Finite Gain $\mathcal{L}_p$ Stability for Hybrid Dynamical Systems ⋆

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Abstract

We characterize finite gain $\mathcal{L}_p$ stability properties for hybrid dynamical systems. By defining a suitable concept of hybrid $\mathcal{L}_p$ norm, we introduce hybrid storage functions and provide sufficient Lyapunov conditions for $\mathcal{L}_p$ stability of hybrid systems, which cover the well-known continuous-time and discrete-time $\mathcal{L}_p$ stability notions as special cases. We then focus on homogeneous hybrid systems and prove a result stating the equivalence among local asymptotic stability of the origin, global exponential stability, existence of a homogeneous Lyapunov function with suitable properties for the hybrid system with no inputs, and input-to-state stability, and we show how these properties all imply $\mathcal{L}_p$ stability. Finally we characterize systems with direct and reverse average dwell time properties and establish parallel results for this class of systems. We also make several connections to the existing results on dissipativity properties of hybrid dynamical systems.

Key words: Input-output stability, $\mathcal{L}_p$ stability, hybrid systems, homogeneous hybrid systems.

1 Introduction

The influence of inputs, noise or disturbances acting on continuous-time and discrete-time nonlinear systems [31, §6] can be measured by properly defining the gains from these inputs to the state or the output. The stability concept related to the input-output properties is often referred to as *external stability* as opposed to the concept of *internal stability* (or Lyapunov stability). The definition of the gain of a system essentially comprises a convenient bound on a suitable norm of the system signals, based on the norm of its input. In the literature (see, e.g., [31, §6.2 and 6.7] for the continuous-time and the discrete-time cases, respectively) the norm which is used for this concept is the $\mathcal{L}_p$ norm and, therefore, the concept of input-output stability is often referred to as $\mathcal{L}_p$ stability (see, e.g, [15]).

The notion of $\mathcal{L}_p$ stability is especially useful in addressing properties of interconnected systems and has been extensively used in the literature (see, for example the results related to the $\mathcal{L}_2$ case in [32, 33]). Moreover, in the absolute stability framework, suitable properties of linear systems interconnected to nonlinearities satisfying certain input-output relations were considered in [35]. Finally, it is well known that for LTI systems exponential stability (or Lyapunov stability) is equivalent to finite gain $\mathcal{L}_p$ stability (see, e.g., [31, §6.3]), and this result is useful when characterizing properties of interconnected systems based on the properties of the disconnected systems with zero inputs.

⋆ Research supported in part by AFOSR grant FA9550-09-1-0203 and NSF grants ECCS-0925637 and CNS-0720842. The first author is supported by the Australian Research Council under the Discovery Project and Future Fellowship schemes.

When focusing on hybrid dynamical systems, the notion of input-to-state stability (ISS) has been investigated in [3] (see also references therein for earlier solutions in similar directions) where the ISS concept introduced by Sontag in the late 1980's, and well developed in the past two decades both for the continuous-time (CT) and the discrete-time (DT) cases, is extended to the hybrid context within the general framework described in [6] (see also [7]).

Despite the ISS results cited above, there seems to be a lack of results on $\mathcal{L}_p$ stability properties of hybrid systems in the literature. However, special classes of hybrid systems were considered in the literature; in particular, $\mathcal{L}_2$ stability of switched systems was considered in [10, 11, 12, 13, 25, 36]; $\mathcal{L}_p$ stability of networked control systems in [18]; $\mathcal{L}_2$ stability of reset systems in [14, 21] and $\mathcal{L}_p$ and input-to-state stability for a class of quantized control systems in [28]. The goal of this paper is to provide some results on $\mathcal{L}_p$ stability of general hybrid systems.

Computing the finite $\mathcal{L}_p$ gain from a control input or an exogenous disturbance can help to define the performance of interconnected systems by analyzing its components separately. In particular, since these results are obtained within the hybrid systems framework of [6], the dissipativity results that recently appeared in [24, 26] can be applied by using the specific supply rates introduced here, thereby providing small gain conditions for interconnected $\mathcal{L}_p$ stable hybrid systems.

In this paper we first introduce the concept of hybrid $\mathcal{L}_p$ norms by incorporating sums and integrals in them, so that the well-known continuous-time and discrete-time norms are recovered as special cases. Then, we illustrate the use of suitable Lyapunov-like conditions involving $\mathcal{L}_p$ storage functions in order to assess finite gain input-to-output $\mathcal{L}_p$ and $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stability of a given hybrid system. Then we focus on homogeneous hybrid systems and establish for this special class of systems a result stating the equivalence among local asymptotic stability, global exponential stability (this is a straightforward consequence of the results in [8]) and ISS and that all these properties imply $\mathcal{L}_p$ stability. These results are important especially in light of the recent results in [8] about homogeneous approximations of hybrid systems. Indeed we establish here that, for these homogeneous approximations, local asymptotic stability of the system with no inputs implies ISS and finite gain $\mathcal{L}_p$ stability which can be usefully exploited for establishing properties of their interconnections. Finally, we study the effect of average dwell-time properties of the hybrid system by incorporating suitable timer dynamics to impose direct or reverse average dwell time on its solutions. Exploiting these dwell-time properties, we illustrate how the Lyapunov inequalities can be weakened and we also establish similar results to the ones proven in the case without average dwell-time, when focusing on homogeneous dynamics.

Some preliminary statements along similar directions to the ones of this paper have been reported in [20] with reference to temporally regularized homogeneous systems. Those results were instrumental to proving suitable stability properties of interconnected reset systems. In [20] the following special case was addressed: 1) systems with temporal regularization (or dwell time) and 2) external inputs only appearing in the flow map. For this special case, there was no need to introduce hybrid $\mathcal{L}_p$ norms and the results were stated in terms of classical continuous-time $\mathcal{L}_p$ norms. Finally, a preliminary version of this paper appeared in [17] where some of our results have been discussed without proofs. As compared to [17], here we provide the missing proofs and we discuss the extension to hybrid systems with average dwell time.

The paper is organized as follows. Preliminaries are presented in Section 2. In Section 3 we show how to obtain finite gain $\mathcal{L}_p$ stability bounds from storage functions for hybrid systems. Then, in Section 4 we focus on homogeneous systems and characterize a number of equivalent properties, including ISS, LAS, GES and existence of suitable Lyapunov functions. In Section 5 we establish parallel properties for systems satisfying (direct and reverse) average dwell-time properties. Concluding remarks are presented in Section 6.

Notation: $\mathcal{B}(r)$ denotes the ball of radius r. $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ $X + Y = \{x + y : x \in X,\ y \in Y\}$. Moreover, $X \subset Y$ means that $x \in Y \Rightarrow x \in X$. A function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\lim_{s\to\infty} \alpha(s) = +\infty$. The vector $[x'\ d']'$ is denoted (x, d).

2 Preliminaries

A solution to a hybrid system is defined on a hybrid time domain, which is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. A *compact hybrid time domain* is any subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ that can be written as $\cup_{i\in\{1,\ldots,J\}}([t_i, t_{i+1}] \times \{i\})$ where $J \in \mathbb{Z}_{\geq 0}$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_{J+1}$. A *hybrid time domain* is any set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $(T_D, J_D) \in E$ implies that $E \cap ([0, T_D] \times \{0, \ldots, J_D\})$ is a compact hybrid time domain. A *hybrid signal* is a function defined on a hybrid time domain. A *hybrid arc* is a hybrid signal x such that $t \mapsto x(t, j)$ is locally absolutely continuous for each j.

Definition 1 ($\mathcal{L}_p$ norm) *For a hybrid signal z, with domain $\mathrm{dom}(z) \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and a scalar $T \in \mathbb{R}_{\geq 0}$ the T-truncated $\mathcal{L}_p$-norm of z is given by*[1]

$$\|z_{[T]}\|_p := \left(\sum_{i=1}^{j(T)} |z(t_i, i-1)|^p + \sum_{i=0}^{j(T)} \int_{t_i}^{\sigma_i} |z(s,i)|^p ds \right)^{\frac{1}{p}} \tag{1}$$

where $t_0 = 0$, $j(T) = \max\{k : (t,k) \in \mathrm{dom}(z),\ \ t+k \leq T\}$ and for all $i \in \{0, \ldots, j\}$, $\sigma_i = \min(t_{i+1}, T-i)$. Based on (1), the $\mathcal{L}_p$-norm of z is defined as

$$\|z\|_p = \lim_{T \to T^\star} \|z_{[T]}\|_p. \tag{2}$$

where $T^\star = \sup\{t + j : (t,j) \in \mathrm{dom}(z)\}$. Moreover, we say $z \in \mathcal{L}_p$ whenever the limit above exists and is finite.

Remark 1 The $\mathcal{L}_p$ norms for continuous-time and discrete time systems are particular cases of the above defined norm. Indeed, if the solution only flows, we have for any T, $j(T) = 0$ and the first sum in (1) disappears, so that the hybrid norm becomes the continuous-time $\mathcal{L}_p$ norm [15, Chapter 5]. Moreover, if the solution only jumps, then $t_k = \sigma_k = 0$ for all k, and all the integral terms in (1) disappear so that (2) corresponds to the discrete-time ℓ_p norm [31, Section 6.7]. ⌟

Remark 2 In (1) the value of the hybrid arc just before the jump $z(t_i, i-1)$ is considered. An alternative definition can be given in terms of the values of the hybrid arc after the jump, that is, $z(t_i, i)$. Then parallel computations to the ones reported in this paper can be carried out. We choose this option here for consistency with the approach in [3] reported below (see, in particular, the definition of $\Gamma(z)$ below). ⌟

Definition 2 ($\mathcal{L}_\infty$ norm) *For a hybrid signal z, with domain $\mathrm{dom}(z) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, the T-truncated $\mathcal{L}_\infty$ norm is given by*

$$\|z_{[T]}\|_\infty := \max \left\{ \begin{array}{c} \operatorname*{ess.\,sup}_{(t,j) \in \mathrm{dom}(z) \setminus \Gamma(z),\ t+j \leq T} |z(t,j)|, \\ \sup_{(t,j) \in \Gamma(z), t+j \leq T} |z(t,j)| \end{array} \right\}. \tag{3}$$

And the $\mathcal{L}_\infty$ norm of z is given by

$$\|z\|_\infty = \lim_{T \to T^\star} \|z_{[T]}\|_\infty \tag{4}$$

with $T^\star$ defined below (2) and where $\Gamma(z)$ denotes the set of all (t,j) such that $(t,j) \in \mathrm{dom}(z)$ and $(t, j+1) \in \mathrm{dom}(z)$. Moreover, we say $z \in \mathcal{L}_\infty$ whenever the above limit exists and is finite.

Consider the following nonlinear hybrid system

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x,d), \quad (x,d) \in \mathcal{C} \\ x^+ &=& g(x,d), \quad (x,d) \in \mathcal{D} \\ y &=& h(x,d) \end{array} \right. \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector, $d \in \mathbb{R}^m$ is an exogenous input, $y \in \mathbb{R}^q$ is the output vector, $f(\cdot,\cdot)$ is the flow map, $g(\cdot,\cdot)$ is the jump map, $h(\cdot,\cdot)$ is the output map, $\mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^m$ is the flow set and $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$ is the jump set. Similar to [3, 24, 26], we say that the pair (x,d) is a solution pair to (5) if $\mathrm{dom}(x) = \mathrm{dom}(d)$ and

- for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t,j) \in \mathrm{dom}(x)$, the pair satisfies $(x(t,j), d(t,j)) \in \mathcal{C}$ and $\dot{x}(t,j) = f(x(t,j), d(t,j))$;
- for all $(t,j) \in dom(x)$ such that $(t, j+1) \in dom(x)$, the pair satisfies $(x(t,j), d(t,j)) \in \mathcal{D}$ and $x(t, j+1) = g(x(t,j), d(t,j))$.

We assume the following regularity condition for the parameters of system (5).

[1] We adopt the convention $\sum_{i=1}^{0} f(i) := 0$.

Assumption 1 *The sets $\mathcal{C}$ and $\mathcal{D}$ are closed sets and $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are continuous in both their arguments.*

In this paper we study the finite gain $\mathcal{L}_p$ stability for system (5) which is defined as follows.

Definition 3 *Given $p \in [1,\ +\infty)$, system (5) is finite gain $\mathcal{L}_p$ stable from d to y with gain (upper bounded by) $\gamma_p \geq 0$ if there exists a scalar $\beta \geq 0$ such that any solution to (5) satisfies*

$$\|y\|_p \leq \beta|x(0,0)| + \gamma_p\|d\|_p. \tag{6}$$

for all $d \in \mathcal{L}_p$. Moreover, it is finite gain $\mathcal{L}_{p,\infty}$ ($\mathcal{L}_p$ to $\mathcal{L}_\infty$) stable from d to y with gain $\gamma_{p,\infty} > 0$ if there exists a scalar, $\beta \geq 1$ such that any solution to (5) satisfies

$$\|y\|_\infty \ \leq \beta|x(0,0)| + \gamma_{p,\infty}\|d\|_p. \tag{7}$$

for all $d \in \mathcal{L}_p$.

Remark 3 Notice that, because of the comments in Remark 1, the $\mathcal{L}_p$ (ℓ_p) stability definitions of [15, Chapter 5] and [31, Section 6.7] respectively for continuous and discrete-time systems correspond to particular cases of inequality (6). ⌟

Definition 4 *The origin of (5) with $d = 0$ is (locally) asymptotically stable (LAS) if there exists a ball $\mathcal{B} \subset \mathbb{R}^n$, centered at the origin and a class $\mathcal{KLL}$ function β such that for any initial condition $x(0,0) = x_0 \in \mathcal{B}$, all solutions satisfy:*

$$|x(t,j)| \leq \beta(|x_0|, t, j)\ \forall (t,j) \in \mathrm{dom}(x).$$

Definition 5 (Exponential ISS) *System (5) is exponentially finite gain input-to-state stable from d if there exist positive scalars m, ℓ and γ_∞ such that for any initial conditions $x(0,0)$ and any $d \in \mathcal{L}_\infty$, all solutions to (5) satisfy*

$$|x(t,j)| \leq \max\left\{me^{-\ell(t+j)}|x(0,0)|, \gamma_\infty\|d\|_\infty\right\}, \tag{8}$$

for all $(t,j) \in \mathrm{dom}(x)$. Moreover the origin of (5) is globally exponentially stable (GES) if (8) holds with $d = 0$.

3 Storage functions for $\mathcal{L}_p$ gain computation

In this section we establish $\mathcal{L}_p$ stability of system (5) by using storage functions.

Definition 6 ($\mathcal{L}_p$ storage function) *Given $p \in [1,\infty)$, a positive semidefinite continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a finite gain $\mathcal{L}_p$ storage function for system (5) if there exist positive constants c_2, k_{yf} and k_{yg} and non-negative constants k_{dg}, k_{df} such that*

$$0 \leq V(x) \leq c_2|x|^p\ \forall (x,d) \in \mathcal{C} \cup \mathcal{D} \tag{9}$$

$$\langle \nabla V(x), f(x,d)\rangle \leq -k_{yf}|h(x,d)|^p + k_{df}|d|^p\ \forall (x,d) \in \mathcal{C} \tag{10}$$

$$V(g(x,d)) - V(x) \leq -k_{yg}|h(x,d)|^p + k_{dg}|d|^p\ \forall (x,d) \in \mathcal{D} \tag{11}$$

Proposition 1 *Consider system (5) and assume that there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (9)-(11). Then the system is finite-gain $\mathcal{L}_p$ stable with gain upper-bounded by $\gamma_p = \sqrt[p]{\dfrac{k_d}{k_y}}$ where $k_d = \max\{k_{df}, k_{dg}\}$, $k_y = \min\{k_{yf}, k_{yg}\}$. Moreover, if $\exists\ c_{yV} > 0,\ c_{yd} > 0$ such that*

$$|h(x,d)|^p \leq c_{yV}V(x) + c_{yd}|d|^p,\ \forall (x,d) \in \mathcal{C} \cup \mathcal{D}, \tag{12}$$

then the system is $\mathcal{L}_{p,\infty}$ stable with gain $\gamma_{p,\infty} \leq \sqrt[p]{c_{yV}k_d + c_{yd}}$.

Remark 4 Even with $h(x,u)=x$, conditions (9)-(10) do not automatically rule out finite escape times for $d\in\mathcal{L}_p$, though such behavior could be excluded (even with $h(x,u)=0$) by strengthening the lower bound on V in (9) to be a class $\mathcal{K}_\infty$ function of $|x|$. An example where finite escape time occurs is provided by scaling the vector field in [15, Exercise 4.8] (see also [9]) by a smooth, positive, nondecreasing scalar function of $|x|$ so that, for some $\varepsilon>0$, $\langle\nabla V(x),f(x)\rangle\le-\varepsilon|x|^2$ for all $x\in\mathbb{R}^2$ for the Lyapunov function V given in [15, Exercise 4.8]. Such a scaling function exists since such a bound on the derivative of V holds near $x=0$ for the original vector field. The scaling function affects only the speed along which the original solutions are followed. As proven in [9], for some initial conditions the original solutions grow unbounded. For the scaled vector field, these solutions must escape to infinity in finite time. Indeed, if this were not the case then the unboundedness of the solution would contradict the fact that the $\mathcal{L}_2$ norm of the solution's state is finite. ⌟

Remark 5 In [26], the concept of "mixed dissipativity" was introduced to assess the stability of interconnected hybrid system. The dissipativity concepts were illustrated by considering quadratic supply rates. Here, the "mixed dissipativity" is obtained with supply rate functions of degree p and can be established by $\mathcal{L}_p$ storage functions $V(x)$ satisfying inequalities (10) and (11). ⌟

Proof of Proposition 1. Consider the function $U(t,j)=V(x(t,j))$ with $x(t,j)$ being a solution to system (5) having domain $\mathrm{dom}(x)$. From inequalities (10) and (11) we get that if $[t_k,t_{k+1}]\times\{k\}\subset\mathrm{dom}(x)$, then

$$\dot U(t,k)\le -k_{yf}|y(t,k)|^p+k_{df}|d(t,k)|^p \quad \text{for almost all } t\in[t_k,t_{k+1}]\,, \tag{13}$$

$$U(t_k,k)-U(t_k,k-1)\le -k_{yg}|y(t_k,k-1)|^p+k_{dg}|d(t_k,k-1)|^p \text{ if } k\ge 1. \tag{14}$$

Consider now any $T\in\mathbb{R}_{\ge0}$ and define $j(T):=\max k$ such that $(t,k)\in\mathrm{dom}(x)$, $t+k\le T$ and $\forall i\in\{0,\dots,j(T)\}$, $\sigma_i:=\min(t_{i+1},T-i)$ and, for simplicity, denote $\sigma_{j(T)}=t$. Then integrate (13) on each interval $[t_k,\sigma_k]$, $k\le j(T)$, to obtain

$$0\le -U(t_{k+1},k)+U(t_k,k)-k_{yf}\int_{t_k}^{\sigma_k}|y(s,j)|^p ds + k_{df}\int_{t_k}^{\sigma_k}|d(s,j)|^p ds\ \forall k\in\{0,\dots,j(T)\} \tag{15}$$

and rearrange the terms in (14) to get

$$0\le -U(t_k,k)+U(t_k,k-1)-k_{yg}|y(t_k,k-1)|^p + k_{dg}|d(t_k,k-1)|^p,\ \forall k\in\{1,\dots,j(T)\}\,. \tag{16}$$

Summing up the $j(T)+1$ terms in (15) and the $j(T)$ terms in (16), we obtain

$$\begin{aligned}0\le U(0,0)-U(\sigma_{j(T)},j(T))&\\ -k_{yg}\sum_{k=1}^{j(T)}|y(t_k,k-1)|^p-k_{yf}\sum_{k=0}^{j(T)}\int_{t_k}^{\sigma_k}|y(s,k)|^p ds&\\ +k_{dg}\sum_{k=1}^{j(T)}|d(t_k,k-1)|^p+k_{df}\sum_{k=0}^{j(T)}\int_{t_k}^{\sigma_k}|d(s,k)|^p ds.&\end{aligned}$$

Defining $k_d=\max\{k_{df},k_{dg}\}$ and $k_y=\min\{k_{yf},k_{yg}\}$, we have

$$\begin{aligned}k_y\|y_{[T]}\|_p^p&\le -U(\sigma_{j(T)},j(T))+U(0,0)+k_d\|d_{[T]}\|_p^p\\ &\le U(0,0)+k_d\|d\|_p^p\end{aligned} \tag{17}$$

where $\|y_{[T]}\|$ is the T-truncated $\mathcal{L}_p$ norm according to Definition 1, and we used the fact that $U(t,j)\ge0,\ \forall(t,j)\in\mathrm{dom}(x)$.

$\mathcal{L}_p$ *gain*: Using (17) and the upper bound for $U(t,j)$ from (9), we obtain $\|y_{[T]}\|_p^p \leq \frac{1}{k_y} c_2 |x(0,0)|^p + \frac{k_d}{k_y}\|d\|_p^p$, namely

$$\|y_{[T]}\|_p \leq \sqrt[p]{\frac{1}{k_y} c_2}|x(0,0)| + \sqrt[p]{\frac{k_d}{k_y}}\|d\|_p$$

which holds for all $0 \leq T < T^\star$ with $T^\star = \sup\{t+j : (t,j) \in \mathrm{dom}(z)\}$. Hence, taking the limit as $T \to T^\star$ the bound for the finite $\mathcal{L}_p$ gain is given by $\sqrt[p]{\frac{k_d}{k_y}}$.

$\mathcal{L}_{p,\infty}$ *gain*: Consider the first inequality in (17). Since $\|y_{[T]}\|_p^p \geq 0$, we have $U(\sigma_{j(T)}, j(T)) \leq U(0,0) + k_d\|d\|_p^p$. Using the bound in (12) and the upper bound in (9), we obtain for all $(\sigma_{j(T)}, j) \in \mathrm{dom}(x)$

$$\begin{aligned} |y(\sigma_{j(T)}, j)|^p &\leq c_{yV} U(0,0) + (c_{yV} k_d + c_{yd})\|d\|_p^p \\ &\leq c_{yV} c_2 |x(0,0)|^p + (c_{yV} k_d + c_{yd})\|d\|_p^p \end{aligned}$$

which implies,

$$|y(\sigma_{j(T)}, j)| \leq \sqrt[p]{c_{yV} c_2}|x(0,0)| + \sqrt[p]{(c_{yV} k_d + c_{yd})}\|d\|_p, \qquad \forall (\sigma_{j(T)}, j) \in \mathrm{dom}(x)$$

therefore according to Definition 2, we have

$$\|y\|_\infty \leq \sqrt[p]{c_{yV} c_2}|x(0,0)| + \sqrt[p]{(c_{yV} k_d + c_{yd})}\|d\|_p,$$

namely the finite $\mathcal{L}_{p,\infty}$ gain of system (5) is upper-bounded by $\sqrt[p]{(c_{yV} k_d + c_{yd})}$. ■

We illustrate now an application of Proposition 1 to the following linear hybrid system

$$\begin{cases} \dot{x} = A_f x + B_{fd} d, & (x,d) \in \mathcal{C} \\ x^+ = A_g x + B_{gd} d, & (x,d) \in \mathcal{D} \\ y = Hx + Ld \end{cases} \tag{18a}$$

$$\begin{aligned} \mathcal{C} &= \left\{ (x,d) : \begin{bmatrix} x \\ d \end{bmatrix}' M_C \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \\ \mathcal{D} &= \left\{ (x,d) : \begin{bmatrix} x \\ d \end{bmatrix}' M_D \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \end{aligned} \tag{18b}$$

with $A_f, A_g, M_C, M_D \in \Re^{n\times n}$, $B_{fd}, B_{gd}, \in \Re^{n\times m}$, $H \in \Re^{q\times n}$ and $L \in \Re^{q\times m}$, $M_C = M_C'$ and $M_D = M_D'$.

The following corollary of Proposition 1 allows one to compute an upper-bound of the $\mathcal{L}_2$ gain of system (18) using a quadratic storage function V.

Corollary 1 *If there exist a positive-semidefinite symmetric matrix $P \in \Re^{n\times n}$, non-negative scalars γ, τ_C and τ_D*

satisfying the next linear matrix inequalities

$$\left[\begin{array}{cc|c} A_f'P+PA_f & PB_{fd} & H' \\ B_{fd}'P & -\gamma I & L' \\ \hline H & L & -\gamma I \end{array}\right] + \left[\begin{array}{c|c} \tau_C M_C & 0 \\ \hline 0 & 0 \end{array}\right] \leq 0; \tag{19a}$$

$$\left[\begin{array}{cc|c} A_g'PA_g-P & A_g'PB_{gd} & H' \\ B_{gd}'PA_g & -\gamma I+B_{gd}'PB_{gd} & L' \\ \hline H & L & -\gamma I \end{array}\right] + \left[\begin{array}{c|c} \tau_D M_D & 0 \\ \hline 0 & 0 \end{array}\right] \leq 0 \tag{19b}$$

then (9)-(11) *are satisfied with* $p=2$*, and* $V(x)=x'Px$ *and the finite input-output* $\mathcal{L}_2$ *gain of* (18) *is bounded by* γ*. Moreover, if* $P>0$ *then* (12) *holds with* $c_{yV}=\frac{|H|^2}{\lambda_{\min}(P)}$*,* $c_{yd}=|L|^2$ *and* (18a) *is* $\mathcal{L}_{p,\infty}$ *stable from* d *to* y*. Finally, if* $P>0$ *and the inequalities in* (19) *are strict, then the origin of* (18) *is GES.*

Proof. Applying the Schur complement to (19a) we arrive at

$$\begin{bmatrix} A_f'P+PA_f & PB_{fd} \\ B_{fd}'P & -\gamma I \end{bmatrix} - \frac{1}{\gamma}\begin{bmatrix} H' \\ L' \end{bmatrix}\begin{bmatrix} H & L \end{bmatrix} + \tau_C M_C \leq 0, \tag{20}$$

recalling that $y=\begin{bmatrix} H & L \end{bmatrix}\begin{bmatrix} x \\ d \end{bmatrix}$, and from the $\mathcal{S}$-procedure, we obtain the following quadratic inequality, which is implied by (20)

$$\langle 2Px, A_f x + B_{fd} d\rangle + \frac{1}{\gamma}|y|^2 - \gamma|d|^2 \leq 0 \ \forall (x,d)\in\mathcal{C}.$$

where $\mathcal{C}$ is given in (18b). The above inequality is equivalent to (10) with the choices $p=2$, $V(x)=x'Px$, $k_{yf}=\frac{1}{\gamma}$, $k_{dg}=\gamma$. Similarly, we have that (19b) implies (11) with $k_{yg}=\frac{1}{\gamma}$, $k_{dg}=\gamma$. Since we have a quadratic V, (9) holds with $c_2=\lambda_{\max}(P)$ and by Proposition 1, the finite input-output gain is given by γ.

The fact that $P>0$ implies (12) follows from $x'Px\geq\lambda_{\min}(P)|x|^2$ and then $\mathcal{L}_{p,\infty}$ stability follows from Proposition 1.

The proof of GES when inequalities (19) hold strictly arises from the fact that considering $d=0$ one gets from the upper left blocks of (19) that the function $V(x)=x'Px$ is a strict Lyapunov function for the hybrid system and [27, Theorem 1] applies. ■

Remark 6 Corollary 1 combines the bounded real lemma for continuous- and for discrete-time systems to obtain a bound of the $\mathcal{L}_2$ gain for hybrid systems when using hybrid norms. In [34, Theorem 2] and [1, Theorem 13], similar results were stated for the special case of systems with dwell time (which ensures unbounded hybrid domains in the ordinary time direction). In that case, since only continuous-time norms are used, it is necessary that d do not affect the jump and flow sets and the jump map, and (19b) reduces to $A_g'PA_g-P+\tau_D M_D\leq 0$. ⌟

Remark 7 Corollary 1 corresponds to the selection of a quadratic V, $k_{yf}=k_{yg}=\gamma^{-1}$, and $k_{df}=k_{dg}=\gamma$ in (10)-(11), and noticing the convexity of the arising conditions. It is emphasized that the selection above for the scalars is not conservative for the gain estimation, due to the following.
• Assume that (10)-(11) are satisfied by different values of k_{df} and k_{dg} or k_{yf} and k_{yg} providing the gain estimate $\gamma_p=\sqrt[p]{\frac{k_d}{k_y}}$ with $k_d=\max\{k_{df},k_{dg}\}$ and $k_y=\min\{k_{yf},k_{yg}\}$. Then, (10)-(11) are clearly also satisfied with k_d replacing both k_{df} and k_{dg} and with k_y replacing both k_{yf} and k_{yg}. Moreover the same gain estimate is obtained by these new inequalities.
• We showed above that it is not conservative to use k_d, k_y. For the quadratic case, it is also not conservative to have $k_d=\frac{1}{k_y}$. Indeed, assume that (19) are satisfied with $\gamma_y=\frac{1}{k_y}$ on the $(2,2)$ elements of the two leftmost matrices and with $\gamma_d=k_d$ on the $(3,3)$ elements. Then the same estimate can be established by setting $k_d=\frac{1}{k_y}=\gamma=\sqrt{\gamma_y\gamma_d}$

on all the $(2,2)$ and $(3,3)$ diagonal terms and selecting $P^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}P$, $\tau_C^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}\tau_C$ and $\tau_D^\star = \sqrt{\frac{\gamma_y}{\gamma_d}}\tau_D$, which satisfy both inequalities. ⌟

Remark 8 While the reasoning of Remark 7 shows no conservativeness in the selection of $k_{df} = k_{dg} = k_{yf}^{-1} = k_{yg}^{-1} = \gamma$, there is however conservativeness of the $\mathcal{L}_2$ gain estimate due to the fact that we restrict V to be quadratic. Indeed, unlike the linear continuous- and discrete-time cases where the bounded real lemma is known to give the exact gain, for the hybrid case this is not true and nonconvex Lyapunov function are needed in general [2], even to assess exponential stability. See, e. g., [5] where sum-of-squares Lyapunov functions are used with homogeneous hybrid systems), or [34, Section 4.1] where a simple example fitting (18a) is given for which no quadratic storage function exists. This conservativeness is also well illustrated in Example 1 provided at the end of Section 4. ⌟

4 Homogeneous systems

In this section, using the results of Section 3, we establish finite gain $\mathcal{L}_p$ stability properties of (5) under a homogeneity assumption for the system without disturbances given by

$$\begin{cases} \dot{x} = f(x,0), & x \in \mathcal{C}_0 \\ x^+ = g(x,0), & x \in \mathcal{D}_0 \end{cases} \tag{21}$$

where $\mathcal{C}_0$ and $\mathcal{D}_0$ are suitable projected versions of the sets $\mathcal{C}$ and $\mathcal{D}$ on the direction of the state x, satisfying the following assumption.

Assumption 2 (Flow and jump sets) *The sets $\mathcal{C}_0$ and $\mathcal{D}_0$ are closed and there exist scalars L_C and L_D such that for all $(x,d) \in \mathbb{R}^{n+m}$*

$$(x,d) \in \mathcal{C} \Rightarrow x \in \mathcal{C}_0 + L_C\mathcal{B}(|d|) \tag{22a}$$
$$(x,d) \in \mathcal{D} \Rightarrow x \in \mathcal{D}_0 + L_D\mathcal{B}(|d|) \tag{22b}$$

where $\mathcal{C}_0 \subset \mathbb{R}^n$ and $\mathcal{D}_0 \subset \mathbb{R}^n$ are closed sets satisfying $\mathcal{C}_0 \times \{0\} \supset (\mathbb{R}^n \times \{0\}) \cap \mathcal{C}$ and $\mathcal{D}_0 \times \{0\} \supset (\mathbb{R}^n \times \{0\}) \cap \mathcal{D}$.

A homogeneous system is defined as follows:

Definition 7 *System (21) is homogeneous of degree zero if given any scalar $\lambda > 0$, we have*

$$\begin{aligned} f(\lambda x, 0) &= \lambda f(x,0), \ \forall x \in \mathcal{C}_0 \\ g(\lambda x, 0) &= \lambda g(x,0), \ \forall x \in \mathcal{D}_0 \end{aligned} \tag{23}$$

$$\begin{aligned} x \in \mathcal{C}_0 &\Rightarrow \lambda x \in \mathcal{C}_0 \\ x \in \mathcal{D}_0 &\Rightarrow \lambda x \in \mathcal{D}_0. \end{aligned} \tag{24}$$

Remark 9 From (22b) and (22b) we have that $\mathcal{C}_0$ and $\mathcal{D}_0$ must respectively contain the flow and jump sets projected on the space of x, that is, the set of points x such that $(x,0) \in \mathcal{C}$ and $(x,0) \in \mathcal{D}$, respectively. Conic flow and jump sets of the form considered in (18b) of the previous section are homogeneous of degree zero but, in general, (22) do not hold for these sets. The reason why is that the quadratic dependence on d in (18b) can, in some cases, prevent the existence of a linear bound as in (22). For example, consider the case $M_c = diag\,(M_x, -I)$ and note that the condition in the first inequality of (18b) becomes $x'M_x x \geq |d|^2$ for which (22) can not possibly hold. Despite the above limitation, if the conic flow and jump sets are defined as

$$\mathcal{C} = \{(x,d) : (x + S_C d)' M_C (x + S_C d) \geq 0\} \tag{25a}$$
$$\mathcal{D} = \{(x,d) : (x + S_D d)' M_D (x + S_D d) \geq 0\} \tag{25b}$$

with $M_C,\ M_D \in \mathbb{R}^{n\times n}$, $S_C,\ S_D \in \mathbb{R}^{n\times m}$, then (22) holds with $L_C = |S_C|$ and $L_D = |S_D|$. This is the case for the sets $\mathcal{C}$ and $\mathcal{D}$ corresponding to Example 1 presented below. ⌟

The following assumption states some properties for the jump and flow maps in (5).

Assumption 3 (Flow and jump maps) *Consider systems* (5) *and* (21) *satisfying Assumption 2. There exist two positive constants L_{df} and L_{dg} such that for all $d \in \mathbb{R}^m$,*

$$|f(z+v,d) - f(z,0)| \leq L_{df}|d|, \ \forall z \in \mathcal{C}_0, |v| \leq L_C|d| \tag{26a}$$

$$|g(z+v,d) - g(z,0)| \leq L_{dg}|d|, \ \forall z \in \mathcal{D}_0, |v| \leq L_D|d| \tag{26b}$$

Remark 10 If $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are globally Lipschitz in both arguments, then (26) hold. However (26) correspond to weaker assumptions as they are required to hold only in some particular subsets of the state space and with $|v|$ upper-bounded by some function of $|d|$. ⌟

The following theorem states equivalent properties for homogeneous systems which satisfy the above assumptions.

Theorem 1 *If Assumptions 1, 2 and 3 hold, and system* (5) *is homogeneous of degree zero in the sense of Definition 7, then the following statements are equivalent:*

(1) The origin of (5) *with $d = 0$, (namely* (21)*) is (locally) asymptotically stable;*
(2) The origin of (5) *with $d = 0$, (namely* (21)*) is globally exponentially stable;*
(3) System (5) *is finite gain exponentially ISS from d to x;*
(4) for each $p \in [1, +\infty)$, there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth in $\mathbb{R}^n \setminus \{0\}$ and positive constants c_1, c_2, L_1, L_2, μ and $\nu \in [0, 1)$ such that

$$c_1|x|^p \leq V(x) \leq c_2|x|^p, \ \forall x \in \mathbb{R}^n \tag{27a}$$
$$\langle \nabla V(x), f(x,0)\rangle \leq -\mu V(x), \forall x \in \mathcal{C}_0 \setminus \{0\}, \tag{27b}$$
$$V(g(x,0)) \leq \nu V(x), \quad \forall x \in \mathcal{D}_0, \tag{27c}$$
$$|\nabla V(x)| \leq L_1|x|^{p-1}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{27d}$$
$$|V(x+v) - V(x)| \leq 2L_1\left(|x|^{p-1}|v| + |v|^p\right), \tag{27e}$$
$$|\nabla V(x+v) - \nabla V(x)| \leq L_2\left(|x|^{p-2}|v| + |v|^{p-1}\right) \tag{27f}$$
$$\forall x, v \text{ such that } x \neq 0 \text{ and } x + v \neq 0.$$

Moreover if there exist positive scalars c_{hx} and c_{hd} such that $|h(x,y)| \leq c_{hx}|x| + c_{hd}|d|$, $\forall (x,d) \in \mathcal{C} \cup \mathcal{D}$ then each one of the above properties implies that for each $p \in [1, +\infty)$, $\exists V$ satisfying (9)-(12) *and therefore system* (5) *is finite gain $\mathcal{L}_p$ stable and finite gain $\mathcal{L}_{p,\infty}$ stable from d to y.*

Before proving Theorem 1, we discuss an example which illustrates the use of Corollary 1 to estimate the input-output $\mathcal{L}_2$-gain for a homogeneous hybrid system of the form (18) and we illustrate the conservativeness of Corollary 1 (see Remark 8) by way of Theorem 1.

Example 1 This example is inspired by [22, Example 1] where we have introduced disturbances $d = [d_u \ d_y]'$ acting both at the plant input (d_u) and at the measured plant output (d_y). Following [22], the hybrid closed-loop can be written as

$$\begin{aligned} &\begin{cases} \dot{x}_p = a_p x_c + b_p(x_c + d_u) \\ \dot{x}_c = a_c x_c - y \end{cases} \quad (x,d) \in \mathcal{C} \\ &\begin{cases} x_p^+ = x_p \\ x_c^+ = -\kappa_M y \end{cases} \quad (x,d) \in \mathcal{D} \\ &y = x_p + d_y \end{aligned} \tag{28a}$$

$$\begin{aligned} \mathcal{C} &= \left\{ (x,d) : \begin{bmatrix} y \\ x_c \end{bmatrix}' \begin{bmatrix} a_p + b_p\kappa_M & b_p \\ b_p & 0 \end{bmatrix} \begin{bmatrix} y \\ x_c \end{bmatrix} \geq 0 \right\} \\ \mathcal{D} &= \overline{\Re^4 \setminus \mathcal{C}}. \end{aligned} \tag{28b}$$

Notice that the disturbance d_u only affects the flow map while disturbance d_y also affects the jump map and the flow and jump sets. Therefore the results in [34] and [1] can not be used to estimate the gain from the disturbance d_y to the output y. Thanks to the structure of the flow and jump sets, which can be written as in (25), Theorem 1 applies to this system. System (28) can be written in the form (18) using the following values:

$$
\begin{aligned}
A_f &= \begin{bmatrix} a_p & b_p \\ a_c & -1 \end{bmatrix}, \; A_g = \begin{bmatrix} 1 & 0 \\ -\kappa_M & 0 \end{bmatrix}, \\
B_{fd} &= \begin{bmatrix} b_p & 0 \\ 0 & 0 \end{bmatrix}, \; B_{gd} = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa_M \end{bmatrix}, \; M_D = -M_C, \\
M_C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_p + b_p\kappa_M & b_p \\ b_p & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
H &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned}
$$

In [22], it is shown that, with $a_c = 0$, $\kappa_M = 1$ and $b_p = 1$, the system is GES for all $a_p \in (-\infty, 1]$. Therefore, according to Theorem 1, the system is also finite gain $\mathcal{L}_2$ stable for these values. However, the conservativeness introduced by the quadratic Lyapunov function used in Corollary 1 (see Remark 8) does not allow to compute a gain estimate for all the values of a_p in the GES range. Figure 1 shows the values of the $\mathcal{L}_2$ gain estimates computed from Corollary 1 from d_u to y and from d_y to y, for different values of a_p. Notice that the gain γ_{d_y} from d_y to y is constant and the gain γ_{d_u} from d_u to y increases with the increase of a_p up to the value of $a_p = -3$ which is a bound for the feasibility of the LMIs (19). The shaded area in Figure 1 corresponds to the set where the system is not finite-gain $\mathcal{L}_2$ stable and the vertical dashed lines correspond to the value of parameter a_p used in [22]. ⌟

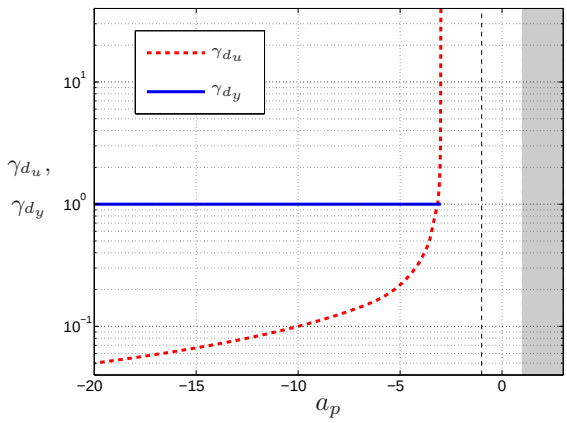

Fig. 1. Example 1: gain estimates from the disturbance inputs d_u and d_y to the output y as a function of parameter a_p.

$$
\begin{array}{ccc}
 & \text{(2) GES} & \\
\nearrow & & \searrow \\
\text{(1) LAS} & & \text{(4) } \exists V \Rightarrow \mathcal{L}_p \text{ and } \mathcal{L}_{p,\infty} \\
\nwarrow & & \swarrow \\
 & \text{(3) ISS} &
\end{array}
$$

Fig. 2. The structure of the proof of Theorem 1.

Remark 11 The proof of Theorem 1 is carried out by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1), and that (4) $\Rightarrow$ $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ properties. Its structure is also graphically shown in Figure 2 ⌟

Proof of Theorem 1. **(1) $\Rightarrow$ (2)**. The proof is similar to the proof of [20, Theorem 7]. Consider a scalar $r > 0$ such that the basin of attraction of the origin $\mathcal{B}_\circ$ contains $\{x : |x| \leq r\}$. Consider now any initial condition $x(0,0) \neq 0$ and denote by $z(t,j)$ the response from the initial condition:

$$z(0,0) := \min\left\{\frac{r}{|x(0,0)|}, 1\right\} x(0,0)$$

From the system's homogeneity assumption, given any solution $x(\cdot,\cdot)$ of (21) and any $\lambda > 0$, $\lambda x(\cdot,\cdot)$ is a solution and, since $z(0,0) \in \mathcal{B}_\circ$ it follows that

$$\begin{aligned} |x(t,j)| &= \max\left\{\frac{|x(0,0)|}{r}, 1\right\} |z(t,j)| \\ &\leq \max\left\{\frac{|x(0,0)|}{r}, 1\right\} \beta(\min\{|x(0,0)|, r\}, t, j) \\ &=: \beta_G(|x(0,0)|, t, j) \end{aligned} \tag{29}$$

where β is the $\mathcal{KLL}$ function arising from local asymptotic stability. Therefore β_G establishes a global $\mathcal{KLL}$ bound on the x response.

Note that (21) satisfies [30, Standing Assumption 1] and from homogeneity, we have $\Gamma_\lambda x = \lambda x$ and $\delta = 0$ in [30, Definition 5]. Moreover, the bound (29) establishes a $\mathcal{KLL}$ bound in the sense of [30, Definition 1] with $\omega(x) = |x|$. Then, all the assumptions of [30, Theorem 2] with $\theta = \kappa = 1$ and $\vartheta = \omega$ are satisfied and by [30, Proposition 3, item 2] and [30, Proposition 4] there exist positive scalars m_f, ℓ_f, respectively m_j, ℓ_j, such that, using $m = \max\{m_f, m_j\}$ and $\ell = \min\{\ell_f, \ell_j\}$, we have

$$|x(t,j)| \leq m|x(0,0)| \min\left\{e^{-\ell t}, e^{-\ell j}\right\}, \ \forall (t,j) \in \mathrm{dom}(x).$$

Then, since $t \geq j$ implies $e^{-\ell t} \leq e^{-\ell\left(\frac{t}{2}+\frac{j}{2}\right)}$ and $t \leq j$ implies $e^{-\ell j} \leq e^{-\ell\left(\frac{t}{2}+\frac{j}{2}\right)}$, we get

$$\begin{aligned} |x(t,j)| &\leq m|x(0,0)| \min\left\{e^{-\ell t}, e^{-\ell j}\right\} \\ &\leq m|x(0,0)| e^{-\frac{\ell}{2}(t+j)} \end{aligned} \tag{30}$$

$\forall (t,j) \in \mathrm{dom}(x)$ for all solutions to (5) with $d = 0$.

(2) $\Rightarrow$ (4). First note that for (5) the results in [30, Theorem 2] apply with the dilation $\Gamma(s,x) = (sx)$, $\omega(x) = |x|$, $\theta = 1$, $\delta = 0$ and $\kappa = p$, with $p \in \mathbb{Z}_{\geq 1}$. Indeed, by Definition 7, all the properties of [30, Definition 5] and of the standing assumption in [30] hold. Then, by [30, Theorem 2] there exist positive constants $\mu > 0$ and $\nu \in (0,1)$ and a smooth function V homogeneous of degree p with respect to the dilation Γ (namely satisfying $V(\lambda x) = \lambda^p V(x)$ for all x and $\lambda > 0$) satisfying (27b) and (27c) and such that for some $\alpha_1(\cdot)$ and $\alpha_2(\cdot) \in \mathcal{K}_\infty$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n. \tag{31}$$

Proof of (27a): Consider $c_1 = \alpha_1(1)$ and $c_2 = \alpha_2(1)$. Then, since $V(\cdot)$ is homogeneous of degree p with respect to Γ, for each $x \in \mathbb{R}^n \setminus \{0\}$ we have $V(x) = V\left(|x|\frac{x}{|x|}\right) = |x|^p V\left(\frac{x}{|x|}\right)$, which, together with (31), leads to (27a).

Proof of (27d): Similar to [23] consider the directional derivative of V at any $x \neq 0$ in the direction w, with $|w| = 1$ given by

$$\langle \nabla V(x), w\rangle = \lim_{h\to 0} \frac{V(x+hw) - V(x)}{h},$$

and define the unit vector $z := \frac{x}{|x|}$ and the scalar $\bar{h} := \frac{h}{|x|}$. Then, since $V(\cdot)$ is homogeneous of degree p, we get

$$\begin{aligned} \frac{V(x+hw) - V(x)}{h} &= \frac{V\left(|x|z + |x|\bar{h}w\right) - V(|x|z)}{|x|\bar{h}} \\ &= |x|^{p-1} \frac{V\left(z + \bar{h}w\right) - V(z)}{\bar{h}}, \end{aligned}$$

therefore, since w is arbitrary, for any $x \neq 0$,

$$\nabla V(x) = |x|^{p-1} \nabla V\left(\frac{x}{|x|}\right). \tag{32}$$

Since $V(\cdot)$ is smooth, and $\{z : |z| = 1\}$ is compact, there exists $L_1 \in \mathbb{R}_{>0}$ such that $\max_{|z|=1} |\nabla V(z)| \leq L_1$, which, together with (32), implies (27d).

Proof of (27e) and (27f): Since V is smooth almost everywhere, then we can write

$$|V(x+v) - V(x)| = \left| \left(\int_0^1 \nabla V(x+tv) dt \right) v \right| \tag{33}$$

which, applying (27d) and for any set of measure zero Ω, can be upper bounded by

$$\begin{aligned} |V(x+v) - V(x)| &\leq \sup_{t \in [0,1] \setminus \Omega} |\nabla V(x+tv)||v| \\ &\leq 2L_1(|x|^{p-1} + |v|^{p-1})|v|, \end{aligned} \tag{34}$$

which implies (27e).

Consider now (27f) and follow a similar reasoning to the one above to obtain

$$\nabla V(x+v) - \nabla V(x) = \left(\int_0^1 JV(x+tv) dt \right) v \tag{35}$$

where $JV(\cdot)$ is the Jacobian of V, for which the following inequality can be derived, for some scalar $\bar{L}_2 > 0$, by exploiting homogeneity, similar to the calculations carried out in the proof of (27d) above:

$$|JV(x)| \leq \bar{L}_2 |x|^{p-2}, \forall x \neq 0. \tag{36}$$

For the case $p \geq 2$, equation (27f) is proven with $L_2 = 2\bar{L}_2$ by carrying out similar calculations carried out above to prove (27e). For the case $p = 1$ we break the analysis in two cases. Consider first $|v| \leq \frac{|x|}{2}$ and note that for all $t \in [0,1]$ we have $|x + tv| \geq \frac{|x|}{2}$. Then, using the previous bound and (35), (36), we get

$$\begin{aligned} |\nabla V(x+v) - \nabla V(x)| &\leq \sup_{t \in [0,1]} \bar{L}_2 |x+tv|^{-1} |v| \\ &\leq \bar{L}_2 \tfrac{2}{|x|} |v|. \end{aligned} \tag{37}$$

Consider now $|v| \geq \frac{|x|}{2}$, which implies $1 \leq 2|v|/|x|$. Then, also using (27d), we get

$$\begin{aligned} |\nabla V(x+v) - \nabla V(x)| &\leq |\nabla V(x+v)| + |\nabla V(x)| \leq 2L_1 \\ &\leq 4L_1 \tfrac{|v|}{|x|}. \end{aligned} \tag{38}$$

Finally, (27f) follows from (37), (38) selecting $L_2 = \max\left\{2\bar{L}_2, 4L_1\right\}$.

(4) $\Rightarrow \mathcal{L}_p, \mathcal{L}_{p,\infty}$. We prove this implication by showing that (9)–(12) hold for system (5) and then applying Proposition 1. The first relation, (9), is implied by (27a). Regarding (12), from the theorem assumption $|h(x,y)| \leq c_{hx}|x| + c_{hd}|d|$ $\forall (x,d) \in \mathcal{C} \cup \mathcal{D}$ and (27a), we have

$$\begin{aligned} |h(x,y)|^p &\leq c_{hx}^p |x|^p + c_{hd}^p |d|^p \\ &\leq \frac{c_{hx}^p}{c_1} V(x) + c_{hd}^p |d|^p \end{aligned}$$

which implies (12) with $c_{yV} = \dfrac{c_{hx}^p}{c_1}$ and $c_{yd} = c_{hd}^p$. We prove (10) and (11) below.

Proof of (10): From Assumption 2 we have that for $(x,d) \in \mathcal{C}$, x may be written as $x = z + v$ with $z \in \mathcal{C}_0$, $z \neq 0$ and v satisfying $|v| \leq 2L_C|d|$. Indeed, if $|d| = 0$, then $v = 0$ and $z = x \neq 0$ by assumption; if $|d| \neq 0$ and the only point satisfying $z \in \mathcal{C}_0 \cap x + L_C\mathcal{B}(|d|)$ is the origin, then by homogeneity of $\mathcal{C}_0$ (see (24)) there exists a nonempty set of points $z \in \mathcal{C}_0 \cap x + 2L_C\mathcal{B}(|d|)$. Then we have for all $(x,d) \in \mathcal{C}$,

$$\begin{aligned}\langle \nabla V(x), f(x,d)\rangle = \langle \nabla V(x), f(z+v,d) + f(z,0) - f(z,0)\rangle \\ &= \langle \nabla V(z+v) + \nabla V(z) - \nabla V(z), f(z,0)\rangle \\ &\quad + \langle \nabla V(x), f(z+v,d) - f(z,0)\rangle \\ &= \langle \nabla V(z), f(z,0)\rangle \\ &\quad + \langle \nabla V(z+v) - \nabla V(z), f(z,0)\rangle \\ &\quad + \langle \nabla V(x), f(z+v,d) - f(z,0)\rangle.\end{aligned} \tag{39}$$

Note also that, from the homogeneity of $f(\cdot,0)$ in (23) and the compactness of the unit ball, we have for all $z \in \mathcal{C}_0$,

$$|f(z,0)| = \left| f\left(|z|\frac{z}{|z|}, 0\right)\right| = |z| \left| f\left(\frac{z}{|z|}, 0\right)\right| \leq c_f|z|, \tag{40}$$

where c_f is finite because it is the maximum over a compact set.

Since $z \in \mathcal{C}_0$, we may use the bounds (27b), (26a) from Assumption 3 and the bounds (27d), (27f) and (40) to obtain from (39),

$$\begin{aligned}\langle \nabla V(x), f(x,d)\rangle \leq -\mu V(z) \\ &+ L_2\left(|z|^{p-2}|v| + |v|^{p-1}\right) c_f|z| \\ &+ L_1|x|^{p-1}L_{df}|d|\end{aligned} \tag{41}$$

Since $|v| \leq L_C|d|$, we get

$$\begin{aligned}\langle \nabla V(x), f(x,d)\rangle \leq -\mu V(z) \\ &+ L_2L_Cc_f|z|^{p-1}|d| + L_2L_C^{p-1}c_f|z||d|^{p-1} \\ &+ L_1L_{df}|x|^{p-1}|d|\end{aligned} \tag{42}$$

Finally, with $z = x - v$, we get $|z| \leq |x| + L_C|d|$, which implies

$$\begin{aligned}\langle \nabla V(x), f(x,d)\rangle \leq -\mu V(x-v) \\ &+ L_2L_Cc_f(|x| + L_C|d|)^{p-1}|d| \\ &\quad + L_2L_C^{p-1}c_f(|x| + L_C|d|)|d|^{p-1} \\ &+ L_1L_{df}|x|^{p-1}|d|.\end{aligned} \tag{43}$$

Consider now the first term at the right hand side of (43). Using (27e) and similar steps to (39)-(43) we get

$$\begin{aligned}-\mu V(x-v) = -\mu\left(V(x-v) + V(x) - V(x)\right) \\ &\leq -\mu V(x) + \mu|V(x) - V(x-v)| \\ &\leq -\mu V(x) + \mu 2L_1\left(|x|^{p-1}|v| + |v|^p\right) \\ &\leq -\mu V(x) + 2\mu L_1\left(|x|^{p-1}L_C|d| + L_C^p|d|^p\right).\end{aligned} \tag{44}$$

Therefore, using the lower bound in (27a), (43)-(44) imply

$$\begin{aligned}\langle \nabla V(x), f(x,d)\rangle \leq & -\mu V(x) \\ & +2\mu L_1\left(|x|^{p-1}L_C|d| + L_C^p|d|^p\right) \\ & +L_2 L_C c_f(|x|+L_C|d|)^{p-1}|d| \\ & \quad +L_2 L_C^{p-1} c_f(|x|+L_C|d|)|d|^{p-1} \\ & +L_1 L_{df}|x|^{p-1}|d| \\ \leq & -\mu c_1|x|^p + c_d|d|^p + \sum_{i=1}^{p-1}\bar{c}_i|x|^{p-i}|d|^i\end{aligned} \tag{45}$$

where $\bar{c}_i$, $i=1,\ldots,p-1$ are the coefficients of the mixed terms and c_d is the coefficient of the terms in $|d|^p$ in (43) and (44). To prove (10) from (45), we use Young's inequality to obtain an upper bound for each term $i=1,\ldots,p-1$ in the sum, as follows:

$$\bar{c}_i|x|^{p-i}|d|^i \leq \frac{1}{p}\left(\epsilon_i\bar{c}_i(p-i)|x|^p + \bar{c}_i i\epsilon_i^{-\left(\frac{p}{i}-1\right)}|d|^p\right). \tag{46}$$

By choosing

$$\epsilon_i = \frac{\mu c_1}{2}\frac{1}{(p-1)}\frac{p}{\bar{c}_i(p-i)},$$

we have

$$\langle \nabla V(x), f(x,d)\rangle \leq -\tfrac{\mu c_1}{2}|x|^p + \tilde{c}_d|d|^p, \forall (x,d)\in\mathcal{C} \tag{47}$$

where

$$\tilde{c}_d = c_d + \sum_{i=1}^{p-1}\frac{1}{p}\bar{c}_i i\epsilon_i^{-\left(\frac{p}{i}-1\right)}. \tag{48}$$

Moreover from the theorem assumption and the triangle inequality we have $-c_{hx}^p|x|^p \leq -h(x,d)^p + c_{hd}^p|d|^p$ which implies (10) with $k_{yf} = \dfrac{\mu c_1}{2c_{hx}^p}$ and $k_{yd} = \tilde{c}_d + \dfrac{c_{hd}^p}{c_{hx}^p}$ as to be proven.

Proof of (11): From Assumption 2 we have that for $(x,d)\in\mathcal{D}$, x may be written as $x = z+v$ with $z\in\mathcal{D}_0$ and v satisfying $|v|\leq L_D|d|$. Then from (27c) we have for all $(x,d)\in\mathcal{D}$,

$$\begin{aligned}V(g(x,d)) &= V(g(z+v,d)) + V(g(z,0)) - V(g(z,0)) \\ &\leq \nu V(z) + |V(g(z+v,d)) - V(g(z,0))|.\end{aligned} \tag{49}$$

Considering the difference $V(g(z+v,d)) - V(g(z,0))$, from the mean value theorem, we have that

$$V(g(z+v,d)) - V(g(z,0)) = \nabla V(\xi)'(g(z+v,d) - g(z,0))$$

with $\xi = \lambda g(z+v,d) + (1-\lambda)g(z,0)$ for some $\lambda\in[0,1]$. Then we get from (26b) and (27d)

$$\begin{aligned}&|V(g(z+v,d)) - V(g(z,0))| \\ &\leq |\nabla V(\xi)||(g(z+v,d) - g(z,0))| \\ &\leq L_1|\lambda g(z+v,d) + (1-\lambda)g(z,0))|^{p-1} \\ &\qquad\qquad |(g(z+v,d) - g(z,0))| \\ &\leq L_1 L_{dg}|d||\lambda(g(z+v,d) - g(z,0)) + g(z,0)|^{p-1} \\ &\leq L_1 L_{dg}|d|(L_{dg}|d| + |g(z,0)|)^{p-1}.\end{aligned} \tag{50}$$

From the homogeneity of $g(\cdot,0)$ in (23), we have for all $z\in\mathcal{D}_0$

$$|g(z,0)| = \left|g\left(|z|\frac{z}{|z|},0\right)\right| = |z|\left|g\left(\frac{z}{|z|},0\right)\right| \leq c_g|z|, \tag{51}$$

where c_g is finite because it is the maximum over a compact set.

Finally, using (51) and the inequality $(a+b)^j \leq (2a)^j + (2b)^j$ equation (50) implies

$$\begin{aligned} |V(g(z+v,d)) - V(g(z,0))| \leq L_1 L_{dg}|d|(L_{dg}|d| + c_g|z|)^{p-1} \\ \leq L_1 L_{dg}|d|((2L_{dg}|d|)^{p-1} + (2c_g|x-v|)^{p-1}) \\ \leq c_5|d|^p + c_6|d|(x + L_D|d|)^{p-1}. \end{aligned} \tag{52}$$

where c_5, c_6 are suitable positive constants.

Similar to (44), consider the following bound for the first term at the right hand side of (49):

$$\begin{aligned} \nu V(z) &= \nu V(x-v) \\ &\leq \nu V(x) + \nu L_1 L_D \left(|x|^{p-1} + (|x| + L_D|d|)^{p-1}\right)|d| \end{aligned} \tag{53}$$

then using (52) and (53), (49) becomes

$$\begin{aligned} V(g(x,d)) &\leq \nu V(z) + c_5|d|^p + c_6|d|(x + L_D|d|)^{p-1} \\ &\leq \tfrac{\nu+1}{2}V(x) - \tfrac{1-\nu}{2}V(x) \\ &\quad + \nu L_1 L_D \left(|x|^{p-1} + (|x| + L_D|d|)^{p-1}\right) \\ &\quad + c_5|d|^p + c_6|d|(x + L_D|d|)^{p-1}, \end{aligned} \tag{54}$$

which can be rewritten considering (27a) as

$$\begin{aligned} V(g(x,d)) \leq \tfrac{\nu+1}{2}V(x) - \tfrac{1-\nu}{2}c_1|x|^p \\ + c_d|d|^p + \sum_{i=1}^{p-1}\bar{c}_i|x|^{p-i}|d|^i, \end{aligned} \tag{55}$$

following a similar approach to (45), consider Young's inequality as in (46) and choosing

$$\epsilon_i = \frac{1-\nu}{2c_1}\frac{1}{(p-1)}\frac{p}{c_i(p-i)},$$

we obtain

$$V(g(x,d)) \leq \frac{\nu+1}{2}V(x) + \tilde{c}_d|d|^p$$

with $\tilde{c}_d$ as in (48). Finally we obtain

$$\begin{aligned} V(g(x,d)) - V(x) &\leq \frac{\nu-1}{2}V(x) + \tilde{c}_d|d|^p \\ &\leq \frac{\nu-1}{2}c_2|x|^p + \tilde{c}_d|d|^p. \end{aligned} \tag{56}$$

Bound (56) implies (11) using $-c_{hx}^p|x|^p \leq -h(x,d)^p + c_{hd}^p|d|^p$ just as in the proof of (10) above.

Based on (9)-(12), proven above, the proof of this step is completed by applying Proposition 1 with the bounds (47) and (56).

(4) ⇒ (3). Consider any solution x to (5) and the function $U(t,j) = V(x(t,j))$, $(t,j) \in \mathrm{dom}(x)$. With $p = 1$, using the first lines in (45) and (56), it follows that

$$\begin{aligned} \dot{U}(t,j) &\leq -\mu U(t,j) + \mu c_f |d(t,j)|, \qquad (57a)\\ &\qquad \text{for almost all } t \in [t_j, t_{j+1}],\\ U(t_j,j) &\leq \eta U(t_j, j-1) + c_j |d(t_j, j-1)|, \qquad (57b)\\ &\qquad \forall (t,j) \in \mathrm{dom}(x),\ j \geq 1, \end{aligned}$$

where $\eta = \frac{1-\nu}{2}$, $c_f = c_d$ and $c_j = \tilde{c}_d$. Consider now the comparison lemma in [16, Theorem 1.10.2] (see also [29, Lemma 1]) which can be applied to (57a) because $d \in \mathcal{L}_p$. We get from (57):

$$\begin{aligned} U(t,j) &\leq e^{-\mu(t-t_j)} U(t_j,j) + \mu c_f \int_{t_j}^{t} e^{-\mu(t-s)} |d(s,j)| ds,\\ &\leq e^{-\mu(t-t_j)} U(t_j,j) + c_f \|d\|_\infty, \quad \forall t \in [t_j, t_{j+1}], \qquad (58a)\\ U(t_j,j) &\leq \eta U(t_j, j-1) + c_j \|d\|_\infty,\\ &\qquad \forall (t,j) \in \mathrm{dom}(x),\ j \geq 1, \qquad (58b) \end{aligned}$$

where we have used $\int_{t_j}^{t} e^{-\mu(t-s)} ds \leq \int_0^{+\infty} e^{-\mu s} ds = \frac{1}{\mu}$ for all $t \geq t_j$.

Use now iteratively the bounds in (58) and $v \geq 0 \Rightarrow e^{-v} \leq 1$, to get the following bound for any $(t,j) \in \mathrm{dom}(x)$:

$$\begin{aligned} U(t,j) &\leq e^{-\mu(t-t_j)} U(t_j,j) + c_f \|d\|_\infty\\ &\leq e^{-\mu(t-t_j)} \left(\eta U(t_j, j-1) + c_j \|d\|_\infty\right) + c_f \|d\|_\infty\\ &\leq \eta e^{-\mu(t-t_j)} \Big(e^{-\mu(t_j - t_{j-1})} U(t_{j-1}, j-1)\\ &\quad + c_f \|d\|_\infty \Big) + (c_f + c_j) \|d\|_\infty\\ &\leq \eta e^{-\mu(t-t_{j-1})} \left(\eta U(t_{j-1}, j-2) + c_j \|d\|_\infty\right)\\ &\quad + (\eta c_f + c_f + c_j) \|d\|_\infty\\ &\ \vdots\\ &\leq \eta^{j-1} e^{-\mu(t-t_1)} \left(e^{-\mu(t_1 - t_0)} U(t_0, 0) + c_f \|d\|_\infty \right)\\ &\quad + \sum_{k=0}^{j-1} \eta^k (c_f + c_j) \|d\|_\infty\\ &\leq \eta^{j-1} e^{-\mu(t-t_0)} U(t_0, 0) + \frac{c_f + c_j}{1-\eta} \|d\|_\infty. \end{aligned}$$

Using (27a) (with $p = 1$) and the previous bound, we finally get:

$$|x(t,j)| \leq \beta(|x(0,0)|, t, j) + \gamma_\infty \|d\|_\infty,$$

where $\beta(x,t,j) = \frac{c_2 \eta^{j-1}}{c_1} e^{-\mu t} x$ is a class $\mathcal{KLL}$ function and $\gamma_\infty = \frac{c_f + c_j}{c_1(1-\eta)}$ is the finite ISS gain.

(3) ⇒ (1). With $d = 0$ in (8) we obtain a $\mathcal{KLL}$ function bound for $|x(t,j)|$ which holds in $\mathbb{R}^n$, satisfying therefore the conditions for the LAS of the origin in Definition 4. ■

5 Systems with average dwell time

In this section we focus on the special case where $h(x,d) = x$ and we show that by imposing an average dwell time condition to system (5), the results of the previous sections can be extended to the case when (10),(11) are relaxed to hold with

(i) k_{yg} possibly non positive under a direct average dwell time condition (see Section 5.1);
(ii) k_{yf} possibly non positive under a reverse average dwell time condition (see Section 5.2).

The proof technique in both cases essentially relies on building a strict Lyapunov function from a non-strict one. The restriction to the case $y = x$ is motivated by the fact that we need a strict decrease of the Lyapunov function along flows (for case (i) above) or jumps (for case (ii) above). Extensions of these results to cases where $y \neq x$ are discussed in Remark 16.

5.1 Direct Average Dwell-Time

To enforce a direct average dwell-time constraint on the hybrid system (5), we introduce a timer variable τ, solution to the following hybrid system:

$$\begin{cases} \dot{\tau} = \min\left\{\delta, \delta - \dfrac{(\tau - N)}{\zeta}\right\}, & \tau \in [0, N + \delta\zeta] \\ \tau^+ = \min\{\tau, N\} - 1, & \tau \in [1, N + \delta\zeta]\,. \end{cases} \tag{59}$$

where $\delta > 0$, $N \in \mathbb{Z}_{>0}$ and $\zeta > 0$ is a (typically small) positive scalar whose role is clarified in Remark 12 below. We now embed the hybrid system (59) into the hybrid dynamics (5) in such a way that jumps are not allowed if the average dwell-time constraints fail to hold. The resulting hybrid dynamics corresponds to (59) and

$$\begin{cases} \dot{x} = f(x, d), & (x, d, \tau) \in \mathcal{C} \times [0, N + \delta\zeta] \\ x^+ = g(x, d), & (x, d, \tau) \in \mathcal{D} \times [1, N + \delta\zeta]\,. \end{cases} \tag{60}$$

Remark 12 The average dwell-time logic (59) ensures that τ belongs to a compact set, which is useful to exploit the converse results of [30]. Alternative simpler selections for the dwell-time logic allowing τ to evolve in an unbounded set are also possible. Notice that with (59) we have

$$\dot{\tau} = \delta,\ \forall \tau \in [0, N] \tag{61a}$$

$$0 \leq \dot{\tau} \leq \delta,\ \forall \tau \in [N, N + \delta\zeta]\,. \tag{61b}$$

Moreover, as $\zeta \to 0$ we recover the formulation in [4, Proposition 1.1], where the flow map is a set-valued map at $\tau = N$. Following [4, Proposition 1.1], an average dwell-time constraint with parameters (δ, N), is satisfied by the timer dynamics (59) namely a hybrid time domain E satisfies

$$j - i \leq \delta(t - s) + N,\ \forall (t, j), (s, i) \in E,\ t + j \geq s + i$$

if and only if E is a domain of a solution to (59). ⌟

Remark 13 In [20, Section 3] a discussion about the role of temporal regularization ((59) with $N = 1$) for systems with first order reset elements is presented. In particular, it is remarked that certain non converging Zeno solutions, that exist without temporal regularization, are not allowed if the system is associated to the timer dynamics (59). ⌟

For the case $y = x$, the presence of the dwell-time logic (59) allows conditions (11) to be weakened by relaxing the assumption that $k_{yg} > 0$ and exploiting (59) to build a new Lyapunov function which depends on the timer τ and exhibits reduced decrease along flows and reduced increase across jumps.

Proposition 2 *Consider system* (5) *with* $y = x$ *and assume that there exists* V *satisfying* (9)-(12) *for positive scalars* $p \in [1, \infty)$, c_1, c_2, k_{yf}, k_{df}, k_{dg} *and a (not necessarily positive) scalar* k_{yg}. *Then for each scalar* $\lambda \geq 0$, *there*

exists a function $(x,\tau) \mapsto W(x,\tau)$ *and scalars*

$$\bar{c}_1 := c_1 \exp(-\lambda N), \bar{c}_2 := c_2 \exp(\lambda\delta\zeta) \tag{62a}$$
$$\bar{k}_{df} := k_{df} \exp(\lambda\delta\zeta), \bar{k}_{dg} := k_{dg} \exp(\lambda(\delta\zeta - 1)) \tag{62b}$$
$$\bar{k}_{xf} := k_{yf} \exp(-\lambda N) - \lambda\delta c_2 \exp(\lambda\delta\zeta) \tag{62c}$$
$$\begin{aligned}\bar{k}_{xg} := \exp(-\lambda) \min\{&\exp(\lambda(1-N))k_{yg}, \exp(\lambda\delta\zeta)k_{yg}\} \\ &+ (1-\exp(-\lambda))c_2 \exp(\lambda\delta\zeta)\end{aligned} \tag{62d}$$

satisfying

$$\bar{c}_1|x|^p \le W(x,\tau) \le \bar{c}_2|x|^p \quad \forall (x,d,\tau) \in (\mathcal{C}\cup\mathcal{D}) \times (\mathcal{T}_C \cup \mathcal{T}_D) \tag{63}$$
$$\left\langle \nabla W(x,\tau), \begin{bmatrix} f(x,d) \\ f_\tau(\tau) \end{bmatrix} \right\rangle \le -\bar{k}_{xf}|x|^p + \bar{k}_{df}|d|^p \quad \forall (x,d,\tau) \in \mathcal{C} \times \mathcal{T}_C \tag{64}$$
$$W(g(x,d), g_\tau(\tau)) - W(x,\tau) \le -\bar{k}_{xg}|x|^p + \bar{k}_{dg}|d|^p \quad \forall (x,d,\tau) \in \mathcal{D} \times \mathcal{T}_D, \tag{65}$$

where $\mathcal{T}_C := [0, N+\delta\zeta]$, $\mathcal{T}_D := [1, N+\delta\zeta]$ *and* $f_\tau(\tau)$ *and* $g_\tau(\tau)$ *are shorthand notations for the right-hand sides in* (59). *Moreover if* $\bar{k}_{xf} > 0$ *and* $\bar{k}_{xg} > 0$, *then system* (59), (60) *is finite gain* $\mathcal{L}_p$ *stable and finite gain* $\mathcal{L}_{p,\infty}$ *stable from* d *to* x *with gains* $\gamma_p = \sqrt[p]{\frac{\bar{k}_d}{\bar{k}_x}}$ *and* $\gamma_{p,\infty} \le \sqrt[p]{\frac{\bar{k}_d}{\bar{c}_1}}$ *where* $\bar{k}_d = \max\left\{\bar{k}_{df}, \bar{k}_{dg}\right\}$, $\bar{k}_x = \min\left\{\bar{k}_{xf}, \bar{k}_{xg}\right\}$.

Proof. Consider the function $\varphi(\cdot) : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, $\varphi(\tau) := \exp(\lambda(\tau - N))$. We have

$$\dot{\varphi}(\tau) = \lambda\varphi(\tau)\dot{\tau} \ge 0 \tag{66}$$

Since $0 \le \tau \le N + \delta\zeta$, we have

$$\varphi(0) = \exp(-\lambda N) \le \varphi(\tau) \le \exp(\lambda\delta\zeta) =: \varphi_M \tag{67}$$

and we also obtain

$$\varphi(\tau^+) \le \varphi(\tau - 1) = \exp(-\lambda)\,\varphi(\tau). \tag{68}$$

Consider now the function $W(x,\tau) := \varphi(\tau)V(x)$. From (9), (12) and (67), we have that (63) holds with the definitions in (62a).

For $(x,d,\tau) \in \mathcal{C} \times [0, N+\delta\zeta]$, using the bounds (10) with $h(x,d) = x$, (61a), (66) and the upper bound in (67), we obtain

$$\begin{aligned}\left\langle \nabla W(x,\tau), \begin{bmatrix} f(x,d) \\ f_\tau(\tau) \end{bmatrix} \right\rangle &= \varphi(\tau)\dot{V}(x) + \dot{\varphi}(\tau)V(x) \\ &\le \varphi(\tau)\left(-k_{yf}|x|^p + k_{df}|d|^p\right) + \lambda\varphi(\tau)\dot{\tau}c_2|x|^p \\ &\le \left(-\exp(-\lambda N)k_{yf} + \lambda\delta c_2 \exp(\lambda\delta\zeta)\right)|x|^p \\ &\qquad + \exp(\lambda\delta\zeta)k_{df}|d|^p \\ &= -\bar{k}_{xf}|x|^p + \bar{k}_{df}|d|^p,\end{aligned} \tag{69}$$

with $\bar{k}_{df}$ and $\bar{k}_{xf}$ in (62b), (62c). For $(x, d, \tau) \in \mathcal{D} \times [1, N + \delta\zeta]$, since $\varphi(\tau) \geq \varphi(1)$, we obtain, using (11) with $h(x, d) = x$, (63), (67) and the definitions in (62b), (62d),

$$\begin{aligned} W(g(x,d), g_\tau(\tau)) &= \varphi(\tau^+)V(x^+) \\ &\leq \exp(-\lambda)\varphi(\tau)(V(x) - k_{yg}|x|^p + k_{dg}|d|^p) \\ &\leq \exp(-\lambda)W(x,\tau) + \exp(-\lambda)\varphi_M k_{dg}|d|^p \\ &\quad - \exp(-\lambda)\min\{\varphi(1)k_{yg}, \varphi_M k_{yg}\}|x|^p \\ &\leq W(x,\tau) + (\exp(-\lambda) - 1)W(x,\tau) \\ &\quad + \bar{k}_{dg}|d|^p - \exp(-\lambda)\min\{\varphi(1)k_{yg}, \varphi_M k_{yg}\}|x|^p \\ &\leq W(x,\tau) + (\exp(-\lambda) - 1)c_2\varphi_M|x|^p \\ &\quad + \bar{k}_{dg}|d|^p - \exp(-\lambda)\min\{\varphi(1)k_{yg}, \varphi_M k_{yg}\}|x|^p \\ &= W(x,\tau) - \bar{k}_{xg}|x|^p + \bar{k}_{dg}|d|^p, \end{aligned} \tag{70}$$

which establishes (65). Note that $k_{yg} \geq 0$ implies that $\min\{\varphi(1)k_{yg}, \varphi_M k_{yg}\} = \varphi(1)k_{yg}$ and $k_{yg} \leq 0$ implies that $\min\{\varphi(1)k_{yg}, \varphi_M k_{yg}\} = \varphi_M k_{yg}$.

If $\bar{k}_{xf} > 0$ and $\bar{k}_{xg} > 0$, for each solution (x, τ) to (59), (60), define $U(t, j) = W(x(t, j), \tau(t, j))$ for all $(t, j) \in \mathrm{dom}(x)$ and follow the same steps as in the proof of Theorem 1, to obtain the bounds for the $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ gain of (59), (60). ■

Remark 14 Equations (62) reveal that when $\lambda = 0$, the "overline" constants pertaining to $W(\cdot, \cdot)$ coincide with those of $V(\cdot)$. As λ grows, we see from (62a) a growth and a decrease, respectively, of the upper and lower bounds on $W(\cdot, \cdot)$. From (62d), we see a desirable increase of $\bar{k}_{xg}$ as λ grows and, if $\delta\zeta < 1$, also a desirable decrease of $\bar{k}_{dg}$, shown by (62b) (note that ζ can be selected arbitrarily small without affecting the x component of the solutions, so this is always possible). This two-fold desirable effect is paid by the undesired decrease of $\bar{k}_{xf}$ in (62c). Finally, the undesired increase of $\bar{k}_{df}$ in (62b) can be made arbitrarily small by reducing ζ. With a suitable choice of λ on (62), one can optimize the $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ finite gain estimates at the end of Proposition 2, which involve $\max(\cdot)$ functions. ⌟

Proposition 2 implies $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ stability (if and) only if $\bar{k}_{xf} > 0$ and $\bar{k}_{xg} > 0$. It is then useful to check under what conditions there exists a λ inducing positive $(\bar{k}_{xg}, \bar{k}_{xf})$ in (62c)-(62d). To this aim, from (62b) we get $\bar{k}_{xf} \geq 0$ if

$$k_{yf} > \delta c_2 \lambda \exp(\lambda(\delta\zeta + N)) \tag{71}$$

which establishes an upper bound on λ and N. Moreover, from (62d) we get that $\bar{k}_{xg} > 0$ for all λ if $k_{yg} > 0$, while for $k_{yg} \leq 0$ (note that $k_{yg} \leq 0$ implies that the minimizer is the second term) we need from (62d)

$$c_2(1 - \exp(-\lambda))\exp(N + \lambda\delta\zeta) > |k_{yg}| \tag{72}$$

which establishes a lower bound on λ and N.

The next corollary highlights the special case when $k_{yg} = 0$, so that for any (N, δ, ζ), a sufficiently small positive λ satisfying (71) and (72) always exists.

Corollary 2 *Consider system* (5) *with* $y = x$ *and assume that there exists* V *satisfying* (9)-(12) *for positive scalars* $p \in [1, \infty)$, c_1, c_2, k_{yf}, k_{df}, k_{dg} *and* $k_{yg} = 0$, *then there exists a sufficiently small* $\lambda > 0$ *satisfying* (71)-(72), *namely such that the scalars in* (62) *are all positive. Hence* (59)-(60) *is finite gain* $\mathcal{L}_p$ *and* $\mathcal{L}_{p,\infty}$ *stable from* d *to* x.

Corollary 2 allows us to construct a strict Lyapunov function $(x, \tau) \mapsto W(x, \tau)$ depending on the timer τ from a non-strict Lyapunov function $x \mapsto V(x)$ independent of τ, namely to find a $W(\cdot, \cdot)$ which strictly decreases both during flows and across jumps from a $V(\cdot)$ which strictly decreases during flows and does not increase at jumps.

Remark 15 In Proposition 1 we used the upper-bound (9) to compute the $\mathcal{L}_p$ gain bound and (12) to compute the $\mathcal{L}_{p,\infty}$ gain bound. Instead, in Proposition 2 (and similarly in Proposition 3 below), to save space, we directly used

(9) and (12) to derive (64) and (65), which imply both finite $\mathcal{L}_p$ and finite $\mathcal{L}_{p,\infty}$ gains. Nevertheless, the lower bound in (63) can be removed, similar to Proposition 1, if only finite $\mathcal{L}_p$ gains are required even though, as explained in Remark 4, finite escape times may then occur. ⌟

Remark 16 When $y \neq x$, similar results to those of Proposition 2 can be derived if the flow condition (10) is strengthened to

$$\langle \nabla V(x), f(x,d) \rangle \leq -k_{Vf} V(x) - k_{yf} |h(x,d)|^p + k_{df} |d|^p, \tag{73}$$

for all $(x,d) \in \mathcal{C}$, with $k_{Vf} > 0$. Indeed, the additional term $k_{Vf} V(x)$ can be used to carry out similar calculations to those in (69), (70). Nevertheless, in this case the constants in (62) will change and will depend on k_{Vf}. We didn't address this case in this section because the bound (73) appears somewhat artificial. Moreover, the case with $y = x$ seems better motivated, e.g., by the results in [20, Theorem 7]. Similar comments apply to the next section where, for the case $y \neq x$, one could add a term $-k_{Vg} V(x)$ to the right hand side of (11). ⌟

5.2 Reverse Average Dwell-Time

Inspired by [19], we introduce a reverse average dwell-time constraint on (5) by using the timer variable τ solution to the following hybrid system:

$$\begin{cases} \dot{\tau} = 1, & \tau \in [0,T] \\ \tau^+ = \max\{0, \tau - \delta\}, & \tau \in [0,T]. \end{cases} \tag{74}$$

where $\delta > 0$ and $T > 0$. We now embed the reverse average dwell-time timer dynamics (74) into the hybrid dynamics (5) in such a way that flows are not allowed if the reverse average dwell-time constraints fail to hold. The resulting hybrid dynamics corresponds to (74) and

$$\begin{cases} \dot{x} = f(x,d), & (x,d,\tau) \in \mathcal{C} \times [0,T] \\ x^+ = g(x,d), & (x,d,\tau) \in \mathcal{D} \times [0,T]. \end{cases} \tag{75}$$

Remark 17 Following [19, Proposition 3.2], a reverse average dwell-time constraint with parameters (δ, T), is satisfied by all solutions to hybrid system (74), namely a hybrid time domain E satisfies

$$t - s \leq \delta(j - i) + T, \ \forall (t,j),(s,i) \in E, \ t + j > s + i$$

if and only if E is a domain of a solution to (74). ⌟

For the case $y = x$, the presence of the reverse average dwell-time logic (74) allows conditions (11) to be weakened by relaxing the assumption that $k_{yf} > 0$ and exploiting (74) to build a new Lyapunov function which depends on τ and exhibits reduced decrease across jumps and reduced increase along flows.

Proposition 3 *Consider system* (5) *with* $y = x$ *and assume that there exists* V *satisfying* (9)-(12) *for positive scalars* $p \in [1,\infty)$, c_1, c_2, k_{yg}, k_{df}, k_{dg} *and a (not necessarily positive) scalar* k_{yf}, *then for each scalar* $\lambda \geq 0$, *there exists a function* $(x,\tau) \mapsto W(x,\tau)$ *and scalars*

$$\bar{c}_1 := c_1 \exp(-\lambda T), \bar{c}_2 := c_2, \tag{76a}$$
$$\bar{k}_{df} := k_{df}, \bar{k}_{dg} := k_{dg} \exp(\lambda\delta), \tag{76b}$$
$$\bar{k}_{xf} := \min\{k_{yf}, \exp(-\lambda T) k_{yf}\} + \lambda \exp(-\lambda T) c_2, \tag{76c}$$
$$\bar{k}_{xg} := \exp(-\lambda(T - \delta)) k_{yg} - (\exp(\lambda\delta) - 1) c_2 \tag{76d}$$

satisfying (63)–(65) with $\mathcal{T}_C = \mathcal{T}_D := [0,T]$ *and with* $f_\tau(\tau)$ *and* $g_\tau(\tau)$ *being shorthand notations for the right-hand sides in* (74). *Moreover if* $\bar{k}_{xf} > 0$ *and* $\bar{k}_{xg} > 0$, *then system* (74), (75) *is* $\mathcal{L}_p$ *stable and* $\mathcal{L}_{p,\infty}$ *finite gain stable from* d *to* x *with gains* $\gamma_p = \sqrt[p]{\frac{\bar{k}_d}{\bar{k}_x}}$ *and* $\gamma_{p,\infty} \leq \sqrt[p]{\frac{\bar{k}_d}{\bar{c}_1}}$ *with* $\bar{k}_d = \max\{\bar{k}_{df}, \bar{k}_{dg}\}$, $\bar{k}_x = \min\{\bar{k}_{xf}, \bar{k}_{xg}\}$.

Proof. Consider the function $\varphi(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\varphi(\tau) := \exp(-\lambda\tau)$, with $\lambda \geq 0$. We have

$$\dot{\varphi}(\tau) = -\lambda \varphi(\tau) \dot{\tau} = -\lambda \varphi(\tau) \leq 0. \tag{77}$$

Since $0 \leq \tau \leq T$, we have

$$\varphi_m = \exp(-\lambda T) \leq \varphi(\tau) \leq 1, \tag{78}$$

and we also obtain

$$\begin{aligned} \varphi(\tau^+) &= \min\{1, \varphi(\tau - \delta)\} \\ &= \min\left\{\frac{1}{\varphi(\tau)}, \exp(\lambda\delta)\right\}\varphi(\tau) \\ &\leq \min\{1, \exp(\lambda\delta)\}\,\varphi(\tau) \\ &\leq \exp(\lambda\delta)\varphi(\tau). \end{aligned} \tag{79}$$

Consider now the function $W(x,\tau) := \varphi(\tau)V(x)$. From (9), (12) and (78), we have that (63) holds with the definitions in (76a).

For $(x,d,\tau) \in \mathcal{C} \times [0,T]$, using (74) and the bounds (10) with $h(x,d) = x$, (77) and the upper bound in (78), we obtain

$$\begin{aligned} \left\langle \nabla W(x,\tau), \begin{bmatrix} f(x,d) \\ f_\tau(\tau) \end{bmatrix} \right\rangle &= \varphi(\tau)\dot{V}(x) + \dot{\varphi}(\tau)V(x) \\ &\leq \varphi(\tau)\left(-k_{yf}|x|^p + k_{df}|d|^p\right) - \lambda\exp(-\lambda T)c_2|x|^p \\ &\leq \left(-\min\{k_{yf}, \exp(-\lambda T)k_{yf}\} - \lambda\exp(-\lambda T)c_2\right)|x|^p \\ &\qquad + k_{df}|d|^p \\ &= -\bar{k}_{xf}|x|^p + \bar{k}_{df}|d|^p, \end{aligned} \tag{80}$$

with $\bar{k}_{df}$ and $\bar{k}_{xf}$ in (76b), (76c). Note that $k_{yf} \geq 0$ implies that $\min\{k_{yf}, \exp(-\lambda T)k_{yf}\} = \exp(-\lambda T)k_{yf}$ and $k_{yf} \leq 0$ implies $\min\{k_{yf}, \exp(-\lambda T)k_{yf}\} = k_{yf}$.

For $(x,d,\tau) \in \mathcal{D} \times [0,T]$, we obtain using (11) with $h(x,d) = x$, (63), (78) and the definitions in (76b), (76d)

$$\begin{aligned} W(g(x,d), g_\tau(\tau)) &= \varphi(\tau^+)V(x^+) \\ &\leq \exp(\lambda\delta)\varphi(\tau)(V(x) - k_{yg}|x|^p + k_{dg}|d|^p) \\ &\leq \exp(\lambda\delta)W(x,\tau) + \exp(\lambda\delta)k_{dg}|d|^p \\ &\quad - \exp(-\lambda(T-\delta))k_{yg}|x|^p \\ &\leq W(x,\tau) + (\exp(\lambda\delta) - 1)W(x,\tau) \\ &\quad + \bar{k}_{dg}|d|^p - \exp(-\lambda(T-\delta))k_{yg}|x|^p \\ &\leq W(x,\tau) + (\exp(\lambda\delta) - 1)c_2|x|^p \\ &\quad + \bar{k}_{dg}|d|^p - \exp(-\lambda(T-\delta))k_{yg}|x|^p \\ &= W(x,\tau) - \bar{k}_{xg}|x|^p + \bar{k}_{dg}|d|^p \end{aligned} \tag{81}$$

which establishes (65).

If $\bar{k}_{xf} > 0$ and $\bar{k}_{xg} > 0$, for each solution (x,τ) to (74), (75), define $U(t,j) = W(x(t,j), \tau(t,j))$ for all $(t,j) \in \mathrm{dom}(x)$ and follow the same steps as in the proof of Theorem 1, to obtain the bounds for the $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ gain of (74), (75). ■

Remark 18 Equations (76) reveal that when $\lambda = 0$, the "overline" constants pertaining to $W(\cdot,\cdot)$ coincide with those of $V(\cdot)$. As λ increases, we see from (76a) a decrease in the lower bound on $W(\cdot,\cdot)$. From (76c), we see a desirable increase of $\bar{k}_{xf}$ as λ increases and $T > 0$ decreases. This desirable effect is paid by the undesired decrease of $\bar{k}_{xg}$ in (76d). With a suitable choice of λ, one can optimize the $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ finite gain estimates at the end of Proposition 3, which involve $\max(\cdot)$ functions. ⌟

Proposition 3 implies $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ stability (if and) only if $\bar{k}_{xf} > 0$ and $\bar{k}_{xg} > 0$. It is then useful to check under what conditions there exists a λ inducing positive $(\bar{k}_{xg}, \bar{k}_{xf})$ in (76c)-(76d). To this aim, from (76b) we get $\bar{k}_{xf} > 0$ for all $\lambda > 0$ if $k_{yf} > 0$ while if $k_{yf} \leq 0$ we get $\bar{k}_{xf} > 0$ if

$$c_2 \lambda \exp(-\lambda T) > -k_{yf} \tag{82}$$

which establishes a lower bound on λ and a lower bound on T. Moreover, from (76d) we get that $\bar{k}_{xg} > 0$ if

$$k_{yg} > \exp(\lambda(T-\delta))(\exp(\lambda\delta) - 1)c_2 \tag{83}$$

which establishes an upper bound on λ and T.

The next corollary highlights the special case when $k_{yf} = 0$, so that for any (T, δ), a sufficiently small λ satisfying (82) and (83) always exists.

Corollary 3 *Consider system* (5) *with $y = x$ and assume that there exists V satisfying* (9)-(12) *for positive scalars $p \in [1, \infty)$, c_1, c_2, k_{yg}, k_{df}, k_{dg} and $k_{yf} = 0$. Then there exists a sufficiently small λ satisfying* (82)-(83)*, namely such that the scalars in* (76) *are all positive and hence* (74)-(75) *is finite gain $\mathcal{L}_p$ and $\mathcal{L}_{p,\infty}$ stable from d to x.*

Corollary 3 allows us to construct a strict Lyapunov function $(x, \tau) \mapsto W(x, \tau)$ depending on the timer τ from a non-strict Lyapunov function $x \mapsto V(x)$ independent of τ, namely to find a $W(\cdot, \cdot)$ which strictly decreases both during flows and across jumps from a $V(\cdot)$ which strictly decreases at jumps and does not increase during flows.

5.3 Homogeneous Systems

In [20] a direct dwell-time timer dynamics is considered ((59) is defined with $N = 1$) associated to a linear system where the disturbance d is assumed to affect only the flow map. In contrast to [20, Theorem 7], the following theorem applies to systems with either direct (59)-(60) or reverse (74)-(75) average dwell-time timer dynamics satisfying the homogeneity properties introduced in Section 4, for which the disturbances affect both flow and jump maps and flow and jump sets.

Theorem 2 *If Assumptions 1, 2 and 3 hold, and system* (5) *with $y = x$ is homogeneous of degree zero in the sense of Definition 7, then the following statements are equivalent for system* (59)-(60) *[respectively* (74)-(75)*]*

(1) The origin of the x dynamics of (59)-(60) *[respectively* (74)-(75)*] with $d = 0$, is (locally) asymptotically stable;*
(2) The origin of the x dynamics of (59)-(60) *[respectively* (74)-(75)*] with $d = 0$, is globally exponentially stable;*
(3) The hybrid system is finite gain exponentially ISS from d to x;
(4) for each $p \in [1, +\infty)$, there exists a function $W : \mathbb{R}^n \times \mathcal{T} \to \mathbb{R}_{\geq 0}$ that is smooth in $(\mathbb{R}^n \setminus \{0\}) \times \mathcal{T}$ and positive constants $\tilde{c}_1, \tilde{c}_2, \tilde{\mu}, \tilde{\nu}, L_1, L_2$ such that

$$\tilde{c}_1 |x|^p \leq W(x, \tau) \leq \tilde{c}_2 |x|^p, \ \forall (x, \tau) \in \mathbb{R}^n \times \mathcal{T} \tag{84a}$$

$$\left\langle \nabla W(x, \tau), \begin{bmatrix} f(x, 0) \\ f_\tau(\tau) \end{bmatrix} \right\rangle \leq -\tilde{\mu} W(x, \tau) \tag{84b}$$
$$\forall (x, \tau) \in (\mathcal{C}_0 \setminus \{0\}) \times \mathcal{T}_C,$$

$$W(g(x, 0), g_\tau(\tau)) \leq \tilde{\nu} W(x, \tau), \tag{84c}$$
$$\forall (x, \tau) \in \mathcal{D}_0 \times \mathcal{T}_D,$$

$$|\nabla_x W(x, \tau)| \leq L_1 |x|^{p-1}, \forall (x, \tau) \in (\mathbb{R}^n \setminus \{0\}) \times \mathcal{T}, \tag{84d}$$

$$|W(x + v, \tau) - W(x, \tau)| \leq 2L_1 \left(|x|^{p-1}|v| + |v|^p \right), \tag{84e}$$

$$|\nabla_x W(x + v, \tau) - \nabla_x W(x, \tau)| \leq L_2 \left(|x|^{p-2}|v| + |v|^{p-1} \right)$$
$$\forall x, v \text{ such that } x \neq 0 \text{ and } x + v \neq 0, \tag{84f}$$

where $\mathcal{T}_C = [0, N + \delta\zeta]$ and $\mathcal{T}_D = [1, N + \delta\zeta]$ [respectively $\mathcal{T}_C = \mathcal{T}_D = [0, T]$] and $\mathcal{T} = \mathcal{T}_C \cup \mathcal{T}_D$.

Moreover, each one of the above properties implies that for each $p \in [1, +\infty)$, there exists $W(\cdot, \cdot)$ satisfying (63)–(65) *and, therefore, system* (59)-(60) *[respectively* (74)-(75)*] is finite gain $\mathcal{L}_p$ stable and finite gain $\mathcal{L}_{p,\infty}$ stable from d to x.*

Proof. The proof follows the same structure of the proof of Theorem 1. We only highlight the differences arising from the presence of the average dwell-time.

(1) $\Rightarrow$ (2). Following the same steps as in 1 $\Rightarrow$ 2 of Theorem 1, there exists $\beta_G \in \mathcal{KLL}$ establishing a global $\mathcal{KLL}$ bound on the x component of any solution to (59)-(60) [respectively (74)-(75)]. Moreover, (59)-(60) [respectively (74)-(75)] satisfies [30, Standing Assumption 1] and from the assumption of homogeneity in the sense of Definition 7, and [30, Definition 5] holds with the dilation $\Gamma_\lambda(x,\tau) = \lambda x$ and $\delta = 0$. Moreover, the function β_G discussed above establishes a $\mathcal{KLL}$ bound in the sense of [30, Definition 1] with $\omega(x,\tau) = |x|$. Then, all the assumptions of [30, Theorem 2] with $\theta = \kappa = 1$ and $\vartheta = \omega$ are satisfied and by [30, Propositions 3 and 4] the same steps as those at the end of the proof (1) $\Rightarrow$ (2) of Theorem 1 establish that there exist $m, \ell > 0$ satisfying the bound in (30).

(2) $\Rightarrow$ (4). First note that for (5) the results in [30, Theorem 2] with the dilation $\Gamma_\lambda(x,\tau) = \lambda x$, $\omega(x) = |x|$, $\theta = 1$, $\delta = 0$ and $\kappa = p$, with $p \in \mathbb{Z}_{\geq 1}$ apply. Indeed by Definition 7 all the properties of [30, Definition 5] and of the standing assumption in [30] hold.

The rest of the item is proven following the same steps as in the proof of (2) $\Rightarrow$ (4) of Theorem 1, replacing $V(x)$ by $W(x,\tau)$, μ by $\tilde{\mu}$, ν by $\tilde{\nu}$ and replacing the compact set $\{z : |z| = 1\}$ by the compact set $\{(x,\tau) : |z| = 1, \tau \in \mathcal{T}_C\}$.

(4) $\Rightarrow$ $\mathcal{L}_p, \mathcal{L}_{p,\infty}$. Following the same steps as those of (4) $\Rightarrow$ $\mathcal{L}_p, \mathcal{L}_{p,\infty}$ in Theorem 1, after replacing V by W we can prove this implication by showing that the assumptions of Proposition 2 [respectively, Proposition 3] hold for system

$$\begin{cases} \dot{x} = f(x,0), \\ \dot{\tau} = f_\tau(\tau), \end{cases} (x,\tau) \in \mathcal{C}_0 \times \mathcal{T}_C \qquad \begin{cases} x^+ = g(x,0), \\ \tau^+ = g_\tau(\tau), \end{cases} (x,\tau) \in \mathcal{D}_0 \times \mathcal{T}_D \tag{85}$$

and then the result follows from Proposition 2 [respectively, Proposition 3].

(4) $\Rightarrow$ (3) This proof follows the same steps as (4) $\Rightarrow$ (3) of Theorem 1 with $W(x(t,j),\tau(t,j))$ replacing $V(x(t,j))$ and $\tilde{\mu}$, $\tilde{\nu}$ replacing μ, ν respectively. ■

6 Conclusions

We presented results for the characterization of finite-gain $\mathcal{L}_p$ stability of hybrid systems. By deriving dissipativity inequalities related to flow and jump pairs of map and sets, we generalized to the hybrid case the $\mathcal{L}_p$ stability concepts of both continuous-time and discrete-time systems. When focusing on homogeneous hybrid systems, under suitable regularity assumptions on the jump and flow maps and sets, we proved the equivalence between LAS, GES, ISS and the existence of a storage function. Each one of these properties implies finite-gain $\mathcal{L}_p$ stability of the homogeneous system. Finally, we addressed the case in which one of the dissipativity inequalities fails to hold by incorporating timers on the hybrid dynamics so that the resulting augmented systems satisfies direct or reverse average dwell time constraints. Then with a suitable storage function depending on the timer variable it is possible to certify $\mathcal{L}_p$ stability properties and to derive similar results under weakened conditions. Future extensions of our work may involve establishing parallel results for homogeneous systems of higher orders and providing numerical tools for less conservative estimates of hybrid finite $\mathcal{L}_p$ gains.

References

[1] O. Beker, C.V. Hollot, Y. Chait, and H. Han. Fundamental properties of reset control systems. *Automatica*, 40(6):905 – 915, 2004.

[2] F. Blanchini and C. Savorgnan. Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions. *Automatica*, 44(4):1166 – 1170, 2008.

[3] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems and Control Letters*, 58(1):47 – 53, 2009.

[4] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems part II: (pre)asymptotic stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

[5] F. Forni and A.R. Teel. Stability for a class of homogeneous hybrid systems by annular Lyapunov analysis. In *Proceedings of the 49th IEEE Conference on Decision and Control (CDC)*, pages 3289 – 3294, 2010.
[6] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.
[7] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: modeling, stability, and robustness.* Princeton University Press, 2012.
[8] R. Goebel and A.R. Teel. Preasymptotic stability and homogeneous approximations of hybrid dynamical systems. *SIAM Review*, 52(1):87–109, 2010.
[9] W. Hahn. *Stability of Motion.* Springer-Verlag, 1967.
[10] J.P. Hespanha. Root-mean-square gains of switched linear systems. *IEEE Transactions on Automatic Control*, 48(11):2040–2045, 2003.
[11] K. Hirata and J.P. Hespanha. L_2-induced gain analysis for a class of switched systems. In *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 2138–2143. IEEE, December 2009.
[12] K. Hirata and J.P. Hespanha. L_2-induced gain analysis of switched linear systems via finitely parametrized storage functions. In *2010 American Control Conference*, pages 4064–4069. IEEE, June 2010.
[13] K. Hirata and J.P. Hespanha. L_2-induced gains of switched systems and classes of switching signals. In *49th IEEE Conference on Decision and Control*, pages 438–442. IEEE, December 2010.
[14] A. Baños J. Carrasco and A. van der Schaft. A passivity-based approach to reset control systems stability. *Systems and Control Letters*, 59:18–24, 2010.
[15] H. K. Khalil. *Nonlinear Systems.* Prentice Hall, London, third edition, 2002.
[16] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, volume 1. Academic Press, London, 1969.
[17] D. Nešić, A. R. Teel, G. Valmorbida, and L. Zaccarian. On finite gain $\mathcal{L}_p$ stability for hybrid systems. In *Proceedings of the 4th IFAC conference on Analysis and Design of Hybrid Systems - ADHS12*, june 2012.
[18] D. Nešić and A.R. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.
[19] D. Nešić and A.R. Teel. A Lyapunov-based small gain theorem for hybrid ISS systems. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 3380–3385, 2008.
[20] D. Nešić, A.R. Teel, and L. Zaccarian. Stability and performance of SISO control systems with first order reset elements. *IEEE Transactions on Automatic Control*, 56(11):2567 – 2582, 2011.
[21] D. Nešić, L. Zaccarian, and A.R. Teel. Stability properties of reset systems. *Automatica*, 44(8):2019–2026, 2008.
[22] C. Prieur, S. Tarbouriech, and L. Zaccarian. Improving the performance of linear systems by adding a hybrid loop. In *Proceedings of the 18th IFAC world congress*, pages 6301–6306, 2011.
[23] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19(6):467 – 473, 1992.
[24] R.G. Sanfelice. Results on input-to-output and input-output-to-state stability for hybrid systems and their interconnections. In *Proceedings of the 49th CDC*, pages 2396–2401. IEEE, dec 2010.
[25] K.R. Santarelli and M.A. Dahleh. L_2 gain stability of switched output feedback controllers for a class of lti systems. *IEEE Transactions on Automatic Control*, 54(7):1504–1514, 2009.
[26] A.R. Teel. Asymptotic stability for hybrid systems via decomposition, dissipativity, and detectability. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 7419–7424, 2010.
[27] A.R. Teel, F. Forni, and L. Zaccarian. Lyapunov-based sufficient conditions for exponential stability in hybrid systems. *IEEE Trans. Aut. Cont., to appear.*, 2013.
[28] A.R. Teel and D. Nešić. Lyapunov functions for L_2 and input-to-state stability in a class of quantized control systems. In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 4542–4547. IEEE, December 2011.
[29] A.R. Teel and L. Praly. On assigning the derivative of a disturbance attenuation control Lyapunov function. *Mathematics of Control, Signals and Systems*, 13:95–124, 2000.
[30] S.E. Tuna and A.R. Teel. Homogeneous hybrid systems and a converse Lyapunov theorem. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 6235–6240, 2006.
[31] M. Vidyasagar. *Nonlinear systems analysis.* SIAM, second edition, 2002.
[32] I. Willems. Dissipative dynamical systems. I. general theory. *Arch. Rational Mech. Anal.*, 45(5):321–351, 1972.
[33] I. Willems. Dissipative dynamical systems. II. linear systems with quadratic supply rates. *Arch. Rational Mech. Anal.*, 45(5):352–393, 1972.
[34] L. Zaccarian, D. Nešić, and A.R. Teel. Analytical and numerical Lyapunov functions for SISO linear control systems with first-order reset elements. *International Journal of Robust and Nonlinear Control*, 21(10):1134–1158, 2011.
[35] G. Zames. On the input-output stability of time-varying nonlinear feedback systems, part I: conditions derived

using concepts of loop gain, conicity, and positivity. *IEEE Transactions on Automatic Control*, AC-11(2):228–238, 1966.

[36] J. Zhao and D.J. Hill. On stability, L_2-gain and H_∞ control for switched systems. *Automatica*, 44:1220–1232, 2008.