its open-loop trajectories are unstable.

Below we show that a particular choice of the elements W, R, G, T α, $\underline{\sigma}$ and $\overline{\sigma}$ in (3) satisfies the conditions of Theorem 1, hence leading to a locally stabilizing gain. Consider the following choice:

$$\begin{aligned} &Q(\sigma) = \tfrac{\sqrt{2}}{2}(1 - 4\eta) + 2\sqrt{2}\eta^2\sigma \\ &R(\sigma) = -kQ(\sigma),\ G(\sigma) = -\epsilon kQ(\sigma),\ \alpha = 8\eta^2 \\ &T(\sigma) = k(1-\epsilon)Q(\sigma),\ W(\sigma) = \alpha^{-1}(\epsilon k)^2 Q^2(\sigma) \end{aligned} \tag{9}$$

with $\epsilon \in (0, 1)$ and $k = 4\eta\epsilon^{-1}$. We verify next that relations (3a)-(3d) hold with $\underline{\sigma} = \eta^{-1}$ and $\overline{\sigma} = \underline{\sigma} + \frac{\sqrt{2}-1}{4}\eta^{-2}$.

Relation (3b): since Q is strictly increasing one gets $W(\sigma) \geq W(\underline{\sigma}) = \frac{(\epsilon k)^2}{\alpha}Q^2(\underline{\sigma}) = 2\left(\frac{\sqrt{2}}{2}\right)^2 = 1$. Hence (3b) holds.

[1] In the paper we select unit (vector) saturations to keep the discussion simple. However the results can be readily extended to the case of symmetric saturations and, with some level of conservativeness also to non-symmetric saturations as long as the unsaturated region has nonempty interior including the origin.

Relation (3c): one gets $\frac{dW(\sigma)}{d\sigma} = 2\alpha^{-1}(\epsilon k)^2 Q(\sigma)\frac{dQ(\sigma)}{d\sigma}$. Since $\alpha^{-1}(\epsilon k)^2 Q(\sigma) > 0$ for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, inequality $\frac{dW(\sigma)}{d\sigma} \leq \alpha W(\sigma)$ is equivalent to inequality $2\frac{dQ(\sigma)}{d\sigma} \leq \alpha Q(\sigma)$ on the inteval $[\underline{\sigma}, \overline{\sigma}]$. Since $\alpha Q(\sigma) \geq \alpha Q(\underline{\sigma}) = 4\sqrt{2}\eta^2 = 2\frac{d}{d\sigma}Q(\sigma)$ the inequality is satisfied and (3c) holds.

Relation (3d): the Schur complement of the matrix is given by $W(\sigma) - \frac{G^2(\sigma)}{\alpha}$, which, from the definition of W and G, is identically zero.

Relation (3a): we need to prove $-\mathrm{He}(N(\sigma)) \geq 0$, that is

$$2\sigma Q(\sigma)\begin{bmatrix} k - \left(\eta + \frac{1}{\sigma}\right)\frac{(\epsilon k)^2}{\alpha}Q(\sigma) & k(1-\epsilon) \\ k(1-\epsilon) & k(1-\epsilon) \end{bmatrix} \geq 0$$

which is true if $(1-\epsilon) > 0$ (always true since $\epsilon \in (0,1)$), and the determinant is positive, that is if $\left(\eta + \frac{1}{\sigma}\right)\frac{(\epsilon k)}{\alpha}Q(\sigma) < 1$. This last inequality holds for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. Indeed for all such values of σ we have

$$\left(\eta + \tfrac{1}{\sigma}\right)\tfrac{(\epsilon k)}{\alpha}Q(\sigma) < \left(\eta + \tfrac{1}{\underline{\sigma}}\right)\tfrac{(\epsilon k)}{\alpha}Q(\overline{\sigma}) = 1.$$

From our choice, the implicit equation (4) yields $W^{-1}(\sigma(x)) = \frac{\alpha^{-1}}{x^2}$ which implies that $Q(\sigma(x)) = \frac{\alpha}{\epsilon k}|x|$ and therefore $R(\sigma(x)) = -\frac{\alpha}{\epsilon}|x|$. Using (6) the state feedback is explicitly written as

$$u(x) := \begin{cases} -\frac{1}{\epsilon}\frac{1}{|x|}x & \text{if} \quad x^T W(\sigma)^{-1}x > \alpha^{-1} \\ -\frac{2\sqrt{2}}{\epsilon}\eta x & \text{if} \quad x^T W(\underline{\sigma})^{-1}x \leq \alpha^{-1} \end{cases}, \tag{10}$$

and stabilizes the origin of system (8) for all $x \in \mathcal{E}(\overline{P}, \alpha^{-1})$.

To obtain a solution in the case of the simple system (8), we have used the potential of the generalized sector condition allowing for $R(\sigma) \neq G(\sigma)$ [21, Lemma 1.6]. Moreover, the second relation in (10) yields $u(x) = -\frac{2\sqrt{2}}{\epsilon}\eta x$ and the unsaturated closed loop satisfies $\dot{x} = -\frac{(2\sqrt{2}-\epsilon)}{\epsilon}\eta x$, therefore yielding local exponential stability of the origin (recall that $\epsilon \in (0,1)$).

Note that for the very simple example chosen here, the solution to relation (3) is polynomial in the parameter σ. It is, therefore, natural to investigate polynomial solutions for more general cases and to provide constructive solutions by formulating SOS programs, as addressed in Section IV.

IV. Convex optimization formulation

This section presents a formulation of an optimization problem in order to solve inequalities by considering functions $W(\sigma)$, $T(\sigma)$, $R(\sigma)$ and $G(\sigma)$ that are polynomials in the variable σ. The solution to this problem relies on a line search for the positive scalar α that, when fixed, allows for a solution to a feasibility problem with sum-of-squares constraints. The assumption of polynomial dependence on parameter σ allows us to prove the nonconservativeness of a sum-of-squares formulation through an application of the Positivstellensatz, reported in Appendix B.

Proposition 1: Assume that $W(\sigma) \in \mathcal{R}^{n\times n}[\sigma]$, $R(\sigma) \in \mathcal{R}^{m\times n}[\sigma]$, $G(\sigma) \in \mathcal{R}^{m\times n}[\sigma]$ and $T(\sigma) \in \mathcal{R}^{m\times m}_{diag}[\sigma]$. Then inequalities (3) hold in the set

$$\mathcal{D} := [\underline{\sigma}, \overline{\sigma}] = \{\sigma | g(\sigma) := -(\sigma - \underline{\sigma})(\sigma - \overline{\sigma}) \geq 0\} \tag{11}$$

with $\overline{\sigma} > \underline{\sigma} > 0$ if and only if there exist polynomial matrices $M_1(\sigma) \in \Sigma^{(n+m)\times(n+m)}[\sigma]$, $M_2(\sigma) \in \Sigma^{n\times n}[\sigma]$, $M_{3j}(\sigma) \in \Sigma^{n\times n}[\sigma]$, $j = 1, 2$, $M_{4i}(\sigma) \in \Sigma^{(n+1)\times(n+1)}_{diag}[\sigma]$, $i = 1, \ldots, m$, such that for some $\epsilon > 0$ we have $(T(\sigma) - \epsilon I) - g(\sigma)M_0(\sigma) \in \Sigma^{n\times n}[\sigma]$, and

$$-\mathrm{He}(N(\sigma)) - g(\sigma)M_1(\sigma) \in \Sigma^{(n+m)\times(n+m)}[\sigma] \tag{12a}$$

$$(W(\sigma) - I) - g(\sigma)M_2(\sigma) \in \Sigma^{n\times n}[\sigma] \tag{12b}$$

$$\frac{\partial W(\sigma)}{\partial \sigma} - g(\sigma)M_{31}(\sigma) \in \Sigma^{n\times n}[\sigma] \tag{12c}$$

$$\left(\alpha W(\sigma) - \frac{\partial W(\sigma)}{\partial \sigma}\right) - g(\sigma)M_{32}(\sigma) \in \Sigma^{n\times n}[\sigma] \tag{12d}$$

$$\begin{bmatrix} W(\sigma) & G_i^T(\sigma) \\ G_i(\sigma) & \alpha \end{bmatrix} - g(\sigma)M_{4i}(\sigma) \in \Sigma^{(n+1)\times(n+1)}[\sigma], i = 1, \ldots, m. \tag{12e}$$

Proof: We will use Lemma 2 (given in the Appendix) in order to relate (12) to (3). Notice that the hypothesis of Lemma 2 holds since the quadratic module $\mathcal{M}(g(\sigma))$ generated by $g(\sigma)$ is Archimedean. Indeed, consider any pair (r, N^*), $r \in \mathbb{R}_{>0}$ and $N^* \in \mathbb{N}$, satisfying

$$N^* \geq \frac{1}{4}\frac{r^2}{r-1}(\underline{\sigma} + \overline{\sigma})^2 - \underline{\sigma}\overline{\sigma}.$$

The Archimedean property is then satisfied with $\theta_0(\sigma) = \left(\left(\sqrt{r-1}\right)\sigma - \frac{1}{2}\frac{r}{\sqrt{r-1}}(\underline{\sigma} + \overline{\sigma})\right)^2 + \underline{\sigma}\overline{\sigma} + N^* - \frac{1}{4}\frac{r^2}{(r-1)}(\underline{\sigma} + \overline{\sigma})^2$, $\theta_1(\sigma) = r$. Consider (3a), if $-\mathrm{He}(N(\sigma)) \geq 0$, $\forall \sigma \in \mathcal{D}$ then according to Lemma 2 there exist $M_{10}(\sigma), M_1(\sigma) \in \Sigma^{(n+m)\times(n+m)}[\sigma]$ such that $-\mathrm{He}(N(\sigma)) = M_{10}(\sigma) + g(\sigma)M_1(\sigma)$, yielding (12a). Lemma 2 is applied in a similar fashion to obtain (12b)-(12e). ■

We rely on the following proposition to provide a numerical procedure to solve the inequalities in Theorem 2, related to the global case.

Proposition 2: Assume that $W(\sigma) \in \mathcal{R}^{n\times n}[\sigma]$, $R(\sigma) \in \mathcal{R}^{m\times n}[\sigma]$, $G(\sigma) \in \mathcal{R}^{m\times n}[\sigma]$ and $T(\sigma) \in \mathcal{R}^{m\times m}_{diag}[\sigma]$, then inequalities (3) hold in the set

$$\mathcal{D}_o := [\underline{\sigma}, +\infty) = \{\sigma | g(\sigma) := (\sigma - \underline{\sigma}) \geq 0\} \tag{13}$$

with $\underline{\sigma} > 0$ if there exist polynomial matrices $M_1(\sigma) \in \Sigma^{(n+m)\times(n+m)}[\sigma]$, $M_2(\sigma) \in \Sigma^{n\times n}[\sigma]$, $M_{3j}(\sigma) \in \Sigma^{n\times n}[\sigma]$, $j = 1, 2$, $M_{4i}(\sigma) \in \Sigma^{(n+1)\times(n+1)}_{diag}[\sigma]$, $i = 1, \ldots, m$, such that (12) holds.

Proof: Since the set $\mathcal{D}_o$ is not compact (hence, not possibly Archimedean [14]), we can not apply Lemma 2 to prove necessity of existence of SOS multipliers $M_j(\sigma)$ in (12). However, sufficiency is immediate since $g(\sigma) \geq 0$, $\forall \sigma \in \mathcal{D}_o$, thus for $M_1(\sigma) \in \Sigma[\sigma]$ one verifies $-\mathrm{He}(N(\sigma)) - g(\sigma)M_1(\sigma) > 0 \Rightarrow -\mathrm{He}(N(\sigma)) > 0$ $\forall \sigma \in \mathcal{D}_o$. ■

Although Lemma 2 gives necessary and sufficient conditions for inequalities (3) (with polynomial data) to hold, in order to make these conditions computationally tractable the degree of the sum-of-squares polynomials $M_i(\sigma)$ in (12) must be fixed, hence yielding only sufficient conditions for (3) for a given degree of M_i, on variable σ. Notice that the conditions for local exponential stability or global asymptotic stability in Propositions 1 and 2 are based on the same set of inequalities (12) although with different choices of polynomial

$g(\sigma)$ (given by (11) or (13) for the local and the global case respectively).

The computation of polynomial matrices $W(\sigma)$, $R(\sigma)$, $G(\sigma)$, $T(\sigma)$ that solve (12) for fixed values of α can be formulated as a convex feasibility problem, since an optimization problem with SOS constraints is a semi-definite program [16] for which efficient and reliable solvers are available [19].

According to item 2 of Theorem 1, to enlarge the basin of attraction for the local case and to impose a bound on the Lispschitz constant for the derivative of W with respect to α (according to (3c)) we can solve the optimization problem

$$\text{minimize } \alpha \quad \text{subject to (12)} \tag{14}$$

Note that (14) is non convex due to the product between W and α in (12d) but α could be found by bi-section as it enters in a quasi-convex way, just as in a maximum decay generalized eigenvalue problem.

V. Numerical Examples

Example 1: Consider the unstable system (8) discussed in Section III with $\eta = 1$. We solve (12) with $g(\sigma)$ as in (11), $\underline{\sigma} = 1.6$, $\overline{\sigma} = 20$, and $\alpha = 10$, by setting $W(\sigma)$ and $R(\sigma)$ to be polynomials of degree 4. The nonlinear control gain corresponding to these values is $K(x) = \frac{n(\sigma^*(x))}{d(\sigma^*(x))}$ where $x \mapsto \sigma^*$ is defined in (4)-(5) and

$$n(\sigma) = -0.9662\sigma^4 + 0.5832\sigma^3 + 0.02296\sigma^2 + 0.2977\sigma + 0.125,$$

$$d(\sigma) = 0.0006743\sigma^3 - 0.05204\sigma^2 + 1.0196\sigma - 0.3741.$$

Figure 1 shows trajectories starting from $x(0) = 0.89$ (on the boundary of the set $\mathcal{E}(\overline{P}, \alpha^{-1})$). The solution to (4)-(5) reaches the lower bound $\sigma(x(t^*)) = \underline{\sigma} = 1.6$ at $t^* = 1.8$, establishing the maximal *local* exponential convergence rate, thus freezing the gain $K(\sigma(x(t))) = K(\underline{\sigma})$ for all $t \geq t^* = 1.8$. To illustrate Remark 1, the closed loop is affected by measurement noise. Interestingly, this does not affect $\text{sat}(u)$ in the initial time due to clipping effects of saturation, nor it affects σ^* in the tail, when the state enters the set $\mathcal{E}(\underline{P}, \alpha^{-1})$. ◦

Example 2: This example illustrates an ANCBI plant with an underdamped mode and a pair of poles at the origin. It corresponds to a seek control model of a hard-disk drive as presented in [9] and studied in [12]

$$P_f(s) = K_p \left(\frac{1}{s^2} - \frac{\kappa}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \tag{15}$$

with $K_p = 3.74 \times 10^9 [trkA^{-1}s^{-2}]$, $\kappa = 0.7$, $\zeta = 0.31$ and $\omega_n = 2\pi(4.1 \times 10^3)[rad\ s^{-1}]$, which, with a saturating input having saturation levels magnitude of 0.5, has the following state-space representation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_n \\ 0 & 0 & -\omega_n & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\kappa \end{bmatrix} K_p(\tfrac{1}{2} sat(u)) \\ y &= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} x. \end{aligned} \tag{16}$$

We obtain $\alpha = 1.2$ by solving Problem (14), where the parameters in (11) are selected as $[\underline{\sigma}, \overline{\sigma}] = [2, 2000]$. We compute trajectories from the initial condition $x(0) = x_0 = [\ 1000\ 0\ 0\ 0\ ]^T$ (selected to satisfy $\sigma(x_0) \leq 2000$). The time histories for the state variables and the control are depicted in Figure 2. Figure 3 shows comparative evolutions of y, starting from the same x_0, obtained with the proposed nonlinear control law (blue solid curve), with the linear control defined by the constant gain $K(\overline{\sigma})$ (blue dashed curve), and with the nonlinear control solutions proposed in [1], [11] (black dotted curve) and in [12] (red solid curve). We have followed the indications of [11, Section 4], where some free parameters are chosen by trial and error and with the goal of improving the speed of convergence for this particular initial condition. The linear control obtained with $K(\overline{\sigma})$ (blue dash-dotted curve) also guarantees convergence to the origin from this initial condition (dashed curve) but the nonlinear gain (blue solid curve) results in a faster convergence rate, which is expected when comparing to the linear solution. Notably, the convergence time is essentially the same as the excellent performance of [12] (red dash-dotted curve). While matching

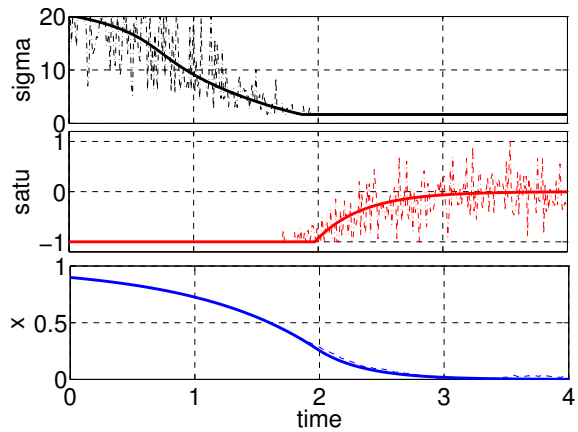

Fig. 1. Example 1 - The solid lines represent the time-evolution of $\sigma^*(x(t))$ (top), trajectory of the state for initial condition $x_0 = 0.89$ (bottom), with the nonlinear gain obtained by solving Problem (14) with $g(x)$ as in (11). The dotted lines in the figure present the same closed loop with a band limited white noise w (of power 1×10^{-4} and sample time $T = 0.01s$) affecting the feedback (10) as $u(x + w)$, illustrating the robustness properties mentioned in Remark 1.

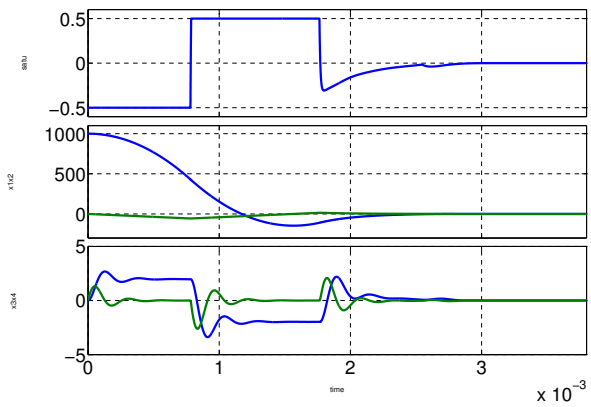

Fig. 2. Example 2 - Trajectories of the input and the state for initial condition $x_0 = [\ 1000\ 0\ 0\ 0\ ]^T$ with the nonlinear gain obtained by solving Problem (14) with $g(x)$ as in (11). From $t^* = 2.56 \times 10^{-3}s$, $\sigma^*(x(t^*)) = \underline{\sigma}$ and the gain is constant and given by $K(\underline{\sigma})$.

that extreme performance on this ANCBI plant, we emphasize that our solution is (more broadly) applicable to any linear saturated plant. ◦

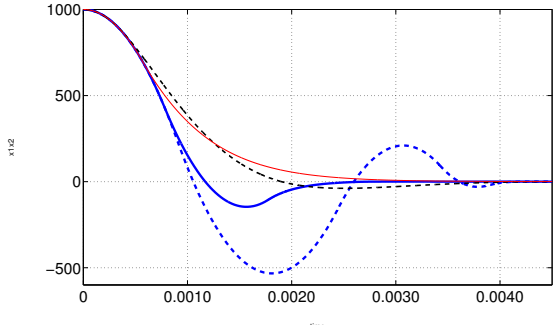

Fig. 3. Example 2 - Output y for the closed loop between (16) and the nonlinear gain $K(\sigma^*(x(t)))$ (blue solid) compared with the output response with the linear control given by the constant gain $K(\overline{\sigma})$ (blue dashed), and with the nonlinear control strategies of [1] (black dash-dotted) and [12] (red solid).

For a globally stabilizing nonlinear control law of the form (6) with $W(\sigma) \in \mathcal{R}[\sigma]$ and $R(\sigma) \in \mathcal{R}[\sigma]$, refer to [24] where we have obtained solutions with set as (13) for the single and the double integrator and a system with a pair of imaginary eigenvalues. The solution to (12) of this section were obtained with SOSTOOLS [15] and SeDuMi [19].

VI. Conclusion

The local exponential stabilization and global asymptotic stabilization problems have been addressed for linear plants with saturating inputs. In the local case, the proposed control law has a nonlinear structure associated with a guaranteed stability region and a guaranteed convergence rate. In the global case, with the same controller structure, only asymptotic stability may be achieved. The proposed method combines parametric quadratic Lyapunov functions and a generalized sector condition. The nonlinear static state-feedback control law depends on a parameter σ, solution to an implicit equation. Constructive solutions to the control law were then proposed by considering polynomial dependence of the matrices on the parameter σ and SOS relaxations for designing the controller.

Appendix

A. Proofs of the Theorems

The proof of Theorem 1 uses the following result that can be found as a special case of [4, Thm 2] (see also the construction in [25, §3.5] and [21, Ch 3]).

Lemma 1: Assume that, for a given scalar $\lambda > 0$, the following holds for suitable matrices A, B, $\bar{K}$, $\bar{H}$, $\bar{P} = \bar{P}^T > 0$, and $\bar{U} > 0$ diagonal:

$$\mathrm{He}\left(\begin{bmatrix} \bar{P}A + \bar{P}\bar{B}\bar{K} + \lambda\bar{P} & \bar{P}B \\ U\bar{K} - U\bar{H} & -\bar{U} \end{bmatrix}\right) \leq 0, \tag{17}$$

$$\begin{bmatrix} \bar{P} & \bar{H}_i^T \\ \bar{H}_i & \alpha \end{bmatrix} \geq 0,\ i = 1, \ldots, m, \tag{18}$$

then, for all $x \in \mathcal{E}(\bar{P}, \alpha^{-1})$, the following holds:

$$\langle 2\bar{P}x, Ax + B\mathrm{sat}(\bar{K}x)\rangle \leq -2\lambda x^T\bar{P}x. \tag{19}$$

Proof of Theorem 1. Well posedness of the algebraic loop (4) follows from the implicit function theorem and the fact that by (3c), one has $\dfrac{\partial W(\sigma)}{\partial \sigma} > 0$ for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. Indeed, for each $x \neq 0$, we have, from (4), $\frac{\partial}{\partial\sigma}\varphi(x,\sigma) = -x^TW^{-1}(\sigma)\frac{\partial W(\sigma)}{\partial\sigma}W^{-1}(\sigma)x$, which implies $\frac{\partial}{\partial\sigma}\varphi(x,\sigma) < 0$ for all x satisfying $x^TW^{-1}(\overline{\sigma})x \leq \alpha^{-1}$. This is a sufficient condition for the implicit function theorem to apply and prove the existence of a unique Lipschitz solution to the implicit equation for all $x \in \mathcal{E}(\overline{P}, \alpha^{-1}) \setminus \mathcal{E}(\underline{P}, \alpha^{-1})$ (which does not include the origin). The fact that σ^* in (5) is Lipschitz directly follows from its continuity at the patching surface, which is easily established by noticing that the solultion to $\varphi(x,\sigma) = 0$ is well defined also at the patching surface where $x^T\underline{P}x = \alpha^{-1}$ and corresponds to $\sigma_\varphi = \underline{\sigma}$.

Consider now the following Lyapunov function candidate:

$$V(x) = \underbrace{x^TW^{-1}(\sigma^*(x))x}_{:=V_{loc}(x)} + \underbrace{\sigma^*(x) - \underline{\sigma}}_{:=V_{glob}(x)} \tag{20}$$

and note that for all $x \in \mathcal{E}(\overline{P}, \alpha^{-1})$ we have $V(x) = 0$ if and only if $x = 0$ and $V(x) > 0$ otherwise. Moreover, since σ^* is a Lipschitz function, function V is differentiable everywhere except for the set where $x^TW^{-1}(\underline{\sigma})x = \alpha^{-1}$. To prove exponential stability we may apply the Lyapunov theorem in [23, page 99] which establishes that the Lyapunov decrease condition must be verified only in the set where function V is differentiable, that is for x satisfying $x^TW^{-1}(\underline{\sigma})x < \alpha^{-1}$ and for x satisfying $x^TW^{-1}(\underline{\sigma})x > \alpha^{-1}$.

To analyse the derivative of V along the solutions to (1), (2), we first multiply inequality (3a) by σ^{-1} and we apply the congruence transformation $\begin{bmatrix} P(\sigma) & 0 \\ 0 & U(\sigma) \end{bmatrix} := \begin{bmatrix} W^{-1}(\sigma) & 0 \\ 0 & T^{-1}(\sigma) \end{bmatrix}$, which is well defined because, from (3b) and by the assumed positive definiteness of T, $W(\sigma)$ and $T(\sigma)$ are nonsingular for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. Then we obtain the following inequality:

$$\mathrm{He}\left(\begin{bmatrix} P(\sigma)A + P(\sigma)BK(\sigma) + \sigma^{-1}P(\sigma) & -P(\sigma)B \\ U(\sigma)K(\sigma) - U(\sigma)H(\sigma) & -U(\sigma) \end{bmatrix}\right) \leq 0, \tag{21}$$

where $K(\sigma)$ is defined in (6) and $H(\sigma) = G(\sigma)W(\sigma)^{-1}$. Now we can split the analysis in two subcases.

Case 1. If x satisfies $x^TW^{-1}(\underline{\sigma})x < \alpha^{-1}$, so that from (5) one has $\sigma^*(x) = \underline{\sigma}$, we may apply Lemma 1 with $\lambda = \underline{\sigma}^{-1}$, $\bar{P} = P(\underline{\sigma}) = W^{-1}(\underline{\sigma})$, $\bar{K} = K(\underline{\sigma})$, $\bar{U} = U(\underline{\sigma})$, and $\bar{H} = H(\underline{\sigma})$, to get

$$\begin{array}{rcl} \dot{V}_{loc}(x) & \leq & -2\underline{\sigma}^{-1}x^TP(\underline{\sigma})x < 0 \\ \dot{V}_{glob}(x) & = & \frac{d(\sigma^*(x))}{dt} = 0, \end{array} \tag{22}$$

where the second inequality follows from the fact that $\sigma^*(x)$ is constant along trajectories for any x in the open set where $x^TW^{-1}(\underline{\sigma})x < \alpha^{-1}$.

Case 2. If x satisfies $x^TW^{-1}(\underline{\sigma})x > \alpha^{-1}$, so that from (5) with $\bar{P} = P(\underline{\sigma}) = W^{-1}(\underline{\sigma})$ one may apply (4) to get $V_{loc}(x) = x^TW^{-1}(\sigma^*(x))x = x^TW^{-1}(\sigma_\varphi(x))x = \alpha^{-1}$, first notice that

$$\begin{array}{rl} 0 & = \frac{d}{dt}\left(x^TW^{-1}(\sigma^*(x))x - \alpha^{-1}\right) \\ & = \langle 2W^{-1}(\sigma^*(x))x, \dot{x}\rangle + x^T\frac{dW^{-1}(\sigma^*(x))}{d\sigma^*}x\frac{d(\sigma^*(x))}{dt}. \end{array} \tag{23}$$

Consider now the right bound in equation (3c), multiply both sides by $W^{-1}(\sigma^*(x))$ and rearrange to get $\alpha W^{-1}(\sigma^*(x)) \geq W^{-1}(\sigma^*(x))\frac{dW(\sigma^*(x))}{d\sigma^*}W^{-1}(\sigma^*(x)) = -\frac{dW^{-1}(\sigma^*(x))}{d\sigma^*}$ Then, equation (23) implies:

$$\begin{aligned}\frac{d\left(\sigma^*(x)\right)}{dt} &= -\left(x^T\frac{dW^{-1}(\sigma^*(x))}{d\sigma^*}x\right)^{-1}\langle 2W^{-1}(\sigma^*(x))x, \dot{x}\rangle \\ &\leq \underbrace{\left(\alpha x^T W^{-1}(\sigma^*(x))x\right)^{-1}}_{=1}\langle 2W^{-1}(\sigma^*(x))x, \dot{x}\rangle \\ &\leq -\sigma^*(x)^{-1}x^T P(\sigma^*(x))x, \end{aligned} \tag{24}$$

where we applied again Lemma 1 with $\lambda = \sigma^*(x)^{-1}$, $\bar{P} = P(\sigma^*(x))$, $\bar{K} = K(\sigma^*(x))$, $\bar{U} = U(\sigma^*(x))$, and $\bar{H} = H(\sigma^*(x))$. Then, we have by using also (24):

$$\begin{aligned}\dot{V}_{loc}(x) &= 0 \\ \dot{V}_{glob}(x) &\leq -\sigma^*(x)^{-1}V_{loc}(x).\end{aligned} \tag{25}$$

Combining equations (22) and (25) of cases 1 and 2 above

$$\dot{V}(x) \leq -\sigma^*(x)^{-1}V_{loc}(x), \quad \forall x \in \mathcal{E}(\overline{P}, \alpha^{-1}). \tag{26}$$

Let us now prove item 3 of the theorem. Since the set $[\underline{\sigma}, \overline{\sigma}]$ is compact and bounded away from zero, we may find two nondecreasing functions c_1 and c_2 such that for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$,

$$x^T P(\sigma)x + \sigma - \underline{\sigma} \leq c_2(\sigma)|x|^2 \qquad c_1(\sigma)|x|^2 \leq x^T P(\sigma)x$$

Then it follows that the function in (20) satisfies $c_1(\underline{\sigma})|x|^2 \leq V(x) \leq c_2(\overline{\sigma})|x|^2$, $\forall x \in \mathcal{E}(\overline{P}, \alpha^{-1})$. That can be used in combination with (26) and with the property that (from (24)), $\sigma^*(x)$ is nonincreasing along solutions, to prove the exponential bound (7) from standard Lyapunov arguments, thus proving item 3. As a consequence, item 2 easily follows from the exponential bound computed in item 3 for the maximum value of $\sigma = \overline{\sigma}$. ■

Proof of Theorem 2. The proof follows the steps of the proof of Theorem 1 except for item 2 that establishes asymptotic stability rather than exponential stability. Indeed, the estimate of the region of attraction is not compact in the global case and no uniform exponential bound is guaranteed. ■

B. Positivstellensatz

For a set of polynomials $\bar{g} = \{g_1(x), \ldots, g_m(x)\}$, $m \in \mathbb{N}$, the *quadratic module* generated by $\bar{g}$ is $\mathcal{M}(\bar{g}) := \{\theta_0 + \sum_{i=1}^m \theta_i g_i | \theta_i \in \Sigma[x]\}$. A quadratic module $\mathcal{M} \in \mathcal{R}[x]$ is said *Archimedean* if $\exists N \in \mathbb{N}$ such that $N - \|x\|_2^2 \in \mathcal{M}(\bar{g})$. An Archimedian set is always compact [14]. We then state

Lemma 2 (Putinar Positivstellensatz): Suppose $\mathcal{M}(\bar{g})$ is Archimedian. Then for every $f \in \mathcal{R}[x]$, $f > 0$, $\forall\ x \in \{x | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \Rightarrow f \in \mathcal{M}(\bar{g})$.

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