Model recovery anti-windup for output saturated SISO linear closed loops

M. Sassano[a,*], L. Zaccarian[b,c,]

[a]Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133 Roma, Italy

[b]CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France and Univ de Toulouse, LAAS, F-31400 Toulouse, France

[c]Dipartimento di Ingegneria Industriale, University of Trento, Italy

Abstract

We define model recovery anti-windup for SISO linear control systems with output saturation. We address the problem by relying on a hybrid modification of the linear closed loop which employs a suitable logic variable to activate/deactivate various components of a control scheme. The scheme relies on a finite-time observation law, an open-loop observer and an open-loop input generator which is capable of driving the plant output within the saturation limits. Then the control scheme is based on suitable (hybrid) resetting laws allowing the controller to operate on the artificial output signal generated by the open-loop observer when the actual plant output is outside the saturation limits. Unlike existing results, not only we prove uniform global asymptotic stability of the closed loop, but we also prove the local preservation and global recovery properties, typical of model recovery anti-windup paradigms. We also illustrate the proposed technique on an example study.

Keywords: Output Saturation, Hybrid Systems, Anti-Windup Scheme

1. Introduction

In practical applications the input and/or output signals generated by the controller or by the plant itself are subject to a saturation bound. Input saturation, in particular, has received increasing attention in the past few decades and is often tackled by employing the so-called *anti-windup* techniques [13], whose main objective consists in the design of *auxiliary* compensators to be combined with standard controllers designed by disregarding input saturation. Anti-windup for input-saturated linear systems can be divided in two big families (see [3]): one called Direct Linear Anti-Windup (DLAW) which mainly focuses on proving asymptotic stability of the origin (possibly global or local with guaranteed regions of attraction), the other one called Model Recovery Anti-Windup (MRAW) focusing

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[*]Corresponding Author

Email addresses: mario.sassano@uniroma2.it (M. Sassano), zaccarian@laas.fr (L. Zaccarian)

on recovering the so-called "unconstrained response" in the presence of non-zero inputs, namely the response that one would have experienced in the absence of saturation. Both approaches guarantee the so-called *local preservation* property, that is the unconstrained linear response is left unchanged by the compensation scheme as long as it never exceeds the saturation limits.

The parallel problem of *sensor or output saturation* has not been addressed as extensively as the input saturation problem, even though it is an important problem in several applications (see, e.g., [9] where it is related to Wiener-type nonlinear models or the edited book [12]). Despite its reduced popularity, some relevant works can be found in the literature and are very clearly surveyed in the excellent introduction of [10]. While apparently similar, the two anti-windup problems are radically different as output saturation poses a nontrivial observability problem, which is nonexistent in the input saturation case. Conversely certain global stabilizability issues appearing in the input saturation case (see, e.g., [11]) are absent in the output saturation problem, where global stabilization of the origin can be achieved in SISO [8] and MIMO [7] output saturated linear plants. Finally, the same problem is approached in [2] for a, possibly unstable, planar system in the presence of quantized and saturated input and output. Anti-windup design for output saturated systems is somewhat more challenging than mere stabilization because one has also to take into account the above mentioned local preservation constraint induced by a pre-existing controller. So far, all existing anti-windup methods only focus on asymptotic stability and follow the DLAW paradigm. In particular, the schemes in [14] provide global results for exponentially stable MIMO plants (see [14, Remark 4]), while the work in [4] addresses local results for any type of MIMO plant. Perhaps the most powerful existing approach is that in [10] (see also [1] where a similar construction is given without semiglobal results), where semiglobal results are obtained for any MIMO controllable and detectable plant (see [10, Prop. 2]). However, it is admitted in [10, §IV.B] that large domains of attraction lead to unacceptably large controller gains.

In this paper we adopt a different paradigm closer in spirit to the MRAW solution to input saturated anti-windup design. First of all, as compared to the works above, we establish global asymptotic stability results of our scheme without needing increasingly large gains. More importantly, beyond stability, we establish a suitable unconstrained response recovery property that has never been previously considered in this field. This property enables us to track references that cause the plant output to spend most of the time outside the saturation limits, as long as they come back within the limits once in a while (this is made more precise in Section 2). The price to be paid for this important extra feature is the complexity of the scheme, which is presented for SISO output saturated systems. Indeed, for effective unconstrained response recovery, suitable switching logics need to activate when the plant output exceeds the saturation limits in such a way that the controller is well behaved in spite of output saturation. We describe these switching logics using the hybrid systems formalism in [5]. We give formal proofs of the main results and we propose a modified scheme for which we can prove Uniform Global Asymptotic Stability (UGAS) of a compact attractor by relying on a proposition pertaining general hybrid systems, which may be of separate interest and is stated in our appendix.

The rest of the paper is organized as follows. Section 2 formalizes the design goal, together with some basic notation and preliminary results. The proposed control architecture is described in Section 3 by detailing its building blocks and their properties. The statement and the proof of the main result is provided in Section 4. The effectiveness of the control scheme is validated by means of a numerical example in Section 5, before drawing some conclusions and future outlook in Section 6. Finally, 6 contains a stability results for hybrid dynamical systems necessary for the proof of the main theorem and also of independent interest.

2. Problem statement

Consider a single-input single output linear plant $\mathcal{P}$ described by equations of the form

$$\dot{x} = Ax + Bu + B_d d, \qquad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^{n_p}$ denotes the plant state, $u \in \mathbb{R}$ denotes its control input, $y \in \mathbb{R}$ denotes its measurement output and $d \in \mathbb{R}^p$ is a vector disturbance. In the paper, we denote by $\alpha \in \mathbb{R}$ the maximum real part among all the eigenvalues of matrix A. Assume that the following controller $\mathcal{C}$ has been designed for plant (1)

$$\dot{x}_c = A_c x_c + B_c u_c + B_r r, \qquad y_c = C_c x_c + D_c u_c + D_r r, \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $r \in \mathbb{R}^q$ denotes a reference input and $u_c \in \mathbb{R}$ is the control input for the feedback interconnection. Following steps similar to the typical anti-windup approaches for the case of input saturation (see, e.g., [13]) we introduce two notable closed-loop systems. The first one is named *unconstrained closed-loop system*, corresponding to the interconnection of (1) and (2) via

$$u = y_c, \quad u_c = y. \tag{3}$$

The second one is named *constrained closed-loop system* and corresponds to recognizing that only a saturated version $\mathrm{sat}(y)$ of the actual output y is available for measurement, where $\mathrm{sat}(y) = \max(-1, \min(y, 1))$ is the scalar unit saturation function. This second closed loop corresponds to (1) and (2) interconnected via

$$u = y_c, \quad u_c = \mathrm{sat}(y). \tag{4}$$

For the unconstrained closed-loop system, we will require the following standing assumption.

Assumption 1. *The unconstrained closed-loop system (1), (2), (3) with zero input $(r, d) = (0, 0)$ is internally stable. Moreover, plant (1) is reachable and observable.*

Due to the presence of the output saturation nonlinearity in (4), the constrained closed-loop system may exhibit undesired responses including instability. In this paper we address the problem of designing an anti-windup compensation scheme aimed at recovering – in some sense to be specified – the desirable response produced by the *unconstrained closed-loop system* also in the case when output saturation is present. Before the formal definition of the problem, the following remark anticipates the main design objectives together with an overview of the structure of the proposed solution, depicted in Figure 1.

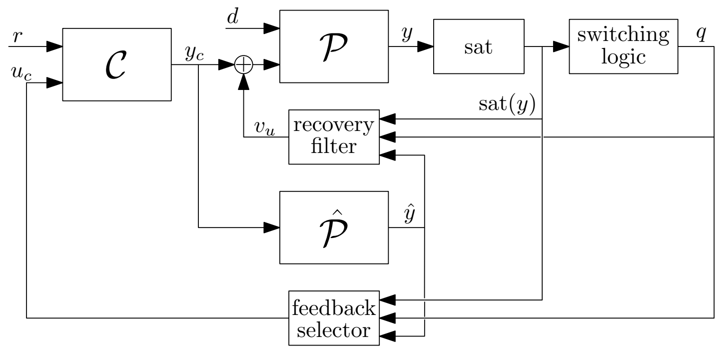

Figure 1: Block diagram of the proposed anti-windup architecture.

Remark 1. Together with the unavoidable objective of (asymptotic) stability of the resulting closed loop, two additional tasks will be considered. The *local preservation* property consists in ensuring that as long as the plant output remains within saturation levels, the anti-windup action does not inject any modification into the unconstrained closed-loop. Therefore, we must design a switching logic (see the scheme in Figure 1) that is capable of detecting such circumstance and of selecting the appropriate signal to be fed to (2). Instead, the second property requires that *asymptotic recovery* be guaranteed as long as the unconstrained response that we wish to recover persistently comes back *sufficiently close* to zero. This is achieved, whenever the behaviors of the anti-windup and of the unconstrained closed loops exhibit a significant mismatch - certified by the observer $\hat{\mathcal{P}}$ in Figure 1 - by activating a *recovery filter*, that steers the output of the plant within the saturation limits in finite time. ⌟

To this aim, we will propose an augmentation of the feedback interconnection (1), (2) capable of solving the following problem.

Goal 1. *Given a pair of positive (typically close to* 1*) scalars* $\delta_o \in (0,1)$ *and* $\delta_i \in (0, \delta_o)$*, given the plant-controller pair (1), (2) and the corresponding* unconstrained closed-loop system *(1), (2), (3), design a compensation scheme injecting a pair of signals* (v_u, v_y) *in the anti-windup interconnection:*

$$u = y_c + v_u, \quad u_c = \text{sat}(y) + v_y, \tag{5}$$

such that the corresponding anti-windup closed-loop system[1] *with (1), (2) and (5) guarantees:*

[1] The *anti-windup closed loop* may encompass additional dynamics (to be designed) required to generate v_u and v_y.

1. Local preservation*: if the initial conditions* $(x(0), x_c(0))$ *and the external inputs* (r, d) *generate a response of the unconstrained closed-loop system satisfying:*

$$|y(t)| < \delta_o, \quad \forall t \geq 0, \tag{6}$$

then the anti-windup closed-loop system generates the same response;

2. η-asymptotic recovery*: if the initial conditions* $(x(0), x_c(0))$ *and the external inputs* (r, d) *generate a response of the unconstrained closed-loop system satisfying:*

$$\begin{aligned} &\exists\ t_d \in \mathbb{R} \text{ such that } d(t) = 0,\ \forall t \geq t_d, \\ &\nexists\ t_y \in \mathbb{R} \text{ such that } |y(t)| \geq \delta_i - \eta,\ \forall t \geq t_y, \end{aligned} \tag{7}$$

where $0 < \eta < \delta_i$*, then the anti-windup closed-loop system generates a response converging to the unconstrained response as* $t \to \infty$*.*

3. Uniform global asymptotic stability*: for any initial condition, the anti-windup closed-loop system is uniformly globally asymptotically stable to a compact attractor* $\mathcal{A}$ *whose projection in the* (x_p, x_c) *directions of the closed-loop state corresponds to the origin.* [2]

Clearly, smaller values of η lead to larger sets of "η-recoverable" unconstrained trajectories. On the other side, small values of η make the scheme more sensitive to noise, in terms of activation of the recovery filter (see Section 3.3) and so the choice of η should be performed as a trade-off taking into account the type of trajectories that one expects to experience from the closed loop. Note that item 2 does not require that r becomes zero and this is a unique and novel feature of our approach (model recovery) as compared to existing works (stabilization only). In particular, as illustrated in our simulations of Section 5 our scheme is capable of tracking references that cause persistent activation of the output saturation nonlinearity, while all existing works only focus on solutions eventually reaching the linearity domain of the saturation (and eventually the origin).

3. Proposed architecture

In this section we propose an anti-windup architecture corresponding to the scheme represented in Figure 1, while the statement of the main result is postponed to Section 4. To suitably describe its dynamical behavior, we split the scheme in the following three main components, which are described in details in Sections 3.1, 3.2 and 3.3, respectively:

1. a *switching logic*, described by a hybrid system and a logic variable $q \in \{0, 1\}$, which governs the behavior of the anti-windup scheme by detecting saturation thereby enforcing jumps of certain

[2] In item 3 of Goal 1 we do not insist that the GAS equilibrium is the origin of the overall state space because the hybrid solution proposed next comprises logical variables whose value is irrelevant within the compact attractor $\mathcal{A}$.

states to suitably chosen reset values, in addition to selecting the appropriate feedback signal to be delivered to the controller input, corresponding to:

$$v_y = (1-q)(\hat{y} - \mathrm{sat}(y)), \tag{8}$$

which, using (5), gives $u_c = q\,\mathrm{sat}(y) + (1-q)\hat{y}$;

2. an *observer* $\hat{\mathcal{P}}$ providing the output estimate $\hat{y}$, based on the internal states z, ξ and on a timer τ; the observer acts as a finite-time convergent observer when the saturation detection signal is active (namely $q = 1$) and acts as an open-loop observer when the saturation detection signal is inactive (namely $q = 0$);
3. a *recovery filter* comprising an internal variable y_R, a logic variable q_R and another timer τ_R which is capable of driving the plant output y back into the region where it is not saturated; this filter is activated when the observer output and the plant output exhibit a significant mismatch.

The following subsections provide a detailed description of each of the above building blocks.

3.1. Switching logic

We adopt the hybrid dynamical systems framework of [5], together with the notation introduced therein (see also [6, §2.1] for a short notation overview), to suitably represent the switching phase between the different operation modes of the remaining two components above, namely the finite-time/open-loop observer and the recovery filter. The switching logic is in charge of the timing and reset values of the jumps induced on the hybrid system. On the other hand, the continuous-time evolutions of the observer and the recovery filter are described in the following sections.

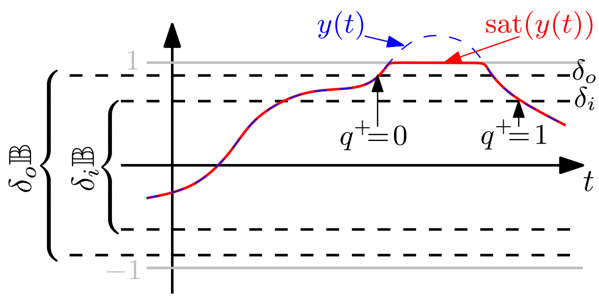

Figure 2: Graphical representation of the *switching logic* for the proposed anti-windup architecture for output saturated closed loops.

According to Figure 2, we use a logic variable q that is set to 0 when the plant output $\mathrm{sat}(y)$ exits the set $\delta_o\mathbb{B}$ (where "o" stands for "out") and toggles back to 1 when it enters the (smaller) set $\delta_i\mathbb{B}$ (where "i" stands for "in") with δ_i and δ_o being design parameters satisfying $0 < \delta_i < \delta_o < 1$, as specified in Goal 1.

Remark 2. As mentioned in the introduction, output saturation induces nontrivial observability issues within the control loop, because the actual plant output y is not accessible (things are different for input saturation anti-windup [3, 13] where both input u and output $\text{sat}(u)$ of the saturation block are accessible and one can implement the typical deadzone-driven or mismatch driven compensation relying on the signal $dz(u) = u - \text{sat}(u)$). The presence of suitable observers in our scheme, as well as in that of [14] and many others, is motivated by the need of suitably estimating the otherwise unaccessible signal y. In addition to this, it is emphasized that if $sat(y) = 1$, from the only knowledge of $\text{sat}(y)$ we cannot detect whether $y = 1$ or $y > 1$. Therefore we introduce the threshold δ_o, in such a way to ensure that $sat(y) = \delta_o$ implies $y = \delta_o$. Finally, the use of an inner threshold δ_i allows us to take suitable actions with our finite-time observer sufficiently before the output y hits the threshold δ_o. Note that this "anticipatory" structure of this compensation law may remind the reader about the recently proposed "anticipatory anti-windup" for input saturated plants [15]. However, the phylosophy behind our approach arises from radically different observability issues. $\lrcorner$

Using the hybrid notation of [5], the hysteretic switching logic for q illustrated in Figure 2 can be represented by defining the following closed subsets of $\{0, 1\} \times \mathbb{R} \times \mathbb{R}$:

$$
\begin{aligned}
\mathcal{D}_{q0} &:= \{0\} \times \delta_i\mathbb{B} \times [0, \tau_M] \\
\mathcal{D}_{q1} &:= \{1\} \times \overline{([-1, 1] \setminus \delta_o\mathbb{B})} \times [0, \tau_M] \\
\mathcal{D}_{\tau_M} &:= \{1\} \times \delta_o\mathbb{B} \times \{\tau_M\} \\
\mathcal{D} &:= \mathcal{D}_{q0} \bigcup \mathcal{D}_{q1} \bigcup \mathcal{D}_{\tau_M} \\
\mathcal{C} &:= \overline{((\{0, 1\} \times [-1, 1] \times [0, \tau_M]) \setminus \mathcal{D})}
\end{aligned}
\tag{9a}
$$

for a given timeout value $\tau_M > 0$, which is employed to *reset* a finite-time convergent observer every τ_M seconds to enforce an upper bound on the maximum finite-time observation interval (technically, this upper bound is necessary to ensure that the attractor introduced later in (26) be compact), and with the following discrete dynamics for q:

$$
(q, \text{sat}(y), \tau) \in \mathcal{D}_{q0}, \qquad \begin{cases} q^+ = 1,\ q_R^+ = 0 \\ z^+ = 0,\ \xi^+ = 0,\ y_R^+ = 0 \\ \tau^+ = 0,\ \tau_R^+ = 0 \end{cases} \tag{9b}
$$

$$
(q, \text{sat}(y), \tau) \in \mathcal{D}_{q1}, \qquad \begin{cases} q^+ = 0,\ q_R^+ = q_R \\ z^+ = \hat{x},\ \xi^+ = \xi,\ y_R^+ = y_R \\ \tau^+ = \tau,\ \tau_R^+ = \tau_R, \end{cases} \tag{9c}
$$

$$
(q, \text{sat}(y), \tau) \in \mathcal{D}_{\tau_M}, \qquad \begin{cases} q^+ = q,\ q_R^+ = q_R \\ z^+ = z,\ \xi^+ = v_\xi,\ y_R^+ = y_R \\ \tau^+ = \vartheta\tau_M,\ \tau_R^+ = \tau_R, \end{cases} \tag{9d}
$$

where $\vartheta \in (0, 1)$ may be chosen small to reduce the frequency of the timeout event and where v_ξ is to be specified below. Note that the sets $\mathcal{D}_{q0}$ and $\mathcal{D}_{q1}$ in (9b) and (9c) correspond to the circumstances in which the output y enters the region $\delta_i \mathbb{B}$ and leaves $\delta_o \mathbb{B}$, respectively. Set $\mathcal{D}_{\tau_M}$ in (9d) characterizes the *reset time instants* of the finite-time observer. The flow dynamics is $\dot{q} = 0$ and is enabled in the closure of the complement of the jump set specified by (9), namely:

$$(q, \mathrm{sat}(y), \tau) \in \mathcal{C}, \quad \dot{q} = 0. \tag{9e}$$

The logic variable q plays a fundamental role in the proposed anti-windup scheme because its dynamics (9) ensures that during flows $q = 1$ implies $y = \mathrm{sat}(y)$. In particular, an important property that one derives from the dynamics (9) and equations (5), (8) is that when the output is within the saturation limits we can write:

$$u_c = (1 - q)\hat{y} + qy, \tag{10}$$

which will turn out to be a useful equation when establishing desirable properties of the proposed scheme. Note that equations (9) comprise all the variables involved in the definition of the anti-windup closed-loop system. In addition from (9b), (9c) it is evident that a large number of state variables are reset to zero when q jumps to 1 (that is, the plant output is well defined within the non-saturated region) and are left unchanged when q jumps to 0. The precise meaning of this type of behavior is motivated and illustrated in the two following sections.

3.2. Finite-time/open-loop observer $\hat{\mathcal{P}}$

Focusing on the plant (1), in this section we introduce an observer of dimension $2n$ that provides a finite-time convergent estimate $\hat{x}$ of the state x when the output y is not saturated and $q = 1$, and an open-loop estimate when the output y is saturated. The latter estimate, however, is not affected by the action of the recovery filter. The variables z and ξ constitute the internal state of the observer, which is reset to zero whenever the output enters the region $\delta_i \mathbb{B}$. In particular, z represents the state of an open-loop observer, which is modified by an additional term depending on the auxiliary state ξ, to provide a finite-time convergent estimate of the state x of (1) whenever the output $\mathrm{sat}(y)$ is equal to y and with $q = 1$. Conversely, whenever $q = 0$, z acts as an open-loop observer initialized at the actual value of the state of (1), yielded by the finite-time convergent observer.

The flow dynamics of the proposed hybrid observer is given by

$$\dot{z} = Az + By_c, \quad \dot{\xi} = -A^\top \xi + C^\top (\mathrm{sat}(y) - Cz), \quad \dot{\tau} = q, \tag{11a}$$

which is enabled in the flow set of the overall anti-windup scheme. Conversely, the jump dynamics of the observer states, as mentioned above, is entirely governed by the switching logic introduced in Section 3.1. Moreover, the output $\hat{x}$ of the observer is selected, based on the matrix-valued function $Q : \mathbb{R} \to \mathbb{R}^{n \times n}$ and the scalar function $\beta : \mathbb{R} \to \mathbb{R}$ defined as

$$Q(\tau) \triangleq \int_0^\tau e^{A^\top(\sigma - \tau)} C^\top C e^{A(\sigma - \tau)} d\sigma \,, \tag{11b}$$

$$\beta(s) = \mathrm{dist}(s, [-1,1] \setminus \delta_o \mathbb{B}), \tag{11c}$$

as follows:

$$\hat{x} = q(Q(\tau) + \beta(\mathrm{sat}(y)) \cdot I)^{-1}\xi + z, \tag{11d}$$

$$\hat{y} = C\hat{x}. \tag{11e}$$

Remark 3. The matrix $Q(\tau)$ defined as in (11b) may be practically computed by adding an extra state Q, whose flow is governed by the standard Lyapunov differential equation $\dot{Q} = -A^\top Q - QA + C^\top C$ and whose jump equation is given by $Q^+ = 0$ to suitably initialize the solution. ⌟

The following result shows that (11) behaves, in the absence of disturbances, as a finite-time convergent observer whenever the output is inside the saturation limits.

Lemma 1. *(Finite-time convergent estimate). Given plant (1), the logic for q in (9) and the hybrid observer (11), consider any hybrid solution to (1), (2), (5), (8), (9), (10), (11), with $v_u = 0$ and*

$$v_\xi = Q(\vartheta\tau_M)Q(\tau_M)^{-1}\xi. \tag{12}$$

Denote its domain as $E = \bigcup_{j\in\mathbb{N}} I^j \times \{j\}$ (possibly with some I^j being empty for large enough j if the solution is eventually continuous), and assume that $\exists \underline{j} \geq 1$ such that $q(t_{\underline{j}}, \underline{j}) = 1$. Define $\bar{j} = \sup\{i \in \mathbb{N} : q(t,j) = 1, \forall t \in I^j, \forall \underline{j} \leq j \leq i\}$ and assume that $d(t,j) = 0$, for all $(t,j) \in E$ satisfying $\underline{j} \leq j \leq \bar{j}$. Define the function

$$\phi(t,j) \triangleq Q(\tau(t,j))\,(x(t,j) - z(t,j)) - \xi(t,j). \tag{13}$$

If $\phi(t_{\underline{j}}, \underline{j}) = 0$, then $\phi(t,j) = 0$, for all $(t,j) \in E$ such that $\underline{j} \leq j \leq \bar{j}$. Moreover, [3] *for all such (t,j),*

$$\hat{x}(t,j) - x(t,j) = \Big(Q(\tau(t,j)) + \beta(\mathrm{sat}(y(t,j)) \cdot I)\Big)^{-1} \beta(\mathrm{sat}(y(t,j)))\,(z(t,j) - x(t,j)). \tag{14}$$

If in addition $|\mathrm{sat}(y(t,j))| \geq \delta_o$, then $\beta(\mathrm{sat}(y(t,j))) = 0$ and, consequently, $\hat{x}(t,j) = x(t,j)$.

Proof. Consider any solution to (1), (2), (5), (8), (9), (10), (11), (12) with $v_u = 0$ whose domain is E and any $\underline{j}, \bar{j} \in \mathbb{N}$ satisfying the assumptions. Recalling that for each $(t,j) \in E$ such that $\underline{j} \leq j \leq \bar{j}$ we have $d(t,j) = 0$ by assumption, then we obtain from (1), the first two equations in (11a) and the flow dynamics of Q as in Remark 3,

$$\begin{aligned}
\dot{\phi}(t,j) &= \dot{Q}(\tau(t,j))(x(t,j) - z(t,j)) + Q(\tau(t,j))(\dot{x}(t,j) - \dot{z}(t,j)) - \dot{\xi}(t,j) \\
&= (-A^\top Q(\tau(t,j)) - Q(\tau(t,j))A + C^\top C)(x(t,j) - z(t,j)) \\
&\quad + Q(\tau(t,j))(Ax(t,j) - Az(t,j)) + A^\top \xi(t,j) - C^\top(y(t,j) - Cz(t,j)) \\
&= -A^\top \phi(t,j),
\end{aligned} \tag{15}$$

[3] Note that the matrix inverse in (14) always exists because Q is the (positive definite) controllability Gramian.

for all $(t,j) \in E$, $\underline{\jmath} \le j \le \bar{j}$. Consider now the assumption that $q(t,j)=1$ for all $(t,j) \in E$, $\underline{\jmath} \le j \le \bar{j}$, which implies can jumps can only happen from $\mathcal{D}_{\tau_M}$ in that interval and so, for all $j \in \{\underline{\jmath}, ..., \bar{j}-1\}$, we have $(q(t_j,j), \mathrm{sat}(y(t_j,j)), \tau(t_j,j)) \in \mathcal{D}_{\tau_M}$, which implies $\tau(t_j,j) = \tau_M$ that, combined with (12), leads to

$$\begin{aligned}\phi(t_j,j+1) &= Q(\tau(t_j,j+1))(x(t_j,j+1) - z(t_j,j+1)) - \xi(t_j,j+1) \\ &= Q(\vartheta\tau_M)(x(t_j,j) - z(t_j,j)) - Q(\vartheta\tau_M)Q(\tau_M)^{-1}\xi(t_j,j) = Q(\vartheta\tau_M)Q(\tau_M)^{-1}\phi(t_j,j)\,,\end{aligned} \tag{16}$$

for all $\underline{\jmath} \le j \le \bar{j}-1$. Thus, by (15) and (16), we have that $\phi(t_{\underline{\jmath}},\underline{\jmath}) = 0$ implies $\phi(t,j)=0$, $\forall (t,j) \in E$, $\underline{\jmath} \le j \le \bar{j}$. Note also that for all such (t,j) we have

$$Q(\tau(t,j))x(t,j) = Q(\tau(t,j))z(t,j) + \xi(t,j)\,. \tag{17}$$

By adding and subtracting the term $\beta(\mathrm{sat}(y(t,j)))(z(t,j) - x(t,j))$ to the right-hand side of equation (17), we obtain (we omit the dependence on (t,j) for simplicity):

$$(Q(\tau) + \beta(\mathrm{sat}(y)) \cdot I)\, x = (Q(\tau) + \beta(\mathrm{sat}(y)) \cdot I)\, z + \xi - \beta(\mathrm{sat}(y))(z-x)\,,$$

from which, recalling that $q(t,j)=1$ by assumption, equation (14) follows directly by definition of the estimate $\hat{x}$ in (11d). Finally, the last claim can be easily proved by noting that $\beta = 0$ whenever the output $\mathrm{sat}(y)$ is outside the set $\delta_o\mathbb{B}$ or on its boundary. □

In the following we let ϑ be such that $Q(\vartheta\tau_M)Q(\tau_M)^{-1}e^{-A^\top\tau_M}$ be a Schur matrix - which can be always achieved by choosing ϑ sufficiently small - since this selection is crucial in enforcing stability of the ξ-dynamics, as shown in Section 4. The behavior of the observer (11) when the output $\mathrm{sat}(y)$ is outside the region $\delta_o\mathbb{B}$ may be described as follows. In this scenario the variable q is equal to zero and the estimate $\hat{x}$ as in (11d) coincides with z. Thus, recalling the definition of the z-dynamics, the dynamics $\hat{x}$ becomes an open-loop copy of the dynamics of plant (1) which is, however, not affected by the action of the recovery filter. Finally, it is crucial to point out that the above open-loop observer is initialized – in the absence of disturbances – at the correct value of the state x, because by Lemma 1 $\hat{x} = x$ on the boundary of the set $\delta_o\mathbb{B}$.

Remark 4. Suppose that all the eigenvalues of the matrix A in (1) have negative real part. Then the inverse matrix $Q(t)^{-1}$ converges exponentially to zero as t tends to infinity. This can be easily seen by noticing that $Q(t)^{-1} = e^{At}\bar{Q}(t)^{-1}e^{A^\top t}$, where the matrix

$$\bar{Q}(t) = \int_0^t e^{A^\top\tau}C^\top C e^{A\tau}d\tau \tag{18}$$

denotes the standard *observability Gramian* associated to (1), non-singular for all $t>0$ by observability of the pair (C,A), and recalling that, whenever A is Hurwitz, the observability Gramian converges to a constant matrix $\bar{Q}$ as time tends to infinity. ⌟

3.3. Recovery filter

Consider the plant (1) assuming that no disturbance input affects the plant itself, namely with $d = 0$. Before presenting the structure of the *recovery filter* discussed in this section, we introduce now, in the Laplace domain, two notable objects: the matrix $\Phi(s)$ of transfer functions from the initial condition $x(0)$ to the Laplace transform of the output $y(s)$, *i.e.* $\Phi(s) = C(sI - A)^{-1}$, and the canonical transfer function $W(s)$ from the Laplace transform of the input $u(s)$ to the Laplace transform of the output, *i.e.* $W(s) = C(sI - A)^{-1}B$.

Suppose, for simplicity[4], that the matrix A possesses n distinct real eigenvalues and let $\alpha = \max_i \lambda_i(A)$, where $\lambda_i(A)$, later also referred to as λ_i for simplicity, denotes the i_{th} eigenvalue of matrix A. Let the recovery input $u_R(\tau_R)$ be defined as

$$u_R(\tau_R) = -k_R\,\mathrm{sat}(y_0) R^W_{M,2} e^{\alpha \tau_R}\,, \tag{19}$$

$k_R > 0$, where $R^W_{M,2} \in \mathbb{R}$ denotes the *residual* associated to the factor $(s-\alpha)^2$ in the transfer function $\hat{W}(s) = W(s)/(s-\alpha)$. The following result shows that the control input (19) is capable of steering, in finite time, the output y inside the region where it is not saturated.

Lemma 2. *(Recovery of unsaturated output). Consider plant (1) with $d = 0$. Suppose that $u(t) = u_R(t)$ for all $t \geq 0$. Then, there exists a finite $\bar{t} > 0$ such that $\mathrm{sat}(y(\bar{t})) = y(\bar{t})$, namely $|y(\bar{t})| < 1$.*

Proof. To begin with, note that by standard arguments of systems theory, the Laplace transform of the output $y(s)$ may be written as

$$y(s) = \Phi(s)x(0) + W(s)u_R(s) = \sum_{i=1}^{n_p} \frac{R^\Phi_i}{s-\lambda_i} x(0) + W(s)u_R(s)\,, \tag{20}$$

where $R^\Phi_i \in \mathbb{R}^{1\times n_p}$ for $i = 1, ..., n$ denote the residuals of the matrix $\Phi(s)$. Recall now that the Laplace transform of $u_R(t)$ defined as in (19) is given by

$$u_R(s) = \frac{\gamma}{s-\alpha}\,, \tag{21}$$

with $\gamma = -k_R\,\mathrm{sat}(y(0))R^W_{M,2}$. Substituting now (21) into (20) yields

$$y(s) = \sum_{i=1}^{n_p} \frac{R^\Phi_i}{s-\lambda_i} x(0) + \sum_{i=1}^{n_p-1} \frac{\gamma R^W_i}{s-\lambda_i} + \frac{\gamma R^W_{M,1}}{s-\alpha} + \frac{\gamma R^W_{M,2}}{(s-\alpha)^2}\,, \tag{22}$$

which is equal to

$$y(t) = \sum_{i=1}^{n_p} e^{\lambda_i t} R^\Phi_i x(0) + \gamma \sum_{i=1}^{n_p-1} e^{\lambda_i t} R^W_i + \gamma R^W_{M,1} e^{\alpha t} + \gamma R^W_{M,2} t e^{\alpha t}\,, \tag{23}$$

[4]The assumption on the eigenvalues of the matrix A may be easily relaxed, as detailed in the following Remark 5.

anti-transforming to the time domain. Note now that, by definition of α, there exists a finite time instant $\hat{t} > 0$ such that the last term of (23) *dominates* the first three terms for all $t \geq \hat{t}$, that is

$$\Big|\sum_{i=1}^{n_p} e^{\lambda_i t} R_i^{\Phi} x(0) + \gamma \sum_{i=1}^{n_p-1} e^{\lambda_i t} R_i^{W} + \gamma R_{M,1}^{W} e^{\alpha t}\Big| < |\gamma R_{M,2}^{W} t e^{\alpha t}|$$

for all $t \geq \hat{t}$. Then, since the choice of γ is such that $y(t)$ is initially *steered* towards the origin and $y(t) \in \mathbb{R}$, necessarily there exists $\bar{t} > \hat{t}$ such that $|y(\bar{t})|$ becomes smaller than the saturation limit. $\square$

Remark 5. The above lemma has been stated with the simplifying assumption of having n distinct and real eigenvalues for the matrix A. In the more general case of real eigenvalues with algebraic multiplicity greater than one, it is still possible to mimic the above approach − letting in particular u_R be defined precisely as in (19) − by noticing that the rationale behind the specific choice of u_R is basically to obtain an additional polynomial term in the Jordan block associated to the maximum eigenvalue of A, which dominates, after some finite time, the remaining terms in the time-response $y(t)$. Therefore, in the presence of complex-conjugate eigenvalues, a term of the form $\gamma/(s^2 + 2\zeta\omega_N s + \omega_N^2)$, corresponding to the pair with largest real part, should be injected into the plant. ⌟

Remark 6. The choice of the value for the constant gain k_R in (19) appears to be non influential on the derivation of the results in Lemma 2. However, it may be used to obtain a transient response of the plant (1) in closed loop with the control input (19). The intuitive idea consists in using the gain k_R to reduce the instant $\hat{t}$ after which the additional polynomial term generated by u_R as in (19) *dominates* the remaining terms of the solution. ⌟

Based on the above discussion and on the properties established in Lemma 2, we propose now the following flow dynamics for the recovery filter represented in Figure 1:

$$\dot{q}_R = 0, \qquad \dot{y}_R = 0, \qquad \dot{\tau}_R = q_R - (1 - q_R)\tau_R, \tag{24a}$$

which should be enabled whenever the anti-windup closed-loop system state belongs to the flow set. The jump dynamics of the recovery filter corresponds to the jump rule already presented in (9b), (9c) augmented with the following rule:

$$(q, \mathrm{sat}(y), \mathrm{sat}(y) - \mathrm{sat}(\hat{y})) \in \mathcal{D}_R, \quad \begin{cases} q_R^+ &= 1 \\ y_R^+ &= \mathrm{sat}(y) \\ \tau_R^+ &= 0, \end{cases} \tag{24b}$$

where

$$\mathcal{D}_R = \{0\} \times \mathcal{Y}_R \times \overline{(\mathbb{R} \setminus [-\eta, \eta])}, \tag{24c}$$

with $\mathcal{Y}_R = ([\delta_i, \delta_o] \cup [-\delta_o, -\delta_i])$. Finally, based on (19), the output of the recovery filter is selected as

$$v_u = -k_R q_R y_R R_{M,2}^{W} e^{\alpha \tau_R}\,, \tag{25}$$

where α and $R^W_{M,2}$ should be selected according to Lemma 2. The overall hybrid dynamics (24), (9b), (9c) of the recovery filter corresponds to activating the filter (thereby driving the plant output back into the unsaturated region) whenever $q = 0$ and the mismatch between $\text{sat}(y)$ and $\text{sat}(\hat{y})$ is larger than η, and de-activating the filter whenever q becomes 1 again (thereby indicating that the plant output has come back within the saturation limits).

4. Main Result

4.1. Anti-windup closed-loop system and its properties

Before stating our main result, we summarize next the overall closed loop arising from the interconnection of the plant-controller pair to the proposed anti-windup compensation scheme. With reference to the block diagram of Figure 1, the output saturation anti-windup architecture consists of the following state-space components:

1. $(x_p, x_c) \in \mathbb{R}^{n_p \times n_c}$ arising from the continuous-time plant (1) and the continuous-time controller (2), represented by blocks $\mathcal{P}$ and $\mathcal{C}$ of Figure 1, respectively, in addition to their interconnections (5), (8) represented also by the "feedback selector" in the figure;
2. $q \in \{0, 1\}$ representing the saturation detection logic in Figure 1, mostly responsible for the jump rules in (9), whose intuitive evolution is shown in Figure 2;
3. $(z, \xi, \tau) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \times \mathbb{R}$, which is the state of the finite-time/open-loop observer $\hat{\mathcal{P}}$ in Figure 1 obeying equations (11);
4. $(y_R, q_R, \tau_R) \in \mathcal{Y}_R \times \{0, 1\} \times \mathbb{R}$ associated to (24), (25) and with the "recovery filter" in Figure 1.

The overall system can be represented by the aggregated state

$$X = (x_p, x_c, q, z, \xi, \tau, y_R, q_R, \tau_R) \in \mathbb{R}^{n_c+n_p} \times \{0, 1\} \times \mathbb{R}^{2n_p} \times [0, \tau_M] \times \mathcal{Y}_R \times \{0, 1\} \times [0, \infty) =: \mathcal{X}$$

comprising physical states, observed states, logical variables and timers. We call here *anti-windup closed-loop system* the interconnection of the above components whose flow map is given by (1), (2), (9e), (11), (24a), with the interconnections (5), (8), (25), whose jump map is given by (9b), (9c), (24b), (12) whose jump set corresponds to the union of the jump sets specified in (9a), (24c) and whose flow set is the closed complement of the jump set. More compactly, we may refer to the *anti-windup closed-loop system* by its composing equations (1), (2), (5), (8), (9), (11), (24), (25), (12). From a functionality viewpoint, the overall block diagram of Figure 1 and the dynamical subcomponents listed above evolve according to the following intuitive behavior. Observer (11) provides a finite-time convergent estimate whenever the plant output is within the saturation limits ($q = 1$) and otherwise behaves as an open-loop copy of the plant driven by the controller output y_c. The recovery filter (24) steers the plant output inside the unsaturated region whenever the latter output exhibits a significantly different behavior with respect to the output of the open-loop copy, which is driven by the output y_c

of controller (2). Finally, these two blocks are governed by the switching logic explained in (9). The following theorem is the main result of the paper establishing desirable properties of the proposed anti-windup scheme as stated in Goal 1 with the following compact attractor $\mathcal{A}$ in item 3:

$$\mathcal{A} = \{0_{n_p+n_c}\} \times \{1\} \times \{0_{2n_p}\} \times [0, \tau_M] \times \mathcal{Y}_R \times \{0\} \times \{0\}. \tag{26}$$

Theorem 1. *(Main Result). The* anti-windup closed-loop system *(1), (2), (5), (8), (9), (11), (24), (25), (12) satisfies Goal 1 for the attractor $\mathcal{A}$ in (26).* ⋄

Proof. To prove Theorem 1, Local Preservation, η-Asymptotic Recovery and UGAS must be shown. We prove these three properties next

1. Local Preservation. Due to the strict inequality in equation (6) and from the definition of flow and jump sets in (9a), one has that the state never enters the jump set. Hence, the solution is purely continuous, therefore unique, and satisfies $q = 1$ for all times. Therefore, this response coincides with the unconstrained response, as to be proven.

2. η-Asymptotic recovery. To begin with, the solutions of the anti-windup closed-loop system have unbounded domain in the ordinary time direction because there are no finite escape times (all right hand sides of the flow equations are globally Lipschitz) and the disturbance d only enters the flow equation of the plant (1), whose state is a continuous function in the ordinary time direction and due to hysteresis switching, there cannot be any Zeno solution. Then, by the above considerations, and since we are only interested in the asymptotic behavior of solutions, we can disregard the hybrid times (t, j) (in the domain of the solution) satisfying $(t, j) \leq (t_d, j_d)$, where j_d is a well defined integer, based on the properties of the hybrid domain of the solutions, established above. For all such hybrid times, by the first constraint in (7) we can take $d = 0$ in all of the following considerations. Three cases may then occur:

(i) there exists $(t_1, j_1) \geq (t_d, j_d)$ such that $q(t_1, j_1) = 1$ for all $(t, j) \geq (t_1, j_1)$, which means that the solution is eventually continuous and its flow dynamics corresponds to the exponentially stable linear closed loop which necessarily converges to the unique forced response, coinciding with the unconstrained response.

(ii) there exists $(t_2, j_2) \geq (t_d, j_d)$ such that $q(t_2, j_2) = 0$ and $\hat{x}(t_2, j_2) = x(t_2, j_2)$. Then, the open-loop observer (11a), (11e) is such that $y = \hat{y}$ which ensures $v_u = 0$ and, therefore, combining (5) and (10) (which is an alternative version for interconnection (8)), we have $u_c(t, j_2) = y(t, j_2)$ and $u(t, j_2) = y_c(t, j_2)$, for all $t \in [t_{j_2}, t_{j_2+1}]$, namely until the next jump of the solution. Moreover, using $v_u = 0$ and $d = 0$, we have $u = y_c$ and we may apply Lemma 1 to ensure that observer (11) behaves alternatively as a finite time observer and an open-loop observer, as q toggles between 0 and 1 for all subsequent hybrid times (t, j). In other words, for all subsequent times (t, j) satisfying $q(t, j) = 1$ and $q(t, j + 1) = 0$, we have from Lemma 1 $\hat{x}(t, j) = x(t, j)$ and iterating the reasoning above, we have $u_c(t,j) = y(t,j)$ and $u(t,j) = y_c(t,j)$ for all $(t,j) \geq (t_2, j_2)$. As a consequence, the controller is virtually interconnected to the plant for all times and the proof is completed as in the previous item.

(iii) there is no $(t_1, j_1) \geq (t_d, j_d)$ nor $(t_2, j_2) \geq (t_d, j_d)$ satisfying the properties in the previous items. We will show that this leads to an absurd. To this aim, note that there exits (t_3, j_3) such that $q(t,j) = 0$ for all $(t,j) \geq (t_3, j_3)$, indeed, if this was not the case then either q was coming back to 1 and remaining 1 for all future times (which is impossible because item 1 does not hold), or q would come back from 0 to 1 after (t_d, j_d), thereby resetting the finite-time observer states as in (9b) and then, from Lemma 1, leading to a correct finite-time estimate at the next jump to $q^+ = 0$ (thus falling into item 2 which does not hold). Nevertheless, if q remains constantly at zero, according to (10), regardless of the value of v_u, the closed loop in the coordinates (x_c, z) can be written as

$$\left\{ \begin{array}{rcl} \dot{z} & = & Az + By_c, \qquad\qquad \hat{y} = Cz, \\ \dot{x}_c & = & A_c x_c + B_c \hat{y} + B_r r, \qquad y_c = C_c x_c + D_c \hat{y} + D_r r, \end{array} \right. \tag{27}$$

which coincide with the unconstrained dynamics and therefore converge to the unconstrained response satisfying the second constraint of (7). As a consequence, there must exist some time (t_R, j_R) when $|\operatorname{sat}(y(t_R, j_R)) - \operatorname{sat}(\hat{y}(t_R, j_R))| > \eta$, which triggers the jump equation (24b) and activates the recovery filter. Finally, combining Lemma 2 with the fact that by (7) $\hat{y}$ keeps coming back to the bounded unsaturated region, one gets that necessarily y comes back to the unsaturated region too, thereby triggering $q^+ = 1$ which leads to a contradiction.

3. UGAS of set $\mathcal{A}$ in (26). According to Proposition 1 given in Appendix A, to show UGAS of $\mathcal{A}$ it is enough to establish (local) stability and global convergence to $\mathcal{A}$. Indeed, as noted at the beginning of the proof of item (ii), there are no finite escape times. Note also that all maximal solutions starting in $\mathcal{X}$ are complete because solutions can always be continued. Therefore the *pre-* in Proposition 1 can be dropped. Let us now prove stability. For small enough initial conditions, we have $q = 1$, $q_R = 0$ and $y = \operatorname{sat}(y)$ (indeed the saturation behaves linearly sufficiently close to the origin). As a consequence, the continuous dynamics for plant and controller coincides with the asymptotically stable closed loop (1), (2), (3). The states of the open-loop/finite-time convergent observer are such that z, being a copy of (1), converges to zero, hence are bounded. Moreover, the linear discrete-time equivalent description of ξ - corresponding to a flow in $[0, \tau_M]$ and a jump at τ_M - evolves according to the system matrix $Q(\vartheta\tau_M)Q(\tau_M)^{-1}e^{-A^\top \tau_M}$, which is Schur by construction, hence it is an input-to-state stable discrete-time system perturbed by the bounded input $C^\top(\operatorname{sat}(y) - Cz)$. The continuous-time evolution of ξ, in addition, is allowed in the finite time interval $[0, \tau_M]$ implying boundedness of the intersample solution. The internal clock τ eventually evolves in the compact set $[\vartheta\tau_M, \tau_M]$, due to its reset policy. Finally, the states of the recovery filter are such that y_R remains constant while τ_R evolves according to asymptotically stable dynamics (because $q_R = 0$). Therefore, stability of $\mathcal{A}$ follows. To show convergence, it is enough to follow the proof of item 2 for the special case $(r,d) = (0,0)$. Indeed in this special case the unconstrained trajectory converges to zero, therefore also y converges to zero and as a consequence q eventually becomes 1, thereby also resetting q_R to zero. Convergence of ξ can be shown by noticing that the external bounded input of the discrete-time equivalent description discussed above is vanishing. Therefore, global convergence to $\mathcal{A}$ follows and the proof is completed by applying Proposition 1 given in 6. $\square$

5. Numerical Example

We consider a simple example corresponding to the following input/output relation for plant (1),

$$y(s) = \frac{1}{s(s+2)}(u(s) + d(s)), \tag{28}$$

which resembles the response of a relative degree one minimum phase plant. For example plant (28) may represent the approximated transfer function from the requested current to the obtained position for a DC electrical motor where the electrical time constant has been disregarded because the mechanical one is dominant. We assume that a controller (2) has been designed for plant (28) in such a way to be able to induce asymptotic tracking of a sinusoid with unit frequency. To this aim, the controller is selected as follows:

$$y_c(s) = \frac{4s^2 + 4s + 1}{s^2 + 1}(r(s) - u_c(s)),$$

where the denominator of its transfer function is chosen to guarantee the presence of the internal model and the numerator is selected in such a way to stabilize the negative error feedback interconnection of the unconstrained closed-loop system whose poles are conveniently placed in the negative half plane.

For this closed-loop system, we consider the response to a sinusoidal reference input oscillating between 0 and $2r_{max}$: $r(t) = r_{max} + r_{max}\sin(t)$ and we test the anti-windup action for several values of r_{max}. Since the unconstrained closed-loop system is exponentially stable and has an internal model for the reference input, we know that the asymptotic response coincides with the reference and therefore keeps coming back arbitrarily close to zero for any initial condition of the plant-controller states (x_p, x_c).

Based on the plant dynamics we can select $\alpha = 0$ and $R^W_{M,2} = 0.5$. Moreover, the anti-windup parameters are selected as $\delta_o = 0.9$, $\delta_i = 0.6$, $\eta = 0.55$, $k_R = 3$. The hysteresis range 0.3 between δ_i and δ_o is fixed to allow testing the anti-windup response also in the presence of output noise (even though our theory does not cover that case). The parameter η is selected to satisfy $\eta < \delta_i$ because this property allows us to satisfy requirement (7) in Goal 1, due to the fact commented above that with our selection of the reference all unconstrained responses keep coming back arbitrarily close to zero. Finally, the gain k_R has been manually tuned to induce a desirable and gradual recovery whenever the recovery filter is activated. We emphasize that no other existing anti-windup method can be used to address this problem as all of them merely focus on asymptotic stability thereby not allowing nonzero references. Indeed, employing alternative constructions, such as the ones in [4, 14, 10] within this setting would be inappropriate due to the explicit or implicit observer nature of the underlying linear dynamics that would experience undesirable transients every time the plant output (persistently excited by the action of reference r) exceeds the ouput saturation limits.

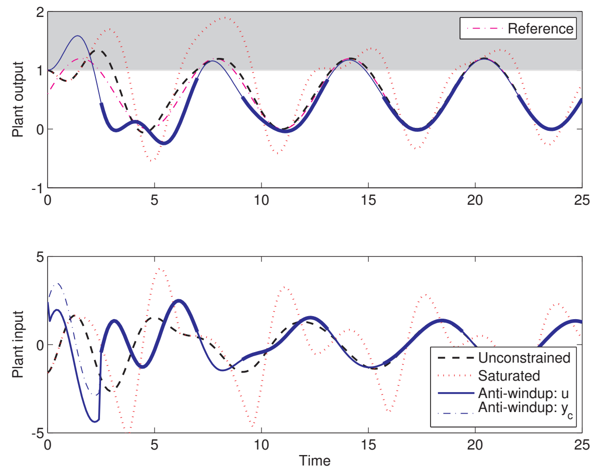

Figure 3: *First simulation test.* Responses of the various closed-loop systems without disturbances and a small amplitude reference defined by $r_{max} = 0.6, \quad x_{c0} = (0, 0), \quad x_{p0} = (0, 1)$.

First simulation test. We first run a simulation of the unconstrained, constrained and anti-windup closed-loop systems with the following parameters: $r_{max} = 0.6, \quad x_{c0} = (0, 0), \quad x_{p0} = (0, 1)$ and $d = 0$, where the plant state space representation is in the observer canonical form so that with the initial condition above we have $y(0, 0) = 1$ and $\dot{y}(0, 0) = 0$. Figure 3 comparatively shows the three responses. Comparing the reference (namely, the dash-dotted thin trace of the upper plot) to the unconstrained and anti-windup output responses (solid and dashed traces in the upper plot) we see that the anti-windup scheme succeeds at recovering asymptotic tracking despite the fact that the plant output keeps exiting the saturation limits (shaded area in the upper plot) and the feedback interconnection is periodically broken by the output nonlinearity. To best appreciate the hysteresis switching of the logic variable q the time intervals where $q = 1$ are characterized in Figure 3 by the intervals where a bold trace is used for y and u, while the time intervals where $q = 0$ are characterized by the intervals showing a thin trace. The constrained closed-loop system output response (red dotted trace in the upper plot) shows that without anti-windup we lose the tracking performance. The lower plot shows the three plant inputs (with the same color codes) and for the anti-windup closed loop we also show the controller output (blue dash-dotted) so that it is evident that in the time interval $t \in [0, 2.4]$ the recovery filter is active and $u_R \neq 0$. After $t = 2.4$, the recovery filter remains at zero because there are no disturbances.

Second simulation test. To test the proposed scheme in the presence of large disturbances, large

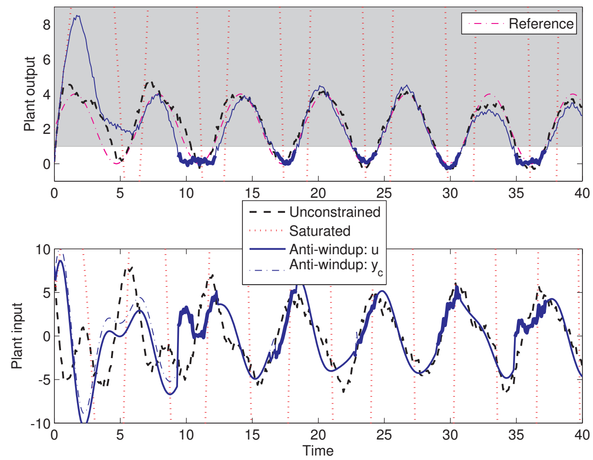

Figure 4: *Second simulation test.* Responses of the various closed-loop systems with disturbances and a large amplitude reference defined by $r_{max} = 2, \quad x_{c0} = (0, 0), \quad x_{p0} = (10, 0)$.

initial conditions and also the presence of measurement noise (which is beyond the theoretical results of this paper), we consider the following situation: $r_{max} = 2, \quad x_{c0} = (0, 0), \quad x_{p0} = (10, 0)$, d is a band limited white noise with sampling time 0.1 and noise power 0.02. We also add a band limited white noise with the same sampling time and noise power 0.0001 to the plant output. Note that in this case the initial condition corresponds to $y(0, 0) = 0$ and $\dot{y}(0, 0) = 10$, so the plant output starts at zero but grows very fast in the initial transient due to the initial condition. The various closed-loop responses are shown in Figure 4 with the same structure and color code as in the previous case commented above. Note that the large reference signal leads to a plant output response which is most of the time outside the saturation limits but it still satisfies the requirements in (7) because the unconstrained response keeps coming out of the saturated region (shaded area in the upper plot). With such a large reference, the constrained closed-loop system generates a diverging response. Instead, the anti-windup closed-loop system is capable of recovering the response despite the large initial condition of the plant, due to the activation of the recovery filter in the time interval $t \in [0.07, 9.4]$ (note that the recovery filter does not activate for the first 0.07 seconds because $y(0, 0) = 0$). Note also that the input disturbance causes a significant tracking error both on the unconstrained and on the anti-windup closed loops. The time intervals when $q = 0$ and the anti-windup closed-loop system operates in feedback from the open-loop observer can be conveniently located by finding the time intervals where the plant input is less noisy, indeed the output of the open-loop observer is not affected by the noise and this leads to a cleaner controller output. Finally, it should be emphasized that around times 16.5 and 22.9, the large mismatch caused by the disturbances activates the recovery filter again, but its effect is almost negligible on the closed loop response.

6. Conclusions

We proposed a novel anti-windup scheme for linear control systems over SISO plants with output magnitude saturation. The scheme, which has a hybrid formulation combines a switching logic detecting entry and exit of the plant output from suitably defined subsets of the unsaturated region, a finite-time/open-loop observer and a recovery filter driving the plant output back into the unsaturated region. For the first time we specified the anti-windup problem for output saturated plants in terms of recovery of the unconstrained trajectory and we proved the corresponding property for the proposed scheme. Simulation results have been given to illustrate the effectiveness of the proposed solution.

Appendix A. A stability result

We state here a result, of separate interest from the focus of the paper, which is needed for the proof of Theorem 1. The result is straightforwardly derived from the results in [5]. We consider a general hybrid system:

$$\mathcal{H} \begin{cases} \dot{X} \in F(X), & X \in \mathcal{C}, \\ X^+ \in G(X), & X \in \mathcal{D}, \end{cases} \tag{29}$$

or more compactly $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$, with $X \in \mathbb{R}^n$, and show below that, in the absence of finite escape times, local stability plus global (not necessarily uniform) convergence to a compact set $\mathcal{A} \subset \mathbb{R}^n$ implies uniform global asymptotic stability of $\mathcal{A}$. We state our results without assuming completeness of maximal solutions so, following [5], we use the *pre-* prefix to suitably highlight this fact. Let us recall a few definitions from [5].

Definition 1. [5, Assumption 6.5] Hybrid system $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$ satisfies the *hybrid basic conditions* if the following holds.

(i) $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathbb{R}^n$.

(ii) F is outer semicontinuous and locally bounded relative to $\mathcal{C}$, $\mathcal{C} \subset \mathrm{dom}F$ and $F(X)$ is convex for any $X \in \mathbb{R}^n$.

(iii) G is outer semicontinuous and locally bounded relative to $\mathcal{D}$ and $\mathcal{D} \subset \mathrm{dom}G$.

Definition 2. [5, Def. 6.12] Hybrid system $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$ is *pre-forward complete* if every solution is either bounded or complete.

Note that Definition 2 essentially rules out the possibility that some solution may escape in finite time during flows. However it does not rule out the possibility that some solution be not complete as long as it remains bounded. In particular, we may clarify the meaning of Definition 2 by saying that a hybrid system is pre-forward complete if and only if none of its solutions has a finite escape time. In light of Definition 2, we can now state the following proposition.

Proposition 1. *Assume that hybrid system (29) satisfies the hybrid basic conditions and is pre-forward complete. Assume that a compact set $\mathcal{A} \subset \mathbb{R}^n$ is stable for $\mathcal{H}$ and all maximal solutions ϕ to $\mathcal{H}$ satisfy*[5] $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$. *Then set $\mathcal{A}$ is robustly globally $\mathcal{KL}$ pre-asymptotically stable for $\mathcal{H}$, therefore UGpAS.*

Proof. The proof is a simple application of [5, Thm 7.12]. Indeed, stability and local convergence to $\mathcal{A}$ corresponds to local pre-asymptotic stability of $\mathcal{A}$. Then due to pre-forward completeness each non-complete solution is bounded and due to the convergence assumption each complete solution is bounded because it converges to the bounded set $\mathcal{A}$. As a consequence from [5, Def 7.3] the basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$ of $\mathcal{A}$ is $\mathbb{R}^n$. Finally, robust $\mathcal{KL}$ pre-asymptotic stability from $\mathcal{B}^p_{\mathcal{A}} = \mathbb{R}^n$ follows from [5, Thm 7.12], and the fact that global $\mathcal{KL}$ pre-asymptotic stability implies UGpAS is trivial. □

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[5] As customary, we denote the distance of $X \in \mathbb{R}^n$ from $\mathcal{A} \subset \mathbb{R}^n$ as $|X|_{\mathcal{A}} := \inf\limits_{Z\in\mathcal{A}} |X - Z|$.

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