Stability Properties of Reset Systems ⋆

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Abstract

Stability properties for a class of reset systems, such as systems containing a Clegg integrator, are investigated. We present Lyapunov based results for verifying $\mathcal{L}_2$ and exponential stability of reset systems. Our results generalize the available results in the literature and can be easily modified to cover $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. Several examples illustrate that introducing resets in a linear system may reduce the $\mathcal{L}_2$ gain if the reset controller parameters are carefully tuned.

Key words: Hybrid systems, nonlinear systems, L_2 stability, disturbances

1 Introduction

Reset controllers are motivated by the so called Clegg integrator introduced in Clegg (1958), whose precise mathematical model is given in Zaccarian et al. (2005). Reset controllers allow more flexibility in controller design and they may remove some fundamental limitations in linear control Beker et al. (2001). First systematic procedures for controller design exploiting the Clegg integrator were proposed in Krishnan and Horowitz (1974), Horowitz and Rosenbaum (1975). A nice account of these results and their relation to more recent developments in reset control is given in Chait and Hollot (2002). Stability analysis of general reset systems can be found in Beker et al. (2004) where Lyapunov based conditions for asymptotic stability of general reset systems were presented. Moreover, the authors proposed computable conditions for quadratic stability based on linear matrix inequalities (LMIs).

In this paper we present Lyapunov based conditions for $\mathcal{L}_2$ stability of general reset systems. We emphasize that the same proof technique can be used to prove $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. Moreover, a similar Lyapunov condition is presented for exponential stability that generalizes the stability condition in (Beker et al., 2004, Theorem 1) in several directions. First, our results use locally Lipschitz Lyapunov functions, including piecewise quadratic Lyapunov functions, as opposed to differentiable Lyapunov functions that were used in Beker et al. (2004). Second, we use a more precise model of reset systems proposed in Zaccarian et al. (2005) that allows us to relax considerably the Lyapunov conditions. For instance, in (Beker et al., 2004, Theorem 1) the authors require the existence of a Lyapunov function that decreases along solutions of the system in the absence of resets everywhere in the state space. Our condition, on the other hand, requires such a decrease only in a smaller subset of state space. This allows us to obtain sharper stability bounds and input/output gains and as a result we obtain interesting new insights into design of reset systems with Clegg integrators and FOREs (see also Zaccarian et al. (2005)). The results of this

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paper provide a framework for analysis of exponential and input output stability of reset systems and will be useful in the development of systematic reset controller design procedures. For instance, the results of this paper are used in Zaccarian et al. (2005) to derive LMI based tools for the construction of piecewise quadratic Lyapunov functions for $\mathcal{L}_2$ and exponential stability testing of reset systems with Clegg integrators and FOREs. We believe that further such developments will be made possible using the results of this paper.

The paper is organized as follows. In Sections 2 and 3 we present respectively preliminaries and linear reset systems. Section 4 contains the main results with an example and all the proofs are given in Section 5.

2 Preliminaries

The sets of positive integers (including zero) and real numbers are respectively denoted as $\mathbb{N}_0$ and $\mathbb{R}$. Given vectors x_1, x_2 we use the notation $(x_1, x_2) := (x_1^T \; x_2^T)^T$. Given an integer $p \in [1, \infty)$ and a Lebesgue measurable function $d : [t_1, t_2] \to \mathbb{R}^d$, we use the notation $\|d[t_1, t_2]\|_2 := \left(\int_{t_1}^{t_2} |d(\tau)|^2 d\tau\right)^{\frac{1}{2}}$. If $\|d[0, \infty)\|_2$ is bounded, then we write $d \in \mathcal{L}_2$. We use the approach from Goebel et al. (2004); Goebel and Teel (2006) to define the solutions of hybrid systems. The hybrid time domain is defined as a subset of $[0, \infty) \times \mathbb{N}_0$, given as a union of finitely or infinitely many intervals $[t_i, t_{i+1}] \times \{i\}$ where the numbers $0 = t_0, t_1, \ldots,$ form a finite or infinite nondecreasing sequence. The last interval is allowed to be of the form $[t_i, T)$ with T finite or $T = +\infty$. Let two closed sets $\mathcal{F}_\xi$ and $\mathcal{J}_\xi$ be given such that $\mathcal{F}_\xi \cup \mathcal{J}_\xi = \mathbb{R}^{n_\xi}$ and continuous functions $f : \mathcal{F}_\xi \to \mathbb{R}^{n_\xi}$ and $g : \mathcal{J}_\xi \to \mathbb{R}^{n_\xi}$. A solution of the hybrid system $\xi(\cdot, \cdot)$ is a function defined on a hybrid time domain such that $\xi(\cdot, j)$ is continuously differentiable on (t_j, t_{j+1}) for each j in the domain and such that

$$\begin{aligned} \dot{\xi}(t, i) &= f_\xi(\xi(t, i)) \text{ if } \xi(t, i) \in \mathcal{F}_\xi \text{ and } t \in (t_i, t_{i+1}) \\ \xi(t_{i+1}, i+1) &= g_\xi(\xi(t_{i+1}, i)) \text{ if } \xi(t_{i+1}, i) \in \mathcal{J}_\xi \text{ and } i \in \mathbb{N}_0 \ . \end{aligned}$$

We sometimes omit the time arguments and write:

$$\dot{\xi} = f_\xi(\xi) \quad \text{if } \xi \in \mathcal{F}_\xi; \qquad \text{and} \qquad \xi^+ \ = \ g_\xi(\xi) \quad \text{if } \xi \in \mathcal{J}_\xi \ .$$

Given (t, N) such that $t \in [t_N, t_{N+1}]$ we define:

$$\int_0^t \xi(\tau) d\tau := \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \xi(\tau, i) d\tau + \int_{t_N}^t \xi(\tau, N) d\tau \ . \tag{1}$$

Let $\epsilon \geq 0$, $M = M^T \in \mathbb{R}^{n \times n}$ and define:

$$\mathcal{F}_\epsilon := \{x \in \mathbb{R}^n : x^T M x + \epsilon x^T x \geq 0\}, \quad \mathcal{F} := \mathcal{F}_0 \tag{2}$$
$$\mathcal{J}_\epsilon := \{x \in \mathbb{R}^n : x^T M x + \epsilon x^T x \leq 0\}, \quad \mathcal{J} := \mathcal{J}_0. \tag{3}$$

3 Linear reset systems

In the sequel we concentrate on the following class of reset system models:

$$\dot{x} = Ax + Bd; \quad \dot{\tau} = 1 \qquad \text{if } x \in \mathcal{F} \text{ or } \tau \leq \rho \tag{4}$$
$$x^+ = A_R x; \quad \tau^+ = 0 \qquad \text{if } x \in \mathcal{J} \text{ and } \tau \geq \rho \tag{5}$$
$$y \ = \ Cx \ , \tag{6}$$

where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^{n_d}$, $\tau \geq 0$ and $\rho > 0$. The role of the variable τ is to achieve "time regularization" in the sense of Johansson et al. (1999) in order to avoid Zeno solutions. Indeed, it is obvious that the reset times satisfy $t_{i+1} - t_i \geq \rho$ for all $i \in \mathbb{N}_0$ and, hence, Zeno solutions cannot occur. It was shown in Zaccarian et al. (2005) that the class of models (4), (5), (6) can be used to describe general (linear) reset systems, as the following example illustrates.

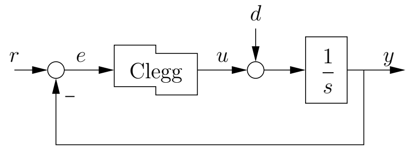

Fig. 1. Clegg integrator controlling an integrator.

Example 1 *The block diagram of the Clegg integrator controlling an integrator via a unity feedback is given in Figure 1.*

The model of the closed loop system is:

$$\dot{x}_r = r - x; \;\; \dot{x} = kx_r + d; \;\; \dot{\tau} = 1; \;\; \textit{if } (x, x_r) \in \mathcal{F} \textit{ or } \tau \le \rho$$
$$x_r^+ = 0; \;\; \tau^+ = 0; \;\; \textit{if } (x, x_r) \in \mathcal{J} \textit{ and } \tau \ge \rho \;.$$

where $\rho > 0$; $\mathcal{F}$ and $\mathcal{J}$ are defined in (2), (3) with $M = \left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$, and $\epsilon = 0$; x_r and x are respectively the (reset) controller and plant states; d and r are the disturbance and reference inputs.

Remark 1 *It is important to note the difference between our model (without disturbances) and the model in Beker et al. (2004). The model in Beker et al. (2004) has the following form:*

$$\dot{x} = A_{cl}x \quad \textit{if } x \notin \mathcal{M}; \qquad x^+ = A_R x \quad \textit{if } x \in \mathcal{M} \;, \tag{7}$$

where $\mathcal{M} := \{x : C_{cl}x = 0, \; (I - A_R)x \neq 0\}$ for some matrix $C_{cl} \in \mathbb{R}^{p \times n}$. There are three main differences between our model (4), (5) and the model (7):
1. *In the model (7) resets are only possible on the hyperplane $C_{cl}x = 0$ (as long as some flow has occurred since the last reset), whereas in our model (4), (5) resets are enforced on a sector $\mathcal{J}$.*
2. *Our model (4), (5) uses time regularization to avoid Zeno solutions whereas there is no time regularization in the model (7). Instead, (Beker et al., 2004, Theorem 1) states a result on existence of solutions for (7). Despite this result, it is not clear what they mean by solution for some states. Indeed, for the reset system (7) without disturbances it is not clear how to define solutions for the initial conditions satisfying $C_{cl}x_0 = 0$, $(I - A_R)x_0 = 0$ and where, following the differential equation for arbitrarily small times yields $Cx(t) = 0$ and $(I - A_R)x(t) \neq 0$. As an example, consider the initial condition $x_0 = (0, a, 0)$, $a > 0$ for the system with $A_{cl} = \left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{smallmatrix}\right]$, $A_R = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $C_{cl} = \left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]$. Note that we have $C_{cl}x_0 = 0$, $(I - A_R)x_0 = 0$ for the given initial condition (thus $x_0 \notin \mathcal{M}$ and the reset is not possible at $t = 0$, which means that the dynamics can only be governed by the flow equation in (7) for small $t \ge 0$). Moreover, integrating the differential equation in (7) from the same initial condition yields $C_{cl}x(t) = 0$ for all t and $(I - A_R)x(t) = \left[\begin{smallmatrix} 0 & 0 & x_3(t) \end{smallmatrix}\right]^T$, which is zero at $t = 0$ but is nonzero for all small $t > 0$ (thus $x(t) \in \mathcal{M}$ for $t > 0$ and thus flowing from the initial condition is not possible). Note that the conditions of (Beker et al., 2004, Theorem 1) hold for this example (use $V(x) = |x|^2$, which yields $\dot{V} = -2V$ and $\Delta V \le 0$).*
3. *The set $\mathcal{M}$ and its complement are not closed whereas the sets $\mathcal{F}$ and $\mathcal{J}$ are always closed. Moreover, the sets $\mathcal{M}$ and its complement are disjoint, whereas the sets $\mathcal{F}$ and $\mathcal{J}$ have a common boundary and, hence, they overlap.*

Assumption 1 *For the system (4), (5), the reset map A_R is such that $x \in \mathcal{J} \implies A_R x \in \mathcal{F}$.*

The condition in Assumption 1 is quite natural to assume for reset systems. This condition guarantees that after each reset time the solutions will be mapped to the set $\mathcal{F}$ where the dynamics are governed by the differential equation (4) so that flowing is possible from there. Without this condition, due to the time regularization, defective trajectories may correspond to solutions that keep flowing and jumping within the set $\mathcal{J}$, so that it would be impossible to establish that all solutions flow only in the set $\mathcal{F}_\epsilon$. This last property is a key tool for exploiting the advantages of resets within the Lyapunov framework, thereby establishing our main results.

Remark 2 *Before we state and prove our main results, we state some properties of the system that are immediate from (4) and (5) and Assumption 1. Consider arbitrary $t \ge 0$, then the following is true: (i) $t_{i+1} - t_i \ge \rho$ for all $i \ge 1$; (ii) $x(t_i, i) \in \mathcal{F}$ for all $i \ge 1$. We can have $x(t_0, 0) \in \mathcal{F}$ or $x(t_0, 0) \in \mathcal{J}$; (iii) If $t_{i+1} - t_i > \rho$, then $x(t, i) \in \mathcal{F}$ for all $t \in [t_i + \rho, t_{i+1}]$; (iv) There exists at most $N := \lfloor \frac{t}{\rho} \rfloor + 1$ reset times t_i on the time interval $[0, t]$ since $t_{i+1} - t_i \ge \rho$ for* *all $i \geq 1$. We always let $t_0 := 0$ and $t_N := t$ even when t_0, t_N are not reset times. It may happen that $t_1 - t_0 \in [0, \rho)$, if $\tau(t_0) > 0$ and $x(t_1, 0) \in \mathcal{J}$. In particular, it is possible that a reset occurs at $t_0 = 0$ if $x(t_0, 0) \in \mathcal{J}$ and $\tau(t_0) \geq 1$, in which case we let $t_0 = t_1$.*

4 Main results

In this section we state our main results. Sufficient $\mathcal{L}_2$ and exponential stability conditions for the system (4), (5) are presented respectively in Theorems 1 and 2.

Theorem 1 *Consider the system (4), (5), (6) with $\mathcal{F}_0$, $\mathcal{J}_0$ as in (2), (3). Suppose that Assumption 1 holds and that there exists a locally Lipschitz Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, strictly positive numbers $a_1, a_2, a_3, a_4, a_5\gamma, \epsilon$ such that the following holds for all $d \in \mathbb{R}^{n_d}$:*

$$a_1|x|^2 \ \leq \ V(x) \leq a_2|x|^2, \ \forall x \in \mathbb{R}^n; \tag{8}$$

$$\frac{\partial V(x)}{\partial x}(Ax + Bd) \leq -a_3|y|^2 + \gamma|d|^2, \text{for a.a. } x \in \mathcal{F}_\epsilon \tag{9}$$

$$V(A_R x) - V(x) \leq 0 \qquad \forall x \in \mathcal{J}; \tag{10}$$

$$\frac{\partial V(x)}{\partial x}(Ax+Bd) \leq a_4 V(x) + a_5|x||d|, \text{for a.a. } x \in \mathbb{R}^n \tag{11}$$

Then, for any $L > 1$ there exists $\rho^* > 0$ such that for all $\rho \in (0, \rho^*)$ the solutions of (4), (5), (6) satisfy:

$$\int_0^t |y(\tau)|^2 d\tau \leq \frac{La_2}{a_3}|x_0|^2 + \frac{\gamma}{a_3}\int_0^t |d(\tau)|^2 d\tau \ ,$$

for all $t \geq 0$, $\tau(0,0) = \tau_0 \geq 0$, $x(0,0) = x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_2$. In particular, we can take: $\rho^* = \min\{\rho_1^*, \rho_2^*, \rho_3^*\}$, where (note that $\varphi_1, \varphi_2, \kappa_1, \kappa_2 \in \mathcal{K}_\infty$):

$$\rho_1^* = \varphi_1^{-1}(\gamma); \ \rho_2^* = \varphi_2^{-1}\left(\frac{\epsilon}{a_2}\right); \ \rho_3^* = \varphi_1^{-1}(L-1) \tag{12}$$

$$\varphi_1(s) := \kappa_1(s) + \kappa_2(s) + \frac{|C|^2 a_3}{a_1} s(1+\kappa_1(s)+\kappa_2(s)) \tag{13}$$

$$\begin{aligned}\varphi_2(s) := {} & L_1 \frac{s}{a_1}(1 + \kappa_1(s) + \kappa_2(s)) \\ & + L_2\sqrt{\frac{s}{a_1}(1 + \kappa_1(s) + \kappa_2(s))}\end{aligned} \tag{14}$$

$$\kappa_1(s) := \exp(\alpha s)\kappa(s) + \frac{2\alpha}{\gamma_1^2}\kappa^2(s) \tag{15}$$

$$\kappa_2(s) := \exp(\alpha s)\kappa(s) + \kappa^2(s) \tag{16}$$

$$\kappa(s) := \gamma_1\sqrt{\frac{\exp(2\alpha s) - 1}{2\alpha}} \tag{17}$$

$$\alpha := \frac{a_4}{2}; \ \gamma_1 := \frac{a_5}{2\sqrt{a_1}}; \tag{18}$$

$$L_1 := |2(M + \epsilon I)A|; \ L_2 := |2(M + \epsilon I)B| \ , \tag{19}$$

where the matrix $M = M^T$ comes from (2), (3). ■

Remark 3 *A result similar to Theorem 1 can be stated for the case of $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. The conditions of Theorem 1 need to be changed slightly and the proofs modified in a straightforward manner. Moreover, similar results can be stated for nonlinear systems that are globally Lipschitz. We did not state these generalizations for simplicity.*

Remark 4 *Sufficient conditions for $\mathcal{L}_\infty$ (bounded input bounded state) stability of reset systems were presented in Beker et al. (2004) for a class of models for reset systems. Theorem 1 presents for the first time results on $\mathcal{L}_2$ stability of reset systems.*

It is instructive to note that $\mathcal{L}_p$ stability from d to x for some $p \in [1, \infty)$ implies exponential stability of the system in the absence of disturbances. Therefore, if we have an appropriate $\mathcal{L}_p$ detectability from y to x, we can conclude $\mathcal{L}_p$ stability from d to x from Theorem 1. Then, under mild technical conditions this implies exponential stability in the absence of disturbances. This result can be proved using results of Teel et al. (2002) and it is very similar to **?**. A special case of the required detectability property is when there exists $\mu > 0$ such that $\mu^2|x|^2 \leq |y|^2$. We formally state this case in the next theorem, while additional results relying on more general detectability conditions will be not covered here.

Theorem 2 *Consider the system (4), (5) without disturbances. Suppose that Assumption 1 holds and that there exists a locally Lipschitz Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, strictly positive numbers $a_1, a_2, a_3, a_4, \epsilon$ such that the following holds:*

$$a_1|x|^2 \;\leq\; V(x) \leq a_2|x|^2, \;\forall x \in \mathbb{R}^n; \tag{20}$$

$$\frac{\partial V}{\partial x}Ax \leq -a_3|x|^2, \; \textit{for almost all } x \in \mathcal{F}_\epsilon; \tag{21}$$

$$V(A_R x) - V(x) \leq 0, \;\forall x \in \mathcal{J}; \tag{22}$$

Then, there exist $\rho^*, K > 0$ such that for all $\rho \in (0, \rho^*)$ the solutions of the system (4), (5) satisfy:

$$|x(t,i)| \leq K \exp\left(-\frac{a_3}{2a_2}t\right)|x_0| \;,$$

for all $t \in [t_i, t_{i+1}]$, $i \geq 0$, $\tau(0,0) = \tau_0 \geq 0$ and $x(0,0) = x_0 \in \mathbb{R}^n$. In particular, we can take: $\rho^* = \varphi^{-1}\left(\frac{\epsilon a_1}{a_2 \cdot |2(M+\epsilon I)A|}\right)$, $K = \frac{a_2}{a_1}\exp\left(\left(|A| + \frac{a_3}{a_2}\right)\frac{\rho}{2}\right)$, where $\varphi(s) := s\exp(|A|s)$ and the matrix $M = M^T$ comes from (2), (3). ■

Remark 5 *Note that conditions (9) and (21) need to hold only on the set $\mathcal{F}_\epsilon$, which is a subset of $\mathbb{R}^n$. Moreover, the closure of $\mathcal{F}_\epsilon$ is typically a proper subset of $\mathbb{R}^n$; hence conditions (9) and (21) are much weaker than requiring stability of $\dot{x} = Ax + Bd$ that was required in (Beker et al., 2004, Theorem 1) to guarantee stability of the reset system. Hence, Theorems 1 and 2 relax the stability conditions used in Beker et al. (2004). Finally, we note that in general we cannot replace $\mathcal{F}_\epsilon$ by $\mathcal{F}$ in (9).*

Remark 6 *Our main results allow for non-differentiable Lyapunov functions $V(\cdot)$, which is another relaxation of the conditions in (Beker et al., 2004, Theorem 1), where continuous differentiability of $V(\cdot)$ was required. This generalization allows us, among other things, to consider piecewise quadratic Lyapunov functions which were not possible to handle using the results of (Beker et al., 2004, Theorem 1). It turns out that piecewise quadratic Lyapunov functions are a key tool for exploiting convex optimization tools such as LMIs when trying to obtain tight estimates of $\mathcal{L}_2$ gains for this class of systems, as illustrated in Zaccarian et al. (2005).*

Theorems 1 and 2 provide a theoretical framework for analysis and design of reset systems. A typical analysis problem consists in finding an appropriate Lyapunov function satisfying the conditions of the theorems for a given system (4), (5). Computational approaches via LMIs that use piecewise quadratic Lyapunov functions are given in Zaccarian et al. (2005). For instance, Theorem 1 can be used to prove the following result on $\mathcal{L}_2$ stability via quadratic Lyapunov functions $V(x) = x^T P x$.

Proposition 1 *(Zaccarian et al. (2005)) Consider the reset control system (4), (5), (6), where the sets $\mathcal{F}$ and $\mathcal{J}$ are defined by the matrix M via (2), (3). If the following linear matrix inequalities in the variables $P = P^T > 0$, τ_F, $\tau_R \geq 0$, $\gamma > 0$ are feasible:*

$$\begin{aligned} &\begin{bmatrix} A^TP + PA + \tau_F M & PB & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \\ &A_R^T P A_R - P - \tau_R M \leq 0, \end{aligned} \tag{23}$$

then, there exists a small enough $\rho > 0$ such that the reset system (4), (5), (6) has a finite $\mathcal{L}_2$ gain from d to y which is smaller than γ. ■

We note that using quadratic Lyapunov functions is often too restrictive for reset systems and more general theorems based on piecewise quadratic Lyapunov functions from Zaccarian et al. (2005) are often needed. In Zaccarian et al. (2005) we presented a method based on Linear Matrix Inequalities to construct piecewise quadratic Lyapunov functions to check $\mathcal{L}_2$ stability for a class of reset systems containing FOREs. We present next an example where we use results from Zaccarian et al. (2005) to analyze the $\mathcal{L}_2$ stability of systems with reset controllers. In particular, we show how changing parameters in the FORE affects the gain of the reset closed-loop system.

Example 2 *Consider an integrator (plant) controlled by a FORE whose continuous equations:*

$$\dot{x}_1 = x_2 + d;\ \dot{x}_2 = -x_1 + \beta x_2;\ \dot{\tau} = 1 \tag{24}$$

are valid on the set $x_1 x_2 \leq 0$ or $\tau \leq \rho$ and:

$$x_2^+ = 0;\ \tau^+ = 0 \tag{25}$$

are valid on the set $x_1 x_2 \geq 0$ and $\tau \geq \rho$. Assume that the output is $y = x_1$. Here, x_1 and x_2 respectively denote the state of the scalar plant and of the FORE. We computed the $\mathcal{L}_2$ gain from d to y for the system (24), (25) using the LMI method from Zaccarian et al. (2005). The gain has been computed for the limit case as $\rho \to 0$. (Larger values of ρ correspond, in general, to larger gains due to the fact that $\mathcal{F}_\epsilon$ would be larger.) The gain is plotted as a function of the parameter β that determines the pole of the FORE. This plot is represented by the dashed line in Figure 2. Moreover, we considered the linear system without resets: $\dot{x}_1 = x_2 + d$; $\dot{x}_2 = -x_1 + \beta x_2$; $y = x_1$. The full line in Figure 2 shows the $\mathcal{L}_2$ gain of the system without resets as a function of the parameter β. Note that adjusting the parameter β in the linear controller cannot produce a gain smaller than ≈ 1.5. Moreover, as β tends to zero the $\mathcal{L}_2$ gain of the linear system tends to infinity. For positive values of β the system without resets is unstable and does not have a well defined $\mathcal{L}_2$ gain. On the other hand, the $\mathcal{L}_2$ gain of (24), (25) is well defined for all values of β. Moreover, as $\beta \to \infty$ the $\mathcal{L}_2$ gain of the reset system approaches zero. This example illustrates that reset controllers may have advantages over linear controllers. Figure 3 illustrates the improved performance for increasing values of β by way of the time histories of the plant output when the system is hit with the disturbance $d(t) = \sqrt{10}$, if $t \in [0, 0.1]$, $d(t) = 0$ otherwise (whose $\mathcal{L}_2$ norm is 1). The $\mathcal{L}_2$ norm of the responses decreases as β increases and corresponds to the following values, respectively: $0.41,\ 0.31,\ 0.26,\ 0.23,\ 0.21,\ 0.1$, which are reported in Figure 2 as asterisks. These show that the LMI-based numerical results from Zaccarian et al. (2005) are not extremely conservative (note that this disturbance selection is not necessarily the worst case).

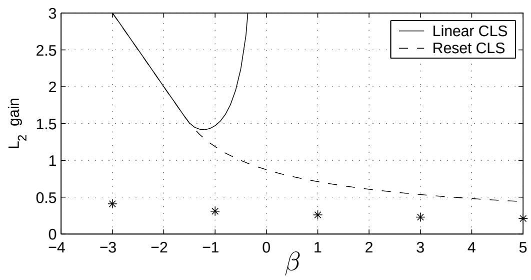

Fig. 2. $\mathcal{L}_2$ gains of linear and reset closed loops for Example 2, as a function of the pole of the FORE.

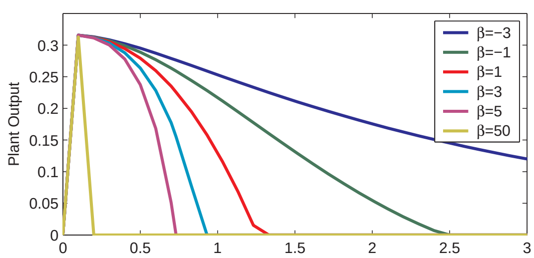

Fig. 3. Time histories for Example 2.

5 Proofs of technical results

This section contains proofs of Theorems 1 and 2. Before stating these proofs, we present three auxiliary lemmas required for the proof of Theorem 1.

Lemma 1 *Suppose that the conditions of Theorem 1 hold. Then, there exists $\rho^* > 0$ such that for all $\rho \in (0, \rho^*)$ we have that if $x(t_i, i) \in \mathcal{F}$ and $d \in \mathcal{L}_2$ then for all $t \in [t_i, t_{i+1}]$:*

$$a_3 \int_{t_i}^{t} |y(\tau, i)|^2 d\tau \leq V(x(t_i, i)) - V(x(t, i)) + \gamma \int_{t_i}^{t} |d(\tau)|^2 \, d\tau \ . \tag{26}$$

In particular, we can take $\rho^* := \min\{\rho_1^*, \rho_2^*\}$, where ρ_1^* and ρ_2^* are defined in (12).

Proof of Lemma 1: Let conditions of Lemma 1 hold. Let $\rho^* = \min\{\rho_1^*, \rho_2^*\}$ and $\rho \in (0, \rho^*)$. We consider two cases: $t \in [t_i, t_i + \rho]$ and $t \in (t_i + \rho, t_{i+1}]$.

Case 1: Suppose that $t \in [t_i, t_i + \rho]$. Using (8), (11) and the definitions in (18), we get for almost all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^{n_d}$:

$$\frac{\partial V}{\partial x}(Ax + Bd) \leq 2\alpha V + 2\gamma_1 \sqrt{V} |d| \ , \tag{27}$$

Defining $W(x) := \sqrt{V(x)}$ and using the calculations as in (Khalil, 2002, p. 271) for all $t \in [t_i, t]$ we have:

$$W(x(t, i)) \leq e^{\alpha(t - t_i)} W(x(t_i, i)) + \gamma_1 \int_{t_i}^{t} e^{\alpha(t - \tau)} |d(\tau)| d\tau. \tag{28}$$

Then, using the Hölder inequality we can write:

$$\begin{aligned}
&W(x(t, i)) \leq e^{\alpha(t - t_i)} W(x(t_i, i)) + \\
&\gamma_1 \left(\int_{t_i}^{t} e^{2\alpha(t - \tau)} d\tau \right)^{1/2} \left(\int_{t_i}^{t} |d(\tau)|^2 d\tau \right)^{1/2} \\
&= e^{\alpha(t - t_i)} W(x(t_i, i)) + \gamma_1 \left(\tfrac{e^{2\alpha(t - t_i)} - 1}{2\alpha} \right)^{1/2} \|d[t_i, t]\|_2 \\
&=: e^{\alpha(t - t_i)} W(x(t_i, i)) + \kappa(t - t_i) \|d[t_i, t]\|_2 \ .
\end{aligned}$$

Squaring this expression and using $2ab \leq a^2 + b^2$ we get:

$$\begin{aligned}
V(x(t, i)) &= W^2(x(t, i)) \\
&\leq e^{2\alpha(t - t_i)} W^2(x(t_i, i)) + \kappa^2(t - t_i) \|d[t_i, t]\|_2^2 + \\
&\qquad 2W(x(t_i, i)) e^{\alpha(t - t_i)} \kappa(t - t_i) \|d[t_i, t]\|_2 \\
&\leq \left(e^{2\alpha(t - t_i)} + e^{\alpha(t - t_i)} \kappa(t - t_i) \right) V(x(t_i, i)) \\
&\qquad + \left(\kappa^2(t - t_i) + e^{\alpha(t - t_i)} \kappa(t - t_i) \right) \|d[t_i, t]\|_2^2 \\
&= (1 + \kappa_1(t - t_i)) V(x(t_i, i)) + \kappa_2(t - t_i) \|d[t_i, t]\|_2^2 \, ,
\end{aligned} \tag{29}$$

where in the last line we used definitions (15) and (16). Integrating this expression and using (8), we can write:

$$\begin{aligned}
a_3 \int_{t_i}^{t} |y(\tau, i)|^2 d\tau \leq \frac{|C|^2 a_3 (t - t_i)}{a_1} &\Big(\kappa_2(t - t_i) \|d[t_i, t]\|_2^2 + \\
&(1 + \kappa_1(t - t_i)) V(x(t_i, i)) \Big) \ .
\end{aligned} \tag{30}$$

As a direct consequence of (29), (30) and the fact that $t - t_i \leq \rho$ and κ_1, κ_2 are strictly increasing functions, we can write that:

$$\begin{aligned} & V(x(t,i)) \leq (1+\kappa_1(\rho))V(x(t_i,i)) + \kappa_2(\rho)\left\|d[t_i,t]\right\|_2^2 \\ & a_3\int_{t_i}^{t}|y(\tau,i)|^2 d\tau \leq \\ & \quad \frac{|C|^2 a_3\rho}{a_1}\left((1+\kappa_1(\rho))V(x(t_i,i)) + \kappa_2(\rho)\left\|d[t_i,t]\right\|_2^2\right) . \end{aligned} \tag{31}$$

Adding and subtracting $\left(1 - \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right)V(x(t_i,i))$ to the first equation in (31) we obtain:

$$\begin{aligned} V(x(t,i)) \leq & \left(1 - \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right)V(x(t_i,i)) \\ & + \left(\kappa_1(\rho) + \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right)V(x(t_i,i)) \\ & + \frac{a_3|C|^2\rho}{a_1}\kappa_2(\rho)\left\|d[t_i,t]\right\|_2^2 \ . \end{aligned} \tag{32}$$

We consider two subcases: **A.** $\left\|d[t_i,t]\right\|_2^2 \geq V(x(t_i,i))$; **B.** $\left\|d[t_i,t]\right\|_2^2 \leq V(x(t_i,i))$.

Subcase A. Using (32) and the condition in Subcase A:

$$\begin{aligned} & \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}V(x(t_i,i)) \leq V(x(t_i,i)) - V(x(t,i)) \\ & + \left(\kappa_1(\rho)+\kappa_2(\rho) + \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right)\left\|d[t_i,t]\right\|_2^2 \end{aligned} \tag{33}$$

Using the second equation in (31), (33) and (13) we have:

$$\begin{aligned} a_3\int_{t_i}^{t}|y(\tau,i)|^2 d\tau \leq & \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}V(x(t_i,i)) + \\ & \frac{a_3|C|^2\rho\kappa_2(\rho)}{a_1}\left\|d[t_i,t]\right\|_2^2 \\ \leq & V(x(t_i,i)) - V(x(t,i)) + \\ & \left(\kappa_1(\rho)+\kappa_2(\rho) + \frac{a_3|C|^2\rho(1+\kappa_1(\rho)+\kappa_2(\rho))}{a_1}\right)\left\|d[t_i,t]\right\|_2^2 \\ = & V(x(t_i,i)) - V(x(t,i)) + \varphi_1(\rho)\left\|d[t_i,t]\right\|_2^2 \\ \leq & V(x(t_i,i)) - V(x(t,i)) + \gamma\left\|d[t_i,t]\right\|_2^2 \ , \end{aligned} \tag{34}$$

where the last step follows from the definition of ρ_1^* and the fact that $\rho \leq \rho_1^*$.

Subcase B. We show now that $x(t,i) \in \mathcal{F}_\epsilon$ for all $t \in [t_i, t]$ and then we obtain directly from the equations (8) and (9) that (26) holds. To this end we introduce $\chi(x) := -x^T M x - \epsilon x^T x$. Note that $\chi(x) \leq 0$ if and only if $x \in \mathcal{F}_\epsilon$. Then, we can write for all $x \in \mathbb{R}^n, d \in \mathbb{R}^{n_d}$:

$$\begin{aligned} \frac{\partial\chi}{\partial x}(Ax+Bd) &= -2(x^T M + \epsilon x^T)(Ax+Bd) \\ &\leq |2(M+\epsilon I)A||x|^2 + |2(M+\epsilon I)B||x||d| \\ &= L_1|x|^2 + L_2|x||d| \ , \end{aligned} \tag{35}$$

where we used the definitions (18). Hence, by integrating (35), we can write for all $t \in [t_i, t]$:

$$\begin{aligned} \chi(x(t,i)) - \chi(x(t_i,i)) \leq & L_1\left\|x[t_i,t]\right\|_2^2 \\ & + L_2\left\|x[t_i,t]\right\|_2\left\|d[t_i,t]\right\|_2 \ . \end{aligned} \tag{36}$$

Moreover, since by Assumption 1 $x(t_i, i) \in \mathcal{F}$, we have that $-x^T(t_i,i)Mx(t_i,i) \le 0$ and using (8) this implies $\chi(x(t_i,i)) = -\epsilon|x(t_i,i)|^2 - x^T(t_i,i)Mx(t_i,i) \le -\epsilon|x(t_i,i)|^2 \le -\frac{\epsilon}{a_2}V(x(t_i,i))$. Hence,

$$\begin{aligned}\chi(x(t,i)) \;\le\; & -\frac{\epsilon}{a_2}V(x(t_i,i)) + L_1\,\|x[t_i,t]\|_2^2 \\ & + L_2\,\|x[t_i,t]\|_2\,\|d[t_i,t]\|_2\;. \end{aligned} \tag{37}$$

Using (8) and the condition in Case B we can integrate (29) from t_i to t to write:

$$\begin{aligned}\|x[t_i,t]\|_2^2 &\le \frac{\rho}{a_1}\left((1+\kappa_1(\rho))V(x(t_i,i)) + \kappa_2(\rho)\,\|d[t_i,t]\|_2^2\right) \\ &\le \frac{\rho}{a_1}\left(1+\kappa_1(\rho)+\kappa_2(\rho)\right)V(x(t_i,i))\;.\end{aligned} \tag{38}$$

Using (38), (37), (14) and (12) we obtain:

$$\begin{aligned}\chi(x(t,i)) &\le L_1\frac{\rho}{a_1}\left(1+\kappa_1(\rho)+\kappa_2(\rho)\right)V(x(t_i,i)) \\ &- \frac{\epsilon}{a_2}V(x(t_i,i)) + L_2\sqrt{\frac{\rho}{a_1}\left(1+\kappa_1(\rho)+\kappa_2(\rho)\right)}V(x(t_i,i)) \\ &= \left(-\frac{\epsilon}{a_2} + \varphi_2(\rho)\right)V(x(t_i,i)) \\ &\le \left(-\frac{\epsilon}{a_2} + \varphi_2(\rho_2^*)\right)V(x(t_i,i)) \;\le\; 0\;.\end{aligned} \tag{39}$$

We have from (39) that $\chi(x(t,i)) \le 0$ for all $t \in [t_i, t]$ which implies that $x(t,i) \in \mathcal{F}_\epsilon, \forall t \in [t_i, t]$. This completes the proof of Subcase B and, hence, of Case 1.

Case 2: Suppose that $t \in (t_i+\rho, t_{i+1}]$. Then:

$$a_3\int_{t_i}^{t}|y(\tau,i)|^2 d\tau = a_3\int_{t_i}^{t_i+\rho}|y(\tau,i)|^2 d\tau + a_3\int_{t_i+\rho}^{t}|y(\tau,i)|^2 d\tau\;.$$

The bound on the first integral in the right hand side was already obtained in Case 1 above with $t = t_i+\rho$. Moreover, note that $x(t,i) \in \mathcal{F}$ for all $t \in [t_i+\rho, t]$ (see item (iii) of Remark 2). Using (8) and integrating (9):

$$\begin{aligned}a_3\int_{t_i+\rho}^{t}|y(\tau,i)|^2 d\tau \;\le\; & V(x(t_i+\rho,i)) - V(x(t,i)) \\ & +\gamma\int_{t_i+\rho}^{t}|d(\tau)|^2 d\tau\;.\end{aligned} \tag{40}$$

This completes the proof of Case 2. Since $t \in [t_i, t_{i+1}]$ was arbitrary, it also completes the proof of Lemma 1. ∎

Lemma 2 *Suppose that all the conditions of Theorem 1 hold. Then, for any $L > 1$ there exists $\rho^* > 0$ such that for any $t_0 \ge 0$, $\rho \in (0,\rho^*)$, $x(t_0,0) = x_0$, $\tau(t_0,0) \ge 0$ and $d \in \mathcal{L}_2$ we have that:*

$$\begin{aligned}a_3\int_{t_0}^{t}|y(\tau,0)|^2 d\tau \;\le\; & LV(x(t_0,0)) - V(x(t,0)) \\ & +\gamma\int_{t_0}^{t}|d(\tau)|^2\,d\tau\;\;\forall t \in [t_0,t_1]\;.\end{aligned} \tag{41}$$

In particular, we can take $\rho^* := \min\{\rho_1^*, \rho_3^*\}$ where ρ_1^* and ρ_3^* are defined respectively in (12) and (12).

Proof of Lemma 2: Let the conditions of Lemma 2 hold. Let $L > 1$ be arbitrary. Let $\rho^* = \min\{\rho_1^*, \rho_3^*\}$ and $\rho \in (0, \rho^*)$. We consider two cases: $t \in [t_0, t_0 + \rho]$ and $t \in (t_0 + \rho, t_1]$.

Case 1: Since possibly $x(t_0, 0) \in \mathcal{J}$, we cannot use the steps in the proof of Lemma 1. Instead, we proceed as follows. Combining (31) with (13) we obtain:

$$\begin{aligned}
& a_3 \int_{t_0}^{t} |y(\tau,0)|^2 d\tau \leq \qquad (42) \\
& \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1} V(x(t_0,0)) + \frac{a_3|C|^2\rho\kappa_2(\rho)}{a_1} \|d[t_0,t]\|_2^2 \\
& = \frac{a_3|C|^2\rho\kappa_2(\rho)}{a_1} \|d[t_0,t]\|_2^2 - \kappa_1(\rho)V(x(t_0,0)) \\
& \quad + \left(\kappa_1(\rho) + \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right) V(x(t_0,0)) \\
& \leq \left(\kappa_2(\rho) + \frac{a_3|C|^2\rho\kappa_2(\rho)}{a_1}\right) \|d[t_0,t]\|_2^2 - V(x(t,0)) \\
& \quad + \left(\kappa_1(\rho) + \frac{a_3|C|^2\rho(1+\kappa_1(\rho))}{a_1}\right) V(x(t_0,0)) \\
& \leq (\varphi_1(\rho)+1)V(x(t_0,0)) - V(x(t,0)) + \varphi_1(\rho) \|d[t_0,t]\|_2^2 \\
& \leq LV(x(t_0,0)) - V(x(t,0)) + \gamma \|d[t_0,t]\|_2^2 \ .
\end{aligned}$$

This completes the proof of Case 1. *Case 2:* Consider (40) with $i = 0$. The bound on the first integral on the right hand side is obtained directly from Case 1 with $t = t_0 + \rho$. The bound on the second integral follows directly from (8), (9) and item (iii) of Remark 2. Hence, (41) holds, which completes the proof. ∎

The proof of the following lemma follows in a straightforward manner from (10), (5) and it is omitted.

Lemma 3 *Under the conditions of Theorem 1 for any $i \geq 0$ we have that $V(x(t_{i+1}, i+1)) \leq V(x(t_{i+1}, i))$.*

Proof of Theorem 1: Consider arbitrary $x(0,0) = x_0 \in \mathbb{R}^n$, $\tau(0,0) = \tau_0 \geq 0$, $d \in \mathcal{L}_2$ and $t \geq 0$. Denote the sequence of the corresponding reset times as t_i where we also denote for convenience $t_0 := 0$ and $t_N := t$ (reset may or may not occur at t_0 and t_N - see item (iv) of Remark 2). Let $L > 1$ be arbitrary and let $\rho^* := \min\{\rho_1^*, \rho_2^*, \rho_3^*\}$ where ρ_i^* are defined in (12) and consider an arbitrary $\rho \in (0, \rho^*)$. Then, using Lemmas 1, 2 and 3 we can write:

$$\begin{aligned}
& a_3 \int_0^t y^2(\tau) d\tau = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} y^2(\tau) d\tau \\
& \leq LV(x(t_0,0)) - V(t_1,0) + \gamma \|d[t_0,t_1]\|_2^2 \\
& \quad + \sum_{i=1}^{N-1} (V(x(t_i,i)) - V(t_{i+1},i) + \gamma \|d[t_i,t_{i+1}]\|_2^2) \\
& \leq LV(x(t_0,0)) + \gamma \left(\|d[t_0,t_1]\|_2^2 + \|d[t_{N-1},t_N]\|_2^2\right) \\
& \quad + V(x(t_{N-1},N-1)) - V(x(t_N,N-1)) - V(t_1,1) \\
& \quad + \sum_{i=1}^{N-2} (V(x(t_i,i)) - V(t_{i+1},i+1) + \gamma \|d[t_i,t_{i+1}]\|_2^2) \\
& = LV(x(t_0,0)) - V(x(t_N,N-1)) + \gamma \|d[t_0,t_N]\|_2^2 \\
& \leq LV(x(t_0,0)) + \gamma \|d[t_0,t_N]\|_2^2 \\
& = LV(x(0,0)) + \gamma \|d[0,t]\|_2^2 \\
& \leq a_2 L |x(0,0)|^2 + \gamma \|d[0,t]\|_2^2 \ ,
\end{aligned}$$

which completes the proof. ∎

Proof of Theorem 2: Let all the conditions of Theorem 2 hold and let $\rho \in (0, \rho^*)$, where ρ^*, K come from the theorem. We show that

$$x(t_i, i) \in \mathcal{F} \implies x(t, i) \in \mathcal{F}_\epsilon \qquad t \in [t_i, t_{i+1}] \ . \tag{43}$$

From (20) and the fact that $|\dot{x}| \leq |A||x|$ we have for all $t \in [t_i, t_i + \rho]$, $i \geq 0$

$$V(x(t,i)) \leq \frac{a_2}{a_1} \exp\left(|A|(t - t_i)\right) V(x(t_i, i)) \tag{44}$$

$$\|x[t_i, t]\|_2^2 \leq \frac{(t - t_i)}{a_1} \exp\left(|A|(t - t_i)\right) V(x(t_i, i)) \ . \tag{45}$$

Introduce $\chi(x) := -x^T M x - \epsilon |x|^2$. Then, we have that for almost all x the following holds $\frac{\partial \chi}{\partial x} Ax \leq |2(M + \epsilon I)A||x|^2$. Integrating this expression along solutions of the system, noting that $\chi(x(t_i, i)) \leq -\epsilon |x(t_i, i)|^2 \leq -\frac{\epsilon}{a_2} V(x(t_i, i))$ and using (45), the definition of φ and the fact that $t - t_i \leq \rho \leq \rho^*$, we can write for all $t \in [t_i, t_i + \rho]$:

$$\begin{aligned} &\chi(x(t,i)) \leq \chi(x(t_i, i)) + |2(M + \epsilon I)A| \, \|x[t_i, t]\|_2^2 \\ &\leq -\tfrac{\epsilon}{a_2} V(x(t_i, i)) + \tfrac{|2(M+\epsilon I)A|}{a_1} \varphi(\rho) V(x(t_i, i)) \leq 0 \ . \end{aligned}$$

If $x(t_i, i) \in \mathcal{F}$ then $x(t, i) \in \mathcal{F}_\epsilon$ for all $t \in [t_i, t_i + \rho]$, which together with item (iii) of Remark 2 implies that (43) holds. Note that (43) and (21) imply that for all $t \geq t_1, i \in \mathbb{N}$:

$$V(x(t,i)) \leq \exp\left(-\frac{a_3}{a_2}(t - t_1)\right) V(x(t_1, 1)) \ . \tag{46}$$

We now show that a similar bound holds for $t \in [t_0, t_1]$. Using items (i) and (ii) of Remark 2 and Lemma 3 we have for all $t \in [t_0 + \rho, t_1]$:

$$\begin{aligned} V(x(t,i)) &\leq e^{-\frac{a_3}{a_2}(t - t_0 - \rho)} V(x(t_0 + \rho, 0)) \\ &\leq e^{\frac{a_3}{a_2}\rho} e^{-\frac{a_3}{a_2}(t - t_0)} V(x(t_0 + \rho, 0)) \\ &\leq e^{\left(\frac{a_3}{a_2} + |A|\right)\rho} e^{-\frac{a_3}{a_2}(t - t_0)} V(x(t_0, 0)) \\ &\leq \frac{a_1}{a_2} K^2 e^{-\frac{a_3}{a_2}(t - t_0)} V(x(t_0, 0)) \ . \end{aligned} \tag{47}$$

Moreover, from (44) we can write for all $t \in [t_0, t_0 + \rho]$:

$$\begin{aligned} V(x(t,0)) &\leq \frac{a_2}{a_1} \exp\left(|A|\rho\right) V(x(t_0, 0)) \\ &\leq \frac{a_2}{a_1} e^{\left(\frac{a_3}{a_2} + |A|\right)\rho} e^{-\frac{a_3}{a_2}(t - t_0)} V(x(t_0, 0)) \\ &= \frac{a_1}{a_2} K^2 e^{-a_3 (t - t_0)} V(x(t_0, 0)) \ . \end{aligned} \tag{48}$$

Combining (46), (47), (48), (20) and (22) completes the proof of Theorem 2. ■

6 Conclusions

We provided Lyapunov like conditions that guarantee $\mathcal{L}_2$ stability and exponential stability of a class of reset systems, such as systems containing Clegg integrators. Our results provide a theoretical framework for systematic analysis and controller design of reset systems and they generalize the corresponding results in Beker et al. (2004). An example illustrates that it is possible to improve the $\mathcal{L}_2$ gain of a linear controller by a simple introduction of resets.

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