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Stability and performance for saturated systems via quadratic and non-quadratic Lyapunov functions*

Tingshu Hu[†] Andrew R. Teel[‡] and Luca Zaccarian[§]

Abstract

In this paper we develop a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems in a general feedback configuration. The only assumptions we make about the system are well-posedness of the algebraic loop and local stability. Problems to be considered include the estimation of the domain of attraction, the reachable set under a class of bounded energy disturbances and the nonlinear $\mathcal{L}_2$ gain. The regional analysis is established through an effective treatment of the algebraic loop and the saturation/deadzone function. This treatment yields two forms of differential inclusions, a polytopic differential inclusion (PDI) and a norm-bounded differential inclusion (NDI) that contain the original system. Adjustable parameters are incorporated into the differential inclusions to reflect the regional property. The main idea behind the regional analysis is to ensure that the state remain inside the level set of a certain Lyapunov function where the PDI or the NDI is valid. With quadratic Lyapunov functions, conditions for stability and performances are derived as linear matrix inequalities (LMIs). To obtain less conservative conditions, we use a pair of conjugate non-quadratic Lyapunov functions, the convex hull quadratic function and the max quadratic function. These functions yield bilinear matrix inequalities (BMIs) as conditions for stability and guaranteed performance level. The BMI conditions cover the corresponding LMI conditions as special cases and hence the BMI results are guaranteed to always be as good as the LMI results. In most examples, the BMI results are significantly better than the LMI results.

keywords: saturation, deadzone, nonlinear $\mathcal{L}_2$ gain, reachable set, domain of attraction, Lyapunov functions.

1 Introduction

1.1 Background

Saturation is an ubiquitous nonlinearity in engineering systems and is the most studied in the literature as compared with other types of nonlinearities. Intensified efforts have been devoted to control systems with saturation since the earlier 1990s due to a few notable breakthroughs (see, e.g., [47, 35, 45]). Saturation exists in different parts of a control system, such as the actuator, the sensor, the controller and components within the plant. Most research has been devoted to addressing actuator saturation, which involves fundamental control problems such as constrained controllability and global/semi-global stabilization. These problems have been discussed in great depth, e.g., in [22, 35, 44, 45, 47, 48] (among

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which [22] considers exponentially unstable systems). Another significant problem arising from actuator saturation is anti-windup compensation, which has attracted tremendous attention over the past decade (see, e.g., [4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 28, 33, 34, 38, 39, 40, 46, 49, 51, 53]).

The approach that is adopted in most of the recent literature to address saturated systems can be categorized as a Lyapunov approach. In this approach, some quantitative measures of stability and performance, such as the size of the domain of attraction, the convergence rate, and the $\mathcal{L}_2$ gain, are characterized by using Lyapunov functions or storage functions. Then the design parameters (e.g., of a controller or of an anti-windup compensator) are incorporated into an optimization problem to optimize these quantitative measures for the closed-loop system. This approach is mostly fueled by the numerical success in solving convex optimization problems with linear matrix inequalities (LMIs) (e.g., see [2]). This is a general approach which can be applied to deal with systems with saturation and deadzone occurring at different locations. The first papers that use LMI-based methods to deal with saturated systems include [21, 41, 34], where [21, 41] consider state feedback design and [34] analyzes anti-windup systems. Since then, extensive LMI-based algorithms have been developed for analysis and design of saturated systems (see, e.g., [4, 5, 6, 10, 13, 22, 25, 26, 16, 17, 18, 38, 39, 46, 53].)

There are mainly two steps involved in the Lyapunov approach. The first step is to include the saturation function or the deadzone function in a sector so that the original system can be cast into the general framework of absolute stability, or can be described with a linear differential inclusion (LDI). The second step applies available tools from absolute stability theory or from general Lyapunov approaches for LDIs, such as the circle criterion or the LMI characterizations of stability and performance in [2]. Roughly speaking, all the analysis tools used in the aforementioned works are obtained by applying quadratic Lyapunov/storage functions to the LDIs except that [38] used a piecewise quadratic function.

Because of the two-step framework, the effectiveness of a particular method depends on how the original system is transformed into LDIs and what kind of analysis tools for LDIs are used. In many works involving anti-windup compensation, global sectors are used to describe saturation/deadzone functions. It is well known that a global sector can be very conservative for regional analysis and can only be applied when the closed-loop system is globally stable or to detect global stability. In some other works, regional LDI descriptions (some based on local sectors) are derived to reduce the conservatism (see, e.g., [4, 5, 13, 21, 10, 25, 26, 34, 41]). Along this direction, the regional LDI description introduced in [25, 26] has proved very effective and easy to manipulate. It has been used successfully for different configurations or for different purposes in [4, 5, 13, 10, 27, 28].

With an effective regional LDI description, there is yet more potential to be explored in the second step about the analysis of LDIs. It is now generally accepted that quadratic Lyapunov functions can be very conservative even for stability analysis of LDIs (see, e.g., [7, 11, 31, 54]). For this reason, considerable attention has been paid to the construction and development of non-quadratic Lyapunov functions (e.g., see [1, 3, 7, 31, 32, 37, 52, 54]).

Recently, a pair of conjugate Lyapunov functions have demonstrated great potential in the analysis of LDIs and saturated linear systems [15, 14, 23, 27]. One is called the convex hull quadratic function since its level set is the convex hull of a family of ellipsoids. The other is called max quadratic function since it is obtained by taking pointwise maximum over a family of quadratic functions and its level set is the intersection of a family of ellipsoids. Some conjugate relationships about these two functions were established in [15, 14]. Since these functions are natural extensions of quadratic functions, they can also be used to perform quantitative performance analysis beyond stability, such as to estimate the $\mathcal{L}_2$ gain, and the reachable set, for LDIs. A handful of dual bilinear matrix inequalities (BMIs) have been derived for these purposes in [14]. As compared with the corresponding LMIs resulting from quadratic Lyapunov functions, these BMIs contain extra degrees of freedom in the bilinear terms, which are injected through the non-quadratic functions. Experience with low order systems shows that these BMIs can be solved effectively with the path-following method in [20]. Although it is possible that numerical difficulties may arise for higher order systems, the great potential of these non-quadratic Lyapunov functions has been demonstrated in [15, 14, 27] through a set of numerical examples.

1.2 Problem formulation

With the recent developments and effective tools mentioned in the previous section, we are now able to address more effectively some stability and performance problems for systems with saturation/deadzone in the following general form:

$$\begin{cases} \dot{x} &= Ax + B_q q + B_w w \\ y &= C_y x + D_{yq} q + D_{yw} w \\ z &= C_z x + D_{zq} q + D_{zw} w \\ q &= \mathrm{dz}(y) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, q, y \in \mathbb{R}^m, w \in \mathbb{R}^r, z \in \mathbb{R}^p$. The deadzone function $\mathrm{dz}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as $\mathrm{dz}(y) := y - \mathrm{sat}(y)$, for all $y \in \mathbb{R}^m$, where $\mathrm{sat}(\cdot)$ is a vector saturation function with the saturation levels given by a vector $\bar{u} \in \mathbb{R}^m$, $\bar{u}_i > 0$, $i = 1, 2, \cdots, m$. In particular

$$\mathrm{sat}(u_i) = \begin{cases} \bar{u}_i, & \text{if } u_i \geq \bar{u}_i, \\ u_i, & \text{if } -\bar{u}_i \leq u_i \leq \bar{u}_i, \\ -\bar{u}_i, & \text{if } u_i \leq -\bar{u}_i. \end{cases} \qquad \mathrm{sat}(u) = \begin{bmatrix} \mathrm{sat}(u_1) \\ \vdots \\ \mathrm{sat}(u_m) \end{bmatrix}. \tag{2}$$

In this paper we consider symmetric saturation functions [1]. System (1) can be graphically depicted in block diagram form as in Fig. 1, where w is the exogenous input or disturbance and z is the output whose performance is under consideration. Many linear systems with saturation/deadzone components

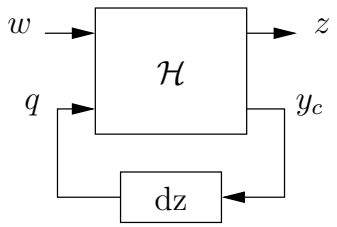

Figure 1: Compact representation of a system with saturation/deadzone.

can be transformed into the above general form through a loop transformation. This general form has been used to study anti-windup systems in [16, 34, 39, 53]. When $D_{yq} = 0$, the system does not contain an algebraic loop, which can simplify the analysis and implementation. However, it was shown in [39] that the algebraic loop can be purposely introduced into the anti-windup configuration to reduce the global $\mathcal{L}_2$ gain. The importance of the parameter D_{yq} will also be illustrated in an example at the end of this paper.

We note that most of the previous works imposed various assumptions on the system, such as exponential stability of the original open-loop plant in an anti-windup configuration (e.g., [16, 39, 53]). In these works, the global sector $[0, I]$ is used to describe the deadzone function. In some other works such as [4, 5, 6, 10, 13, 25, 26, 27, 46] (among which [6, 13] study the $\mathcal{L}_2$ gain), regional LDI descriptions are used to reduce the conservatism. In these works, the algebraic loop is absent ($D_{yq} = 0$) and the disturbance (in [6, 13]) does not enter the deadzone function, i.e., $D_{yw} = 0$. In [30], the algebraic loop has a special structure, namely, D_{yq} is diagonal.

A recent attempt was made in [51] to perform regional analysis on the general form without the assumption on stability of the open-loop plant. The main idea, which had also been suggested in some other works, was to use a smaller sector $[0, K]$ with $K < I$ to bound the deadzone function. However, this idea would not work on the general form if $D_{yw} \neq 0$. As can be seen from the second equation in

[1] Asymmetric saturations can be treated with the methods developed here with some level of conservativeness by taking $\bar{u}_i$ as the minimum absolute value of the negative and positive saturation levels.

(1), y is not necessarily bounded in $\mathcal{L}_\infty$ norm when w is only bounded in the $\mathcal{L}_2$ norm. Hence there exists no $K < I$ to bound the deadzone function even at $x = 0$. After all, as commented in [25, 27], even in the absence of w, this kind of sector description is not only hard to manipulate, but also has a much restricted degree of freedom as compared with the regional LDI description initiated in [25].

In this paper, we will extend the regional LDI description in [25] to deal with the general situation where $D_{yq} \neq 0$ and $D_{yw} \neq 0$, and to address both stability and performance issues.

The only assumptions that we will make about the system (1) is its local stability (A is Hurwitz) and the well-posedness of the algebraic loop, which will be made precise in Section 2. These were also the only assumptions made in our recent paper [28] and they are clearly basic requirements for the system to be functional.

The objective of this paper is to carry out a systematic and comprehensive analysis of system (1) by using quadratic and non-quadratic Lyapunov functions. The following problems will be addressed:

1. Estimation of the domain of attraction (in the absence of w) by using invariant ellipsoids or invariant level sets of the non-quadratic Lyapunov functions.

2. With a given bound on the $\mathcal{L}_2$ norm of w, i.e, $\|w\|_2 \leq s$ for a given s, we would like to determine a set S as small as possible so that under the condition $x(0) = 0$, we have $x(t) \in S$ for all t. This set S will be considered as an estimate of the reachable set.

3. With $\|w\|_2 \leq s$ for a given s, we would like to determine a number $\gamma > 0$ as small as possible, so that under the condition $x(0) = 0$, we have $\|z\|_2 \leq \gamma\|w\|_2$. Performing this analysis for each $s \in (0, \infty)$, we obtain an estimate of the nonlinear $\mathcal{L}_2$ gain.

To address these problems systematically, we will first provide an effective treatment of the algebraic loop and the deadzone function in Section 2. In particular, the necessary and sufficient condition for the well-posedness of the algebraic loop will be made explicit. Moreover, we will derive two forms of differential inclusions to describe the original system (1). The first one is a polytopic differential inclusion (PDI) involving a certain adjustable parameter or nonlinear function. This parameter or nonlinear function offers extra degrees of freedom associated with a local region under consideration. It will be optimized in conjunction with the Lyapunov functions in the final analysis problems. The second differential inclusion is a norm-bounded differential inclusion (NDI) which is derived from the PDI. The NDI is more conservative than the PDI but may be more numerically tractable for some cases.

In Section 3, we will apply quadratic Lyapunov functions via the PDI and the NDI to characterize stability and performance of the original system (1). We note that quadratic functions have been used for these purposes in [4, 5, 6, 10, 13, 25, 26, 46] under the assumption that $D_{yq} = 0$ and $D_{yw} = 0$. In Section 4, we apply the convex hull quadratic function and the max quadratic function respectively via the PDI (It turns out that when these nonquadratics are applied to the NDI, they produce the same results as the quadratics). In Section 5, we use a numerical example to demonstrate the effectiveness of this paper's results and the relationship between them. Section 6 concludes this paper.

Notation

- $|\cdot|_\infty$: For $u \in \mathbb{R}^m$, $|u|_\infty := \max_i |u_i|$.
- $\|\cdot\|_2$: For $u \in \mathcal{L}_2$, $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{\frac{1}{2}}$.
- $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1 + 1, \cdots, k_2\}$.
- $\mathrm{sat}(\cdot)$: The symmetric saturation function with implicit saturation level given by $\bar{u} \in \mathbb{R}^m$ (see (2)).
- $\bar{U} := \mathrm{diag}\{\bar{u}_1, \ldots, \bar{u}_m\}$ where $\bar{u}_i > 0$ is the saturation level for the ith component of $\mathrm{sat}(\cdot)$.
- $\mathrm{dz}(u)$: The deadzone function, $\mathrm{dz}(u) := u - \mathrm{sat}(u)$.
- $\mathrm{co}S$: The convex hull of a set S.
- $\mathcal{K}$: The set of diagonal matrices with 0 or 1 at each diagonal element.
- $\mathrm{He}X$: For a square matrix X, $\mathrm{He}X := X + X^T$.
- $\mathcal{E}(P)$: For $P \in \mathbb{R}^{n\times n}, P = P^T \geq 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : \ x^T P x \leq 1\}$.

- $\mathcal{L}(H)$: For $H \in \mathbb{R}^{m\times n}$, $\mathcal{L}(H) := \left\{x \in \mathbb{R}^n : \ |Hx|_\infty \le 1\right\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(\bar{U}^{-1}H)$, for a given $s > 0$, we have (see, e.g., [25]),

$$s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H) \Longleftrightarrow \begin{bmatrix} \bar{u}_\ell^2/s^2 & H_\ell \\ H_\ell^T & P \end{bmatrix} \ge 0 \quad \forall\, \ell \in I[1,m] \tag{3}$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓ-th diagonal element of $\bar{U}$.

2 Two forms of parameterized differential inclusions

Algebraic loops in linear systems can be easily solved (if they are well-posed). For system (1), the presence of the deadzone function makes the algebraic loop much harder to deal with. Theoretically, an explicit solution can be derived as a piecewise affine function, in terms of both x and w, by partitioning the vector space $\mathbb{R}^m$ into 3^m polytopic regions (see Remark 1). However, the complexity of the partition even for $m = 2$ or 3 makes the solution almost impossible to manipulate. In this paper, we would like to use convex sets to bound all the possible solutions. By doing that, we obtain differential inclusion descriptions for the original system (1) and make it more approachable with Lyapunov methods.

Recall that the deadzone function belongs to the $[0, I]$ sector, i.e., for each y there exists a diagonal $\Delta \in \mathbb{R}^{m\times m}$ satisfying $0 \le \Delta \le I$ and $\mathrm{dz}(y) = \Delta y$. Let $\mathcal{K}$ be the set of diagonal matrices whose diagonal elements are either 1 or 0. Then co$\mathcal{K}$ is the set of diagonal Δ satisfying $0 \le \Delta \le I$. There are 2^m matrices in $\mathcal{K}$ and we number them as $K_i, i = 1, 2, \cdots, 2^m$. Then we have $\mathcal{K} = \{K_i : i \in I[1, 2^m]\}$ and

$$\mathrm{dz}(y) \in \mathrm{co}\{K_i y : \ i \in I[1, 2^m]\}.$$

This relation holds for all $y \in \mathbb{R}^m$ but could be conservative over a local region where the system operates. In [25, 26], a flexible description was introduced for dealing with the saturated state feedback $\mathrm{sat}(Fx)$. This description can be easily adapted for the deadzone function. The main idea behind this description is the following simple fact:

Fact 1 *Suppose $v_i \in [-\bar{u}_i, \bar{u}_i]$ (with $\bar{u}_i$ being the ith saturation level). For any $u_i \in \mathbb{R}$, we have $\mathrm{sat}(u_i) \in \mathrm{co}\{u_i, v_i\}$, i.e., $\mathrm{sat}(u_i) = \delta u_i + (1-\delta)v_i$ for some $\delta \in [0, 1]$, and $\mathrm{dz}(u_i) \in \mathrm{co}\{0, u_i - v_i\}$, i.e., $\mathrm{dz}(u_i) = \delta(u_i - v_i)$ for some $\delta \in [0, 1]$.*

This simple fact has also been used in [13] to analyze the nonlinear $\mathcal{L}_2$ gain for a special case of (1), where D_{yq}, D_{yw}, D_{zq} and D_{zw} are all zero. For the general case where D_{yq} may be nonzero, we have the following algebraic loop,

$$y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw}w. \tag{4}$$

This algebraic loop is said to be well-posed if there exists a unique solution y for each $C_y x + D_{yw}w$. A sufficient condition for the algebraic loop to be well-posed is the existence of a diagonal matrix $W > 0$ such that $2W - D_{yq}W - WD_{yq}^T > 0$ (see, e.g., [16, 43]). In what follows, we give a precise characterization of the well-posedness of the algebraic loop.

Claim 1 *Assume that ϕ is the deadzone function or the saturation function. Then $y = D\phi(y) + v$ has a unique solution for every $v \in \mathbb{R}^m$ if and only if $\det(I - D\Delta) \ne 0$ for all $\Delta \in \mathrm{co}\mathcal{K}$.*

Proof. See the Appendix. □

Remark 1 If the algebraic loop $y = D\phi(y) + v$ is well-posed, then the solution y is a piecewise affine function of v with 3^m polytopic regions. To understand this, consider the function g: $y \mapsto v = y - D\phi(y)$. It is piecewise affine with 3^m polytopic partitions. If there is a unique solution y for each v, then each polytope in the domain of g is uniquely and affinely mapped to a polytope in the range of g. Hence the inverse function of g, i.e., the solution of the algebraic loop, is also piecewise affine, with partition corresponding to that of the original g. ◦

Based on Claim 1 we have the following criterion for the well-posedness of the algebraic loop.

Claim 2 *The algebraic loop (4) is well-posed if and only if the values of* $\det(I - D_{yq}K_i), i \in I[1, 2^m]$, *are all nonzero and have the same sign. In this case, we have*

$$\{(I - \Delta D_{yq})^{-1}\Delta : \ \Delta \in \mathrm{co}\mathcal{K}\} \subseteq \mathrm{co}\{(I - K_i D_{yq})^{-1} K_i : \ i \in I[1, 2^m]\}, \tag{5}$$

Proof. See the Appendix. □

The well-posedness condition in Claim 2 can be easily verified. The relation (5) will be used to bound the solution of the algebraic loop with a polytope.

Throughout this paper, we assume that this well-posedness condition is satisfied. For $i \in I[1, 2^m]$, denote

$$T_i = (I - K_i D_{yq})^{-1} K_i, \tag{6}$$

and

$$A_i = A + B_q T_i C_y, \quad B_i = B_w + B_q T_i D_{yw}, \quad C_i = C_z + D_{zq} T_i C_y, \quad D_i = D_{zw} + D_{zq} T_i D_{yw}.$$

Proposition 1 *Let* $h : \mathbb{R}^n \to \mathbb{R}^m$ *be a given map and let* h_ℓ *be the* ℓ*th component of* h*. Consider system (1). If* $x \in \mathbb{R}^n$ *satisfies* $|h_\ell(x)| \le \bar{u}_\ell$ *for all* $\ell \in I[1, m]$*, then*

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix} : \ i \in I[1, 2^m] \right\}. \tag{7}$$

Proof. Since $|h_\ell(x)| \le \bar{u}_\ell$ for all $\ell \in I[1, m]$, by Fact 1, we have

$$q = \mathrm{dz}(y) = \Delta(y - h(x))$$

for some $\Delta \in \mathrm{co}\mathcal{K}$. Recalling $y = C_y x + D_{yq} q + D_{yw} w$, we obtain $q = \Delta(C_y x + D_{yq} q + D_{yw} w - h(x))$. It follows that $q = (I - \Delta D_{yq})^{-1}\Delta(C_y x + D_{yw} w - h(x))$. By (5) and (6) we have

$$q \in \mathrm{co}\{T_i(C_y x + D_{yw} w - h(x)) : \ i \in I[1, 2^m]\}. \tag{8}$$

Applying this relation to the first and the third equations in (1), we obtain (7). □

By taking $h(x) = 0$ in (7), we obtain a polytopic linear differential inclusion (PLDI) representation which holds globally for the original system (1). A nonzero term $h(x)$ is used to inject additional degrees of freedom in some subset of the state space to reduce conservatism in regional analysis. When we use quadratic Lyapunov functions, we will choose $h(x) = Hx$ where H can be used as an optimizing parameter. When we use non-quadratic Lyapunov functions, a nonlinear $h(x)$ is more effective in general.

The polytopic differential inclusion (PDI) (7) involves 2^m vertices. This may present numerical difficulties when m is large (e.g., $m > 6$) and the order of the system is high. To reduce this computational burden, we may use a more conservative description; namely, to approximate the system (1) we may use a norm bounded differential inclusion (NDI), which is based on the following result.

Claim 3 *Let* M *be a positive diagonal matrix. Suppose that*

$$2I - M^{-1} D_{yq} M - M D_{yq}^T M^{-1} = S^2,$$

where S *is symmetric and nonsingular. Then*

$$\mathrm{co}\{(I - K_i D_{yq})^{-1} K_i : \ i \in I[1, 2^m]\} \subseteq \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} : \|\Omega\| \le 1\}, \tag{9}$$

where $\|\Omega\|$ *is the spectral norm of* Ω *(namely its largest singular value). Furthermore, each vertex of the lefthand side is on the boundary of the righthand side.*

Proof. See the Appendix. □

Proposition 2 *Assume that there exist a diagonal $M > 0$ and a symmetric nonsingular S such that*

$$S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1}.$$

Let $H \in \mathbb{R}^{m\times n}$ be given. For $\Omega \in \mathbb{R}^{m\times m}$, define

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} := \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} \begin{bmatrix} C_y - H & D_{yw} \end{bmatrix}.$$

Consider system (1). If $x \in \mathbb{R}^n$ satisfies $|\bar{U}^{-1}Hx|_\infty \le 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \ \|\Omega\| \le 1 \right\}. \tag{10}$$

Proposition 2 can be proved like in Proposition 1 by applying Claim 3 to (8) with $h(x) = Hx$ (note that $T_i = (I - K_i D_{yq})^{-1}K_i$). Then we obtain

$$q \in \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1}((C_y - H)x + D_{yw}w) : \ \|\Omega\| \le 1\}.$$

Applying this to the original system (1), we obtain (10). We call (10) the norm bounded differential inclusion (NDI) for (1). If $m = 1$, then the two sets in (9) are the same and the NDI is the same as the PDI. If $m > 1$, generally the NDI strictly contains the PDI. We also note that to obtain the NDI, there must exist a positive diagonal matrix M such that $2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1} > 0$, which is a stronger requirement than well-posedness.

3 Analysis with quadratic Lyapunov functions

3.1 Some general results for linear differential inclusions

In [2], extensive results were established for stability and performance analysis of LDIs by using quadratic Lyapunov functions. Consider the LDI

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi \right\}, \tag{11}$$

where Φ is a given convex set of matrices. The following lemma can be established like in the corresponding results in [2] by extending a polytopic Φ to a general Φ.

Lemma 1 *Given $P = P^T > 0, \gamma > 0$, let $V(x) = x^T Px$ and denote by $\dot{V}(x, w)$ the derivative of V in any of the directions of the right hand side of (11). The following holds:*

1. *$\dot{V}(x, w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$, if*

$$A^T P + PA < 0 \quad \forall A \in \left\{ \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} : X \in \Phi \right\}. \tag{12}$$

2. *$\dot{V}(x, w) \le w^T w$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if*

$$\mathrm{He} \begin{bmatrix} PA & PB \\ 0 & -I/2 \end{bmatrix} \le 0 \quad \forall \begin{bmatrix} A & B \end{bmatrix} \in \left\{ \begin{bmatrix} I & 0 \end{bmatrix} X : X \in \Phi \right\}.$$

3. $\dot{V}(x,w) + \frac{1}{\gamma^2} z^T z \le w^T w$ *for all* $x \in \mathbb{R}^n, w \in \mathbb{R}^r$*, if*

$$\mathrm{He}\begin{bmatrix} PA & PB & 0 \\ 0 & -I/2 & 0 \\ C & D & -\gamma^2 I/2 \end{bmatrix} \le 0 \quad \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi. \tag{13}$$

The condition in item 1 guarantees that the ellipsoid $\mathcal{E}(P)$ is contractively invariant in the absence of w. It will be used for the estimation of the domain of attraction. The condition in item 2 guarantees that if $\|w\|_2 \le s$, then under the initial condition $x(0) = 0$, we will have $x(t) \in s\mathcal{E}(P)$ for all $t \ge 0$. This will be used to determine the reachable set under a class of bounded energy disturbances. Item 3 gives a condition for γ to be a bound for the $\mathcal{L}_2$ gain, i.e., $\|z\|_2 \le \gamma \|w\|_2$ for all w and $x(0) = 0$. The result in item 3 can also be found, e.g., in [19]. For the case where Φ is a polytope, we only need to verify the conditions at its vertices.

Combining Lemma 1 with the two differential inclusion descriptions, we will obtain different methods for the analysis of the original system (1). The crucial point is to guarantee that the PDI (7) (or the NDI (10)) is valid for all time under the class of disturbances and the set of initial $x(0)$'s under consideration. We are mainly concerned about the existence of a matrix H, such that $|\bar{U}^{-1}Hx(t)|_\infty \le 1$, i.e., $x(t) \in \mathcal{L}(\bar{U}^{-1}H)$, for all t. To ensure this property, we are going to construct a quadratic function $V(x) = x^T P x$, $P = P^T > 0$, and use Lemma 1 to guarantee that $x(t) \in s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ for all $t \ge 0$.

3.2 Analysis based on the polytopic differential inclusion

When $h(x) = Hx$, the PDI (7) can be written as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i - B_q T_i H & B_i \\ C_i - D_{zq} T_i H & D_i \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : i \in I[1, 2^m] \right\}. \tag{14}$$

which corresponds to (11) with

$$\Phi = \mathrm{co}\left\{ \begin{bmatrix} A_i - B_q T_i H & B_i \\ C_i - D_{zq} T_i H & D_i \end{bmatrix} : i \in I[1, 2^m] \right\}.$$

We will restrict our attention to a certain ellipsoid $s\mathcal{E}(P)$. For the purpose of presenting the results in terms of LMIs, we state the results using $Q = P^{-1}$ and $Y = HQ$. To apply the PDI description within the ellipsoid $s\mathcal{E}(P) = s\mathcal{E}(Q^{-1})$, we need to ensure that $s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ so that $|\bar{U}^{-1}Hx|_\infty \le 1$ (i.e., $|h_\ell(x)| \le \bar{u}_\ell$ for all ℓ) for all $x \in s\mathcal{E}(P)$, which is equivalent to (recall from (3)),

$$\begin{bmatrix} \bar{u}_\ell^2/s^2 & H_\ell \\ H_\ell^T & P \end{bmatrix} \ge 0 \quad \forall \ell \in I[1, m],$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓ-th diagonal element of $\bar{U}$. Multiplying on the left and the right by $\mathrm{diag}\{1, Q\}$, we obtain the equivalent condition

$$\begin{bmatrix} \bar{u}_\ell^2/s^2 & Y_\ell \\ Y_\ell^T & Q \end{bmatrix} \ge 0 \quad \forall \ell \in I[1, m]. \tag{15}$$

Theorem 1 *Given* $Q \in \mathbb{R}^{n\times n}, Q = Q^T > 0$*. Let* $V(x) = x^T Q^{-1} x$*. Consider system (1).*

1. *If there exists* $Y \in \mathbb{R}^{m\times n}$ *satisfying (15) with* $s = 1$ *and*

$$QA_i^T + A_i Q - Y^T T_i^T B_q^T - B_q T_i Y < 0 \quad \forall i \in I[1, 2^m], \tag{16}$$

then $\dot{V}(x,w) < 0$ *for all* $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$ *and* $w = 0$*, i.e.,* $\mathcal{E}(Q^{-1})$ *is a contractively invariant ellipsoid.*

2. *Let $s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (15) and*

$$\mathrm{He}\begin{bmatrix} A_iQ - B_qT_iY & B_i \\ 0 & -I/2 \end{bmatrix} \le 0 \quad \forall\, i \in I[1, 2^m], \tag{17}$$

then $\dot V(x, w) \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3. *Let $\gamma, s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (15) and*

$$\mathrm{He}\begin{bmatrix} A_iQ - B_qT_iY & B_i & 0 \\ 0 & -I/2 & 0 \\ C_iQ - D_{zq}T_iY & D_i & -\gamma^2 I/2 \end{bmatrix} \le 0 \quad \forall\, i \in I[1, 2^m], \tag{18}$$

then $\dot V(x, w) + \frac{1}{\gamma^2} z^T z \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma\|w\|_2$.

Proof. Let $P = Q^{-1}$ and $H = YP$.

1. If we multiply (16) on the left and the right by P, we obtain $(A_i - B_qT_iH)^T P + P(A_i - B_qT_iH) < 0 \quad \forall\, i \in I[1, 2^m]$. Applying item 1 of Lemma 1 to the LDI (14), this guarantees that $\dot V(x, w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$ for (14). Because of (15) with $s = 1$, we have $\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar U^{-1}H)$, i.e., $|\bar U^{-1}Hx|_\infty \le 1$ for all $x \in \mathcal{E}(P)$. By Proposition 1, system (1) satisfies (14) for all $x \in \mathcal{E}(Q^{-1})$. Hence for system (1), we also have $\dot V(x, w) < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$.

2. If we multiply (17) on the left and the right by $\mathrm{diag}\{P, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i \\ 0 & -I/2 \end{bmatrix} \le 0 \quad \forall\, i \in I[1, 2^m].$$

By item 2 of Lemma 1, this ensures that $\dot V(x, w) \le w^T w$ for all x and w for (14). Also, the condition (15) ensures that $s\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar U^{-1}H)$ and hence (14) is valid within $s\mathcal{E}(Q^{-1})$. Therefore, we have $\dot V(x, w) \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$ for system (1). If $x(0) = 0$ and $\|w\|_2 \le s$, then by integrating both sides of $\dot V \le w^T w$, we have $V(x(t)) \le s^2$, i.e., $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3. We note that (18) implies (17). So by item 2, it is ensured that $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$ if $x(0) = 0$ and $\|w\|_2 \le s$. Hence the LDI (14) is valid for system (1) for all $\|w\|_2 \le s$ and $x(0) = 0$. If we multiply (18) on the left and the right by $\mathrm{diag}\{P, I, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i & 0 \\ 0 & -I/2 & 0 \\ C_i - D_{zq}T_iH & D_i & -\gamma^2 I/2 \end{bmatrix} \le 0 \quad \forall\, i \in I[1, 2^m].$$

By Lemma 1, this ensures that $\dot V(x, w) + \frac{1}{\gamma^2} z^T z \le w^T w$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$ for system (14). For system (1), the inequality holds for all $x \in s\mathcal{E}(Q^{-1})$ and $w \in \mathbb{R}^r$. By integrating both sides of the inequality, we have $\|z\|_2 \le \gamma\|w\|_2$ as long as $\|w\|_2 \le s$ and $x(0) = 0$. □

It can be verified that for the special case where $D_{yq} = 0, D_{yw} = 0, D_{zq} = 0$ and $D_{zw} = 0$, items 1 and 3 reduce to the corresponding results in [25] and [13] respectively. The three parts in Theorem 1 can be respectively used to estimate the domain of attraction, the reachable set and the nonlinear $\mathcal{L}_2$ gain for system (1). For these purposes, we may formulate corresponding optimization problems with linear matrix inequality (LMI) constraints. For the estimation of the nonlinear $\mathcal{L}_2$ gain, we need to minimize γ for a selections of s over $[0, \infty)$.

Problem 1: Estimation of the domain of attraction. For the purpose of enlarging the estimation of the domain of attraction, we may choose a shape reference set X_R (see e.g., [22, 25, 26]) and maximize

a scaling $\alpha > 0$ such that $\alpha X_R \subseteq \mathcal{E}(Q^{-1})$, with Q satisfying (15) and (16). The optimizing parameters are Q and Y. When X_R is a polygon or an ellipsoid, the resulting optimization problem has an LMI formulation.

Problem 2: Estimation of the reachable set. Under the condition (15) and (17), an estimate of the reachable set is given by $s\mathcal{E}(Q^{-1})$. Since smaller (or tighter) estimates are desirable, we may formulate an optimization problem to minimize the size of $s\mathcal{E}(Q^{-1})$. There are different measures of size for ellipsoids, such as the trace of Q and the determinant of Q, among which the trace of Q is a convex measure and is much easier to handle. In a practical situation, we may be interested in knowing the size of a certain state or an output during the operation of the system. For instance, given a row vector $C \in \mathbb{R}^{1\times n}$, we would like to estimate the maximal value of $|Cx(t)|$ for all $t \geq 0$. Since $x(t) \in s\mathcal{E}(Q^{-1})$, the maximal value of $|Cx(t)|$ is less than

$$\bar{\alpha} := (\max\{x^T C^T C x : x^T (s^2 Q)^{-1} x \leq 1\})^{1/2}.$$

Given $\alpha > 0$. Consider the set $\mathcal{E}(C^T C/\alpha^2) = \{x : x^T C^T C x \leq \alpha^2\} = \{x : |Cx| \leq \alpha\}$. It is the region between the two hyperplanes $Cx = \alpha$ and $Cx = -\alpha$. It can also be considered as a degenerated ellipsoid corresponding to a positive semidefinite matrix $C^T C$. Hence we have $\alpha \geq \bar{\alpha}$ if and only if $\mathcal{E}((s^2Q)^{-1}) \subset \mathcal{E}(C^T C/\alpha^2)$, which is equivalent to $C^T C/\alpha^2 \leq (s^2 Q)^{-1}$. Thus $\bar{\alpha} = \min\{\alpha : C^T C \leq \alpha^2 (s^2 Q)^{-1}\}$. Note that $C^T C \leq \alpha^2 (s^2 Q)^{-1}$ is equivalent to $Q^{\frac{1}{2}} C^T C Q^{\frac{1}{2}} \leq \alpha^2/s^2 I$ and to $CQC^T \leq \alpha^2/s^2$, we have

$$\bar{\alpha} = \min\{\alpha : CQC^T \leq \alpha^2/s^2\}.$$

To minimize $\bar{\alpha}$, we can minimize α^2 satisfying the linear (in Q and α^2) constraint $CQC^T \leq \alpha^2/s^2$ with Q satisfying (15) and (17). With α determined this way, we have $|Cx(t)| \leq \alpha$ for all $t \geq 0$. We may choose different C's, such as $C_i, i = 1, 2, \cdots, N$, and obtain a bound α_i on $|C_i x(t)|$ for each i. The polytope formed as $\{x \in \mathbb{R}^n : \ |C_i x| \leq \alpha_i, i = 1, \cdots, N\}$ will also be an estimate of the reachable set.

Problem 3: Estimation of the nonlinear $\mathcal{L}_2$ gain. The problem of minimizing a bound on the $\mathcal{L}_2$ gain follows directly from item 3 of Theorem 1 by minimizing γ along with parameters Q and Y satisfying (15) and (18). For each $s > 0$, denote $\gamma^*(s)$ as the minimal γ, then we have

$$\|z\|_2 \leq \gamma^*(\|w\|_2)\|w\|_2,$$

for all w. In other words, $\gamma^*(s)$ serves as an estimate for the nonlinear $\mathcal{L}_2$ gain.

3.3 Analysis based on the norm-bounded differential inclusion

For easy reference, the NDI description for (1) is repeated as follows. If $|\bar{U}^{-1} H x|_\infty \leq 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \ \|\Omega\| \leq 1 \right\}, \tag{19}$$

where

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} := \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1}) M^{-1} \begin{bmatrix} C_y - H & D_{yw} \end{bmatrix}, \tag{20}$$

and $M > 0$ is diagonal, S is symmetric and nonsingular such that $S^2 = 2I - M^{-1} D_{yq} M - M D_{yq}^T M^{-1}$.

The next lemma will be used to handle the norm-bounded differential inclusion (19).

Lemma 2 *Given X, Y, Z, S of compatible dimensions, where S is symmetric and nonsingular. If*

$$\mathrm{He}\begin{bmatrix} Z & X \\ Y & -S^2/2 \end{bmatrix} \leq 0, \tag{21}$$

then $\mathrm{He}(Z + X(S^{-2} + S^{-1}\Omega S^{-1})Y) \leq 0 \quad \forall \|\Omega\| \leq 1.$

This lemma follows directly using Schur complements and from $M\Omega N + N^T\Omega^T M^T \leq MM^T + N^T N$ for all $\|\Omega\| \leq 1$.

Theorem 2 *Given $Q \in \mathbb{R}^{n\times n}, Q = Q^T > 0$. Let $V(x) = x^T Q^{-1} x$. Consider system (1).*

1. *If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (15) with $s = 1$ and*

$$\mathrm{He}\begin{bmatrix} AQ & B_q U \\ C_y Q - Y & -U + D_{yq} U \end{bmatrix} < 0, \tag{22}$$

then $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.

2. *Given $s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (15) and*

$$\mathrm{He}\begin{bmatrix} AQ & B_w & B_q U \\ 0 & -I/2 & 0 \\ C_y Q - Y & D_{yw} & -U + D_{yq} U \end{bmatrix} \leq 0, \tag{23}$$

then $\dot{V}(x,w) \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \geq 0$.

3. *Given $\gamma, s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (15) and*

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_q U \\ 0 & -I/2 & 0 & 0 \\ C_z Q & D_{zw} & -\gamma^2 I/2 & D_{zq} U \\ C_y Q - Y & D_{yw} & 0 & -U + D_{yq} U \end{bmatrix} \leq 0, \tag{24}$$

then $\dot{V}(x,w) + \frac{1}{\gamma^2} z^T z \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $\|z\|_2 \leq \gamma \|w\|_2$.

Proof. The procedure is very similar to the proof of Theorem 1 except we need to establish that the conditions (22), (23) and (24) imply the respective conditions in Lemma 1 for the NDI (19). This is a little more complicated than the counterpart for Theorem 1.

Here we only show that (24) guarantees (13) when the differential inclusion (11) is specified to (19). The other correspondences in item 1 and item 2 are similar and simpler. For system (19), the condition (13) in Lemma 1 can be written as

$$\mathrm{He}\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -I/2 & 0 \\ C_\Omega & D_\Omega & -\gamma^2 I/2 \end{bmatrix} \leq 0 \quad \forall \|\Omega\| \leq 1. \tag{25}$$

¿From (20), we have

$$\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -I/2 & 0 \\ C_\Omega & D_\Omega & -\gamma^2 I/2 \end{bmatrix} = \begin{bmatrix} PA & PB_w & 0 \\ 0 & -I/2 & 0 \\ C_z & D_{zw} & -\gamma^2 I/2 \end{bmatrix}$$
$$+ \begin{bmatrix} PB_q M \\ 0 \\ D_{zq} M \end{bmatrix} (S^{-2} + S^{-1}\Omega S^{-1}) \begin{bmatrix} M^{-1}(C_y - H) & M^{-1} D_{yw} & 0 \end{bmatrix}.$$

By Lemma 2, to guarantee (25), it suffices to have

$$\mathrm{He}\begin{bmatrix} PA & PB_w & 0 & PB_q M \\ 0 & -I/2 & 0 & 0 \\ C_z & D_{zw} & -\gamma^2 I/2 & D_{zq} M \\ M^{-1}(C_y - H) & M^{-1} D_{yw} & 0 & -S^2/2 \end{bmatrix} \leq 0. \tag{26}$$

Multiplying on the left and the right by $\mathrm{diag}\{Q, I, I, M\}$, noticing that $\mathrm{He}(-S^2/2) = \mathrm{He}(-I+M^{-1}D_{yq}M)$, $Q = P^{-1}, Y = HQ$, (26) is equivalent to

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_qM^2 \\ 0 & -I/2 & 0 & 0 \\ C_zQ & D_{zw} & -\gamma^2 I/2 & D_{zq}M^2 \\ C_yQ - Y & D_{yw} & 0 & -M^2 + D_{yq}M^2 \end{bmatrix} \leq 0,$$

which is (24) with $U = M^2$. □

Remark 2 If we take $Y = 0$ in (24), then the inequality reduces to (10a) of [16] (with some permutation). A nonzero parameter Y introduces additional degrees of freedom for regional analysis and makes the results applicable to the case where the system wrapped around the saturation is not globally exponentially stable. ◦

As with Theorem 1, different optimization problems with LMI constraints can be formulated for stability and performance analysis of the original system (1) based on the three parts of Theorem 2. Since the NDI is a more conservative description than the PDI and since Theorems 1 and 2 are developed from the same framework, it is easy to see that the analysis results from using Theorem 2 are more conservative than those from using Theorem 1. Actually, even for the special case $m = 1$ for which the NDI and PDI descriptions are the same, Theorem 2 could still be more conservative than Theorem 1 because of using Lemma 2 to derive (26). The advantage of Theorem 2 is that the conditions involve fewer LMIs (but of larger size, i.e., $+m$ more than those in Theorem 1).

We should note that the results in Theorem 2 were established in [28] through the S-procedure. The approach taken in this paper helps us to understand the relationship between the results based on two different types of differential inclusions.

4 Analysis with non-quadratic Lyapunov functions

In this section, we will use a pair of conjugate functions, the convex hull quadratic function and the max quadratic function to perform stability and performance analysis of system (1). For the PDI (7), significant improvement may be achieved with these non-quadratic functions. However, for the NDI (10), there is no advantage in using these non-quadratic functions over quadratic functions. As a matter of fact, this result also applies to any norm-bounded linear differential inclusion (NLDI) (see Remark 5). We first review some results about this pair of conjugate functions.

4.1 The max quadratic function and the convex hull quadratic function

Given a family of positive definite matrices $P_j \in \mathbb{R}^{n\times n}, P_j = P_j^T > 0, j \in I[1, J]$, the pointwise maximum quadratic function is defined as

$$V_{\max}(x) := \max\{x^T P_j x : \ j \in I[1, J]\}. \tag{27}$$

Given $Q_j \in \mathbb{R}^{n\times n}, Q_j = Q_j^T > 0, j \in I[1, J]$. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \geq 0\right\},$$

the convex hull quadratic function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^{J} \gamma_j Q_j\right)^{-1} x. \tag{28}$$

For simplicity, we say that V_c is composed from Q_j's. It was shown in [15] that $\frac{1}{2}V_{\max}$ is conjugate to $\frac{1}{2}V_c$ if $Q_j = P_j$ for each $j \in I[1, J]$. It is evident that V_c and $V_{\max}$ are homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x), V_{\max}(\alpha x) = \alpha^2 V_{\max}(x)$. Also established in [15, 23] are that V_c is convex and continuously differentiable and that $V_{\max}$ is strictly convex.

The 1-level set of $V_{\max}$ and that of V_c are respectively

$$L_{V_{\max}} := \left\{x \in \mathbb{R}^n : \ V_{\max}(x) \le 1\right\}, \quad L_{V_c} := \left\{x \in \mathbb{R}^n : \ V_c(x) \le 1\right\}.$$

Since $V_{\max}$ and V_c are homogeneous of degree 2, we have

$$sL_{V_{\max}} = \left\{x \in \mathbb{R}^n : \ V_{\max}(x) \le s^2\right\}, \quad sL_{V_c} = \left\{x \in \mathbb{R}^n : \ V_c(x) \le s^2\right\}.$$

It is easy to see that $L_{V_{\max}}$ is the intersection of the ellipsoids $\mathcal{E}(P_j)$'s. In [23], It was established that L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_j^{-1})$'s, i.e.,

$$L_{V_c} = \left\{ \sum_{j=1}^{J} \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma \right\}.$$

For a compact convex set S, a point $x \in S$ is called an extreme point if it cannot be represented as the convex combination of any other points in S. Clearly an extreme point must belong to the boundary of S (denoted as ∂S). For a strictly convex set, such as $L_{V_{\max}}$, every boundary point is an extreme point. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$'s, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1, J]$ (If $x \in \partial L_{V_c} \setminus \cup_{j=1}^{J} \mathcal{E}(Q_j^{-1})$, then x must be the convex combination of at least two points from $\cup_{j=1}^{J} \mathcal{E}(Q_j^{-1})$ and thus not an extreme point of L_{V_c}). Denote

$$E_j := \partial L_{V_c} \cap \partial \mathcal{E}(Q_j^{-1}) = \left\{x \in \mathbb{R}^n : V_c(x) = x^T Q_j^{-1} x = 1\right\}.$$

Then $\bigcup_{j=1}^{J} E_j$ contains all the extreme points of L_{V_c}. The exact description of E_j is given as follows.

Lemma 3 *For each $j \in I[1, J]$, define $F_j = \{x \in \mathbb{R}^n : x^T Q_j^{-1}(Q_k - Q_j)Q_j^{-1} x \le 0 \ \ \forall\, k \in I[1, J]\}$. Then $E_j = \partial L_{V_c} \cap F_j$.*

Proof. See Appendix A. □

It is clear that $\alpha F_j = F_j$ for any $\alpha > 0$. Since L_{V_c} is convex and contains the origin in its interior, we have $L_{V_c} = \bigcup_{\delta \in [0,1]} \delta(\partial L_{V_c})$. It follows from Lemma 3 that $\bigcup_{\delta \in [0,1]} \delta E_j = L_{V_c} \cap F_j$.

The following lemma combines some results from [23, 24].

Lemma 4 *For a given $x_0 \in \mathbb{R}^n$, let $\gamma^* \in \Gamma$ be an optimal γ such that*

$$x_0^T \left(\sum_{j=1}^{J} \gamma_j^* Q_j \right)^{-1} x_0 = \min_{\gamma \in \Gamma} x_0^T \left(\sum_{j=1}^{J} \gamma_j Q_j \right)^{-1} x_0 = V_c(x_0).$$

For simplicity and without loss of generality, assume that $\gamma_j^* > 0$ for $j \in I[1, J_0]$ and $\gamma_j^* = 0$ for $j \in I[J_0 + 1, J]$. Denote

$$Q_0 = \sum_{j=1}^{J_0} \gamma_j^* Q_j, \quad x_j = Q_j Q_0^{-1} x_0, \quad j \in I[1, J_0].$$

Then $V_c(x_j) = V_c(x_0)$ and $x_j \in V_c(x_0)^{\frac{1}{2}} E_j, j \in I[1, J_0]$. Moreover, $x_0 = \sum_{j=1}^{J_0} \gamma_j^* x_j$, and

$$\nabla V_c(x_0) = \nabla V_c(x_j) = 2Q_j^{-1} x_j = 2Q_0^{-1} x_0, \quad j \in I[1, J_0],$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

The following lemma is adapted from a result of [27] to the slightly different definition of V_c and $V_{\max}$ (the two functions in [27] have the coefficient $\frac{1}{2}$ and the saturation levels in $\bar{U}$ are also included here).

Lemma 5 *[27] Let $H \in \mathbb{R}^{m\times n}$, $\bar{U} \in \mathbb{R}^{m\times m}$ be positive definite diagonal and denote the ℓ-th row of H by H_ℓ and the ℓ-th diagonal element of $\bar{U}$ by $\bar{u}_\ell$. We have,*

1) $L_{V_c} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $\frac{1}{\bar{u}_\ell}H_\ell^T \in L_{V_{\max}}$ for all $\ell \in I[1,m]$;

2) $L_{V_{\max}} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $\frac{1}{\bar{u}_\ell}H_\ell^T \in L_{V_c}$ for all $\ell \in I[1,m]$.

4.2 Analysis with convex hull quadratic functions

In this section, we apply the convex hull quadratic function to the analysis of system (1) through the polytopic differential inclusion (7), which is repeated below for easy reference:

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix} : \ i \in I[1,2^m] \right\}. \tag{29}$$

This PDI is a valid description for (1) as long as $|\bar{U}^{-1}h(x)|_\infty \le 1$. We will restrict our attention to a level set sL_{V_c}, where $|\bar{U}^{-1}h(x)|_\infty \le 1$ for all $x \in sL_{V_c}$. As with the case of using quadratic functions, the crucial point is to guarantee that $x(t) \in sL_{V_c}$ under the class of norm-bounded w and the set of initial states under consideration.

It may appear that choosing $h(x)$ as a linear function Hx within sL_{V_c} should lead to simpler results than choosing it as a nonlinear function. However, it turns out that a nonlinear $h(x)$ not only reduces conservatism but also leads to cleaner and numerically more tractable results. As expected, the derivation of the results is more involved than the former cases in Section 3 because of the non-quadratic Lyapunov function and the nonlinear function $h(x)$. For this reason, we present the results separately for the estimation of the domain of attraction, the reachable set and the $\mathcal{L}_2$ gain. Based on technical considerations, we first present the result about the reachable set.

Theorem 3 *(Reachable set by $\mathcal{L}_2$-norm-bounded inputs) Given $Q_j = Q_j^T > 0, j \in I[1,J]$, let V_c be composed from Q_j's as in (28). Given any $s > 0$, system (1) with $x(0) = 0$ satisfies $x(t) \in sL_{V_c}$ for all $t \ge 0$ and for all w such that $\|w\|_2 \le s$ if there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \ge 0$, $i \in I[1,2^m]$, $j,k \in I[1,J]$ such that*

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^J \lambda_{ijk}(Q_j - Q_k) & B_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \quad \forall\, i \in I[1,2^m], j \in I[1,J], \tag{30}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \ge 0 \quad \forall\, \ell \in I[1,m], j \in I[1,J], \tag{31}$$

where $Y_{j,\ell}$ is the ℓth row of Y_j.

Proof. We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, we have $\dot{V}_c(x,w) \le w^Tw$, where $\dot{V}_c(x,w)$ is the time derivative of V_c in the direction of the right hand side of (1), which depends on x and w.

Let $P_j = Q_j^{-1}$, $H_j = Y_jQ_j^{-1}$. Multiplying (30) on the left and the right by $\mathrm{diag}\{P_j, I\}$, we have

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^J \lambda_{ijk}P_j(Q_j - Q_k)P_j & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0. \tag{32}$$

This implies that for all $i \in I[1,2^m], j \in I[1,J]$,

$$2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \le 2\sum_{k=1}^J \lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \quad \forall\, x \in \mathbb{R}^n, w \in \mathbb{R}^r. \tag{33}$$

Given $j \in I[1, J]$ and any $\delta > 0$. Consider $x \in \delta E_j$. By Lemma 3 we have

$$\sum_{k=1}^{J} \lambda_{ijk} x^T P_j (Q_k - Q_j) P_j x \leq 0.$$

It follows from (33) that

$$2x^T P_j (A_i x + B_i w - B_q T_i H_j x) - w^T w \leq 0 \quad \forall\, x \in \delta E_j, w \in \mathbb{R}^r, \;\; \delta > 0. \tag{34}$$

(In view of (29) and condition (31), this actually shows that $\dot{V}_c(x, w) \leq w^T w$ for all $x \in s(L_{V_c} \cap E_j)$, recalling from Lemma 4 that $\nabla V_c(x) = 2P_j x$ for $x \in E_j$. More explanation can be seen below). We proceed to show that $\dot{V}_c(x, w) \leq w^T w$ holds for all $x \in sL_{V_c}$ by exploiting the properties of V_c.

Now consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. By Lemma 4, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1, J_0]$ with $J_0 \leq J$ such that $\sum_{j=1}^{J_0} \gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0} \gamma_j x_j$ (we note that the indices j can always be reordered to make this true for each x_0). Let

$$Q_0 = \sum_{j=1}^{J_0} \gamma_j Q_j, \quad Y_0 = \sum_{j=1}^{J_0} \gamma_j Y_j, \quad H_0 = Y_0 Q_0^{-1}. \tag{35}$$

Then we also have $x_0^T Q_0^{-1} x_0 = V_c(x_0) = \delta^2$ and

$$\nabla V_c(x_0) = 2Q_0^{-1} x_0 = 2Q_j^{-1} x_j, \quad j \in I[1, J_0]. \tag{36}$$

Applying convex combination to the inequalities in (31), we have

$$\begin{bmatrix} \bar{u}_\ell^2/s^2 & Y_{0,\ell} \\ Y_{0,\ell}^T & Q_0 \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} \bar{u}_\ell^2/s^2 & H_{0,\ell} \\ H_{0,\ell}^T & Q_0^{-1} \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1, m].$$

By (3), this implies that $s\mathcal{E}(Q_0^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1} H_0)$. Since $x_0^T Q_0^{-1} x_0 = \delta^2 \leq s^2$, we have $|\bar{U}^{-1} H_0 x_0| \leq 1$. Thus (29) is valid at x_0 with $h(x_0) = H_0 x_0$. Hence we have

$$\dot{x}|_{x=x_0} \in \mathrm{co}\{A_i x_0 + B_i w - B_q T_i H_0 x_0 : \; i \in I[1, 2^m]\} \tag{37}$$

and

$$\dot{V}_c(x_0, w) - w^T w \in \mathrm{co}\{(\nabla V_c(x_0))^T (A_i x_0 + B_i w - B_q T_i H_0 x_0) - w^T w : \; i \in I[1, 2^m]\}. \tag{38}$$

Recalling that

$$x_0 = \sum_{j=1}^{J_0} \gamma_j x_j, \;\; \nabla V_c(x_0) = 2Q_0^{-1} x_0 = 2Q_j^{-1} x_j = 2P_j x_j, \;\; x_j \in \delta E_j. \tag{39}$$

Applying (34) to x_j and replacing $2x_j^T P_j$ with $(\nabla V_c(x_0))^T$, we obtain

$$(\nabla V_c(x_0))^T (A_i x_j + B_i w - B_q T_i H_j x_j) - w^T w \leq 0 \quad \forall\, w \in \mathbb{R}^r. \tag{40}$$

By the definition of Q_0, H_0 and Y_0 in (35),

$$H_0 x_0 = Y_0 Q_0^{-1} x_0 = \left(\sum_{j=1}^{J_0} \gamma_j Y_j \right) Q_0^{-1} x_0 \tag{41}$$

and from (36) we have

$$H_j x_j = Y_j Q_j^{-1} x_j = Y_j Q_0^{-1} x_0, \quad j \in I[1, J_0]. \tag{42}$$

Combining (39), (41) and (42), and noting that $\gamma_1 + \gamma_2 + \cdots + \gamma_{J_0} = 1$, we have

$$\begin{aligned} A_i x_0 + B_i w - B_q T_i H_0 x_0 &= \sum_{j=1}^{J_0} \gamma_j A_i x_j + \sum_{j=1}^{J_0} \gamma_j B_i w - B_q T_i \sum_{j=1}^{J_0} \gamma_j Y_j Q_0^{-1} x_0 \\ &= \sum_{j=1}^{J_0} \gamma_j (A_i x_j + B_i w - B_q T_i H_j x_j) \quad \forall\, w \in \mathbb{R}^r. \end{aligned} \tag{43}$$

Note that this is satisfied for all $i \in I[1, 2^m]$. It follows from (40) that for each $i \in I[1, 2^m]$ and $w \in \mathbb{R}^r$,

$$\begin{aligned} &(\nabla V_c(x_0))^T (A_i x_0 + B_i w - B_q T_i H_0 x_0) - w^T w \\ &= \sum_{j=1}^{J_0} \gamma_j [(\nabla V_c(x_0))^T (A_i x_j + B_i w - B_q T_i H_j x_j) - w^T w] \\ &\leq 0 \end{aligned}$$

By (38), we have $\dot{V}_c(x_0, w) - w^T w \leq 0 \quad \forall\, w \in \mathbb{R}^r$. Note that x_0 is an arbitrary point in sL_{V_c}.

Hence we have that $\dot{V}_c(x, w) \leq w^T w$ for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$. Now suppose $x(0) = 0$ and $\|w\|_2^2 \leq s^2$. Then for any $t_0 > 0$, as long as $x(t) \in sL_{V_c}$ for all $t \in (0, t_0)$, we have $V_c(x(t_0)) \leq \int_0^{t_0} w^T(\tau) w(\tau) d\tau \leq s^2$, i.e., $x(t_0) \in sL_{V_c}$. On the other hand, if there exists $t_0 > 0$ such that $V_c(x(t)) \leq s^2$ for all $t \in (0, t_0)$ and $V_c(x(t_0)) = s^2$ then we must have $\int_{t_0}^{\infty} w^T(\tau) w(\tau) d\tau = 0$ and $\dot{V}_c(x(t), w(t)) \leq 0$ for almost all $t \geq t_0$. Hence $V_c(x(t)) \leq s^2$ for all $t \geq t_0$. Therefore, we conclude that $x(t) \in sL_{V_c}$ for all $t \geq 0$. □

Remark 3 (Optimization issues) With conditions (30) and (31), we may formulate an optimization problem to minimize the estimate of the reachable set as with the quadratic function case. We observe that (30) is a bilinear matrix inequality (BMI) which contains some bilinear terms as the product of a full matrix and a scalar at the (1,1) block of the lefthand-side matrix. Similar bilinear terms are contained in the matrix inequalities in [14, 15, 27] for stability and performance analysis of linear differential/difference inclusions. A direct method to solve BMI problems is to alternatively fix one set of parameters and optimize the other set. In [14, 15, 27], we adopted the path-following method from [20] and our experience with a set of numerical examples shows that the path-following method is much more effective than the straightforward iterative method. We actually implemented a two-step algorithm which combines the path-following method and the direct iterative method. The first step uses the path-following method to update all the parameters at the same time. The second step fixes λ_{ijk}'s and solves the resulting LMI problem which includes Q_j's and Y_j's as variables. This two-step method proves very effective on the BMI problems in [14, 15, 27] and also works well on the example in Section 5. We also see that if we take $Q_j = Q$ and $Y_j = Y$ for all j, then the bilinear terms vanish and the conditions reduce to the LMIs in (15) and (17). In our computation, we first solve the resulting optimization problem with LMI constraints and then use the optimal Q^* and Y^* to start the two-step algorithm, with $Q_j = Q^*$ and $Y_j = Y^*$ for all j and $\lambda_{ijk} \geq 0$ randomly chosen. This approach also proves effective for the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction, which will be addressed in Theorems 4 and 5.

Although there is no guarantee that the global optimal solution can be located, the convergence of the algorithms is satisfactory. Furthermore, since the initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic functions, the algorithms ensure that the results are at least as good as those from using quadratic functions in Theorem 1. The above discussion also applies to the optimization problems resulting from Theorems 4 and 5. ◦

Remark 4 (About the nonlinear function $h(x)$) ¿From the proof of Theorem 3, we see that a nonlinear function $h(x_0) = H_0(x_0) x_0$ is constructed from Q_j's and Y_j's so that $|\bar{U}^{-1} H_0(x_0) x_0| \leq 1$ for all $x_0 \in sL_{V_c}$

(see (35) where H_0 is constructed and the subsequent discussion up to (37)). This makes the proof more complicated than with a linear function Hx but the result turns out to be cleaner and more easily tractable numerically. If we attempt to use a linear function $h(x) = Hx$ such that $|\bar{U}^{-1}Hx|_\infty \leq 1$ for all $x \in sL_{V_c}$, we would have Y_j in (30) replaced with HQ_j and $Y_{j,\ell}$ in (31) replaced with $H_\ell Q_j$. When we formulate an optimization problem to estimate the reachable set by taking H and Q_j's as optimizing parameters, this would result in more complex BMI terms including HQ_j which may cause difficulties in the algorithms, such as slow convergence or getting stuck easily at a local solution. ◦

Remark 5 (Discussion about results based on NDIs) With similar developments as in the proof of Theorem 3, we can obtain a corresponding condition by using the norm-bounded differential inclusion (10) instead of using the PDI (7). The resulting condition involves the existence of Y_j's, $\lambda_{jk} \geq 0$, and a diagonal $U > 0$ satisfying (31) and

$$\mathrm{He}\begin{bmatrix} AQ_j + \sum_{k=1}^{J} \lambda_{jk}(Q_j - Q_k) & B_w & B_q U \\ 0 & -I/2 & 0 \\ C_y Q_j - Y_j & D_{yw} & -U + D_{yq}U \end{bmatrix} \leq 0 \quad \forall\, j \in I[1, J]. \tag{44}$$

The bilinear terms in the first block seem to inject extra degrees of freedom as compared with (23) in Theorem 2 but they actually wouldn't help to reduce the conservatism. In other words, (44) implies the existence of a Q satisfying (23). To see this, we formulate a matrix

$$\Psi = \begin{bmatrix} -\sum_{k=2}^{J} \lambda_{1k} & \lambda_{21} & \cdots & \lambda_{J1} \\ \lambda_{12} & -\sum_{k=1,k\neq 2}^{J} \lambda_{2k} & \cdots & \lambda_{J2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1J} & \lambda_{2J} & \cdots & -\sum_{k=1}^{J-1} \lambda_{Jk} \end{bmatrix}.$$

Then Ψ is a Metzler matrix. Since the sum of each column of Ψ is 0, the eigenvalue with the maximal real part is 0. Hence there exists a vector $c \neq 0$ with $c_i \geq 0$ such that $\Psi c = 0$ (e.g., see [36]) and in particular we assume $\sum_{j=1}^{J} c_j = 1$ (i.e., $c \in \Gamma$). If we let $Q = \sum_{j=1}^{J} c_j Q_j$, and $Y = \sum_{j=1}^{J} c_j Y_j$, then Q and Y will satisfy (23) and (15). Furthermore, $s\mathcal{E}(Q) \subseteq sL_{V_c}$ is a smaller estimate of the reachable set. This means that with the NDI description, using the convex hull quadratic Lyapunov function offers no advantage to using the quadratic Lyapunov function. The same situation occurs for the estimation of the $\mathcal{L}_2$ gain or the domain of attraction, or, when applying a max quadratic function to NDIs.

For the special case where $H = 0$, the regional NDI (10) becomes a global norm-bounded linear differential inclusion (NLDI). Thus we can conclude that for any NLDI, the convex hull quadratic function or the max quadratic function offers no advantage over quadratic functions when these stability and performance issues are concerned. ◦

We next address the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction.

Theorem 4 *($\mathcal{L}_2$ gain for norm-bounded w) Given $Q_j = Q_j^T > 0, j \in I[1, J]$, let V_c be composed from Q_j's as in (28). Consider system (1). Given $s, \gamma > 0$, if there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geq 0$, $i \in I[1, 2^m]$, $j, k \in I[1, J]$ such that*

$$\mathrm{He}\begin{bmatrix} A_i Q_j - B_q T_i Y_j + \sum_{k=1}^{J} \lambda_{ijk}(Q_j - Q_k) & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i Q_j - D_{zq} T_i Y_j & D_i & -\frac{\gamma^2}{2} I \end{bmatrix} \leq 0 \quad \forall\, i \in I[1, 2^m], j \in I[1, J], \tag{45}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1, m], j \in I[1, J], \tag{46}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$.

Proof. We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, $\dot{V}_c(x,w) + \frac{1}{\gamma^2} z^T z \le w^T w$. Since (45) implies (30), by Theorem 3, we have $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \le s$, $x(0) = 0$. Also, all the relationships established in the proof of Theorem 3 are true under the conditions of the current theorem.

Let $P_j = Q_j^{-1}$ and $H_j = Y_j Q_j^{-1}$. Multiplying (45) on the left and the right by $\mathrm{diag}\{P_j, I, I\}$, we have

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^J \lambda_{ijk} P_j(Q_j - Q_k)P_j & P_jB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH_j & D_i & -\frac{\gamma^2}{2} \end{bmatrix} \le 0.$$

By Schur complements, this is equivalent to

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^J \lambda_{ijk} P_j(Q_j - Q_k)P_j & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} (C_i - D_{zq}T_iH_j)^T \\ D_i^T \end{bmatrix}[C_i - D_{zq}T_iH_j \;\; D_i] \le 0. \tag{47}$$

Denote

$$f_{ij}(x,w) = A_ix + B_iw - B_qT_iH_jx,$$
$$g_{ij}(x,w) = C_ix + D_iw - D_{zq}T_iH_jx.$$

Then (47) implies that for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$,

$$2x^TP_jf_{ij}(x,w) + \frac{1}{\gamma^2}g_{ij}^T(x,w)g_{ij}(x,w) - w^Tw \le 2\sum_{k=1}^J \lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx. \tag{48}$$

Consider $x \in \delta E_j$ for $\delta > 0$. Like in the proof of Theorem 3, we have

$$\sum_{k=1}^J \lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \le 0 \quad \forall\, x \in \delta E_j, \;\; w \in \mathbb{R}^r, \;\; \delta > 0.$$

It follows from (48) that

$$2x^TP_jf_{ij}(x,w) + \frac{1}{\gamma^2}g_{ij}^T(x,w)g_{ij}(x,w) - w^Tw \le 0, \quad \forall\, x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0. \tag{49}$$

We note that this is true for all $i \in I[1, 2^m]$ and $j \in I[1, J]$.

Now consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. Like in the proof of Theorem 3, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1, J_0]$ such that $\sum_{j=1}^{J_0} \gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0} \gamma_j x_j$. Let H_0, Q_0, Y_0 be defined as in (35). Then we also have $|\bar{U}^{-1}H_0x_0| \le 1$. Applying Proposition 1 at x_0, we have

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_ix_0 + B_iw - B_qT_iH_0x_0 \\ C_ix_0 + D_iw - D_{zq}T_iH_0x_0 \end{bmatrix} : \; i \in I[1, 2^m]\right\}.$$

Let

$$f_{i0}(x_0,w) = A_ix_0 + B_iw - B_qT_iH_0x_0, \quad g_{i0}(x_0,w) = C_ix_0 + D_iw - D_{zq}T_iH_0x_0.$$

Then

$$\dot{V}_c(x_0,w) + \frac{1}{\gamma^2}z^Tz - w^Tw \le \max\{(\nabla V_c(x_0))^Tf_{i0}(x_0,w) + \frac{1}{\gamma^2}|g_{i0}(x_0,w)|^2 - w^Tw : \; i \in I[1,2^m]\}. \tag{50}$$

Since $2x_j^T P_j = 2x_0^T Q_0^{-1} = (\nabla V_c(x_0))^T$ (see (36)), applying (49) at x_j, we obtain

$$(\nabla V_c(x_0))^T f_{ij}(x_j, w) + \frac{1}{\gamma^2}|g_{ij}(x_j, w)|^2 - w^T w \leq 0, \quad \forall\, w \in \mathbb{R}^r,\ i \in I[1, 2^m].$$

Like in (43), we have

$$f_{i0}(x_0, w) = \sum_{j=1}^{J_0} \gamma_j f_{ij}(x_j, w), \quad g_{i0}(x_0, w) = \sum_{j=1}^{J_0} \gamma_j g_{ij}(x_j, w).$$

It follows that

$$(\nabla V_c(x_0))^T f_{i0}(x_0, w) + \frac{1}{\gamma^2}|g_{i0}(x_0, w)|^2 - w^T w \leq 0,$$

and from (50)

$$\dot{V}_c(x_0, w) + \frac{1}{\gamma^2} z^T z - w^T w \leq 0, \tag{51}$$

which is satisfied for all $x_0 \in sL_{V_c}$ and $w \in \mathbb{R}^r$. Since $x(0) = 0$, $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \leq s$, integrating both sides of (51), we have $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$. This completes the proof. □

Theorem 5 *(Estimation of the domain of attraction) Given $Q_j = Q_j^T > 0$, $j \in I[1, J]$, let V_c be composed from Q_j's as in (28). Consider system (1) with $w \equiv 0$. We have $\dot{V}_c(x) < 0$ for all $x \in L_{V_c} \setminus \{0\}$ if there exist $\lambda_{ijk} \geq 0$, $Y_j \in \mathbb{R}^{m\times n}, i \in I[1, 2^m], j, k \in I[1, J]$ such that*

$$\mathrm{He}(A_i Q_j - B_q T_i Y_j + \sum_{k=1}^{J} \lambda_{ijk}(Q_j - Q_k)) < 0 \quad \forall\, i \in I[1, 2^m], j \in I[1, J],$$

$$\begin{bmatrix} 1 & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1, m], j \in I[1, J].$$

Proof. The proof can be adapted from the proof of Theorem 3 by assuming that $B_i = 0$. Then with the same procedure, it can be shown that $\dot{V}_c(x) < 0$ for all $x \in L_{V_c} \setminus \{0\}$. □

Remark 6 Note that the condition in Theorem 5 is similar to (but less conservative than) that of Theorem 4 in [27], which is developed for a special case without algebraic loops. Similar numerical complexity can be expected. ◦

4.3 Analysis with max quadratic functions

The max quadratic function is not differentiable everywhere. Following the definition of [42] (page 215), a subgradient of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is a vector $v \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) \geq v^T(x - x_0) \ \ \forall\, x \in \mathbb{R}^n, \tag{52}$$

and the subdifferential, denoted as $\partial f(x_0)$ (not to be confused as the boundary of a set) , is the set of all subgradient at x_0. The function $f(x)$ is differentiable at x_0 if and only if $\partial f(x_0)$ is single valued. We use $\partial V_{\max}(x)$ to denote the sub-differential of $V_{\max}$ at x.

Lemma 6 *Consider $x_0 \in \mathbb{R}^n$. Suppose that there exists $J_0 \in I[1, J]$ such that $V_{\max}(x_0) = x_0^T P_j x_0$ for $j \in I[1, J_0]$ and $V_{\max}(x_0) > x_0^T P_j x_0$ for $j > J_0$. Then*

1) $\partial V_{\max}(x) = \mathrm{co}\{2P_j x : j \in I[1, J_0]\}$.

2) For a vector $\zeta \in \mathbb{R}^n$, the directional derivative at x_0 along ζ is,

$$\lim_{t\to 0, t>0} \frac{V_{\max}(x_0+t\zeta)-V_{\max}(x_0)}{t} = \max_{\xi\in\partial V_{\max}(x_0)} \{\xi^T\zeta\}. \tag{53}$$

Proof. See Appendix. □

For simplicity and with some abuse of notation, for $\dot{x}$ given by (1), denote

$$\dot{V}_{\max}(x,w) := \max_{\xi\in\partial V_{\max}(x)} \{\xi^T\dot{x}\} = \max_{\xi\in\partial V_{\max}(x)} \{\xi^T(Ax+B_q q+B_w w)\}.$$

Then by Lemma 6 with $\zeta = \dot{x}$, $V_{\max}$ is decreasing along $\dot{x}$ if and only if $\dot{V}_{\max}(x,w) < 0$.

Theorem 6 *(Reachable set by bounded inputs) Given $P_j = P_j^T > 0, j \in I[1,J]$, let $V_{\max}$ be the max quadratic function formed by P_j's as in (27). Given any $s > 0$, system (1) with $x(0) = 0$ satisfies $x(t) \in sL_{V_{\max}}$ for all $t \geq 0$ and for all w such that $\|w\|_2 \leq s$ if there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^J \alpha_{\ell j} = 1$, and*

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH + \sum_{k=1}^J \lambda_{ijk}(P_j-P_k) & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \leq 0 \quad \forall\, i \in I[1,2^m], j \in I[1,J], \tag{54}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^J \alpha_{\ell j}P_j \end{bmatrix} \geq 0 \quad \forall \ell \in I[1,m]. \tag{55}$$

Proof. By the definition of V_c, condition (55) implies that $V_c(\frac{s}{\bar{u}_\ell}H_\ell) \leq 1$ for all $\ell \in I[1,m]$. By Lemma 5, this implies that $L_{V_{\max}} \subseteq \mathcal{L}(s\bar{U}^{-1}H) = (1/s)\mathcal{L}(\bar{U}^{-1}H)$, i.e., $sL_{V_{\max}} \subseteq \mathcal{L}(\bar{U}^{-1}H)$. Hence $|\bar{U}^{-1}Hx|_\infty \leq 1$ for all $x \in sL_{V_{\max}}$. By Proposition 1, we have

$$\dot{x} \in \mathrm{co}\,\{A_ix + B_iw - B_qT_iHx :\ i \in I[1,2^m]\} \quad \forall\, x \in sL_{V_{\max}}.$$

On the other hand, it can be verified that (54) implies that

$$2x^TP_j(A_ix+B_iw-B_qT_iHx) - w^Tw \leq 2\sum_{k=1}^J \lambda_{ijk}x^T(P_k-P_j)x, \tag{56}$$

for all $j \in I[1,J], i \in I[1,2^m]$.

The state space of x can be partitioned as the following subsets:

$$S_j = \{x \in \mathbb{R}^n : x^T(P_k-P_j)x \leq 0, k \in I[1,J]\}, \quad j \in I[1,J].$$

If $x \in S_j \setminus \cup_{k\neq j} S_k$, then $V_{\max}(x) = x^TP_jx$ and $\partial V_{\max}(x) = 2P_jx$. If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^J S_j$, then $V_{\max}(x) = x^TP_jx, j \in I[1,J_0]$ and $\partial V_{\max}(x) = \mathrm{co}\{2P_jx : j \in I[1,J_0]\}$.

We first consider $x \in S_j \setminus \cup_{k\neq j} S_k$. Then

$$\sum_{k=1}^J \lambda_{ijk}x^T(P_k-P_j)x \leq 0, \tag{57}$$

and

$$\dot{V}_{\max}(x,w) - w^Tw \leq \max_{i\in I[1,2^m]} (2x^TP_j(A_ix+B_iw-B_qT_iHx) - w^Tw).$$

If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^J S_j$, then (57) is satisfied for all $j \in I[1,J_0]$ and we have

$$\dot{V}_{\max}(x,w) - w^Tw \leq \max_{j\in I[1,J_0]} \max_{i\in I[1,2^m]} (2x^TP_j(A_ix+B_iw-B_qT_iHx) - w^Tw).$$

It follows from (56) and (57) that $\dot{V}_{\max}(x,w) - w^Tw \leq 0$. The remaining part of the proof is similar to the proof of Theorem 3. □

Theorem 7 *($\mathcal{L}_2$ gain for norm-bounded w) Given $P_j = P_j^T > 0, j \in I[1,J]$, consider system (1) and $s, \gamma > 0$. If there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^J \alpha_{\ell j} = 1$ and*

$$\mathrm{He}\begin{bmatrix} P_jA_i - P_jB_qT_iH + \sum_{k=1}^J \lambda_{ijk}(P_j - P_k) & P_jB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH & D_i & -\frac{\gamma^2}{2}I \end{bmatrix} \leq 0 \quad \forall\, i \in I[1,2^m], j \in I[1,J], \tag{58}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^J \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1,m], \tag{59}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$.

Proof. Like in the proof of Theorem 6, we have $x(t) \in sL_{V_{\max}}$ for all $t \geq 0$ under the condition $\|w\|_2 \leq s$ and $x(0) = 0$. Also we have $|\bar{U}^{-1}Hx|_\infty \leq 1$ for all $x \in sL_{V_{\max}}$. By Proposition 1,

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} f_i(x,w) \\ g_i(x,w) \end{bmatrix} :\ i \in I[1,2^m] \right\},$$

where $f_i(x,w) = A_ix + B_iw - B_qT_iHx$, $g_i(x,w) = C_ix + D_iw - D_{yq}T_iHx$. Using Schur complements, it can be verified that (58) implies

$$2x^TP_jf_i(x,w) + \frac{1}{\gamma^2}|g_i(x,w)|^2 - w^Tw \leq 2\sum_{k=1}^J \lambda_{ijk}x^T(P_k - P_j)x$$

for all $j \in I[1,J], i \in I[1,2^m]$. With similar arguments as in the proof of Theorem 6, it can be shown that for all $x \in sL_{V_{\max}}$ and $w \in \mathbb{R}^r$, $\dot{V}_{\max}(x,w) + \frac{1}{\gamma^2}z^Tz - w^Tw \leq 0$. The remaining part of the proof is similar to the proof of Theorem 4. □

The following result can be derived by adapting the proof of Theorem 6.

Theorem 8 *(Estimation of the domain of attraction) Given $P_j = P_j^T > 0, j \in I[1,J]$, consider system (1) with $w \equiv 0$. We have $\dot{V}_{\max}(x,0) < 0$ for all $x \in L_{V_{\max}} \setminus \{0\}$ if there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j,k \in I[1,J]$, $i \in I[1,2^m]$, $\ell \in I[1,m]$, such that $\sum_{j=1}^J \alpha_{\ell j} = 1$ and*

$$\mathrm{He}\Big[P_jA_i - P_jB_qT_iH + \textstyle\sum_{k=1}^J \lambda_{ijk}(P_j - P_k) \Big] < 0 \quad \forall\, i \in I[1,2^m], j \in I[1,J], \tag{60}$$

$$\begin{bmatrix} \bar{u}_\ell^2 & H_\ell \\ H_\ell^T & \sum_{j=1}^J \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1,m]. \tag{61}$$

As compared with the counterpart results from using convex hull quadratic functions, the conditions (54), (58) and (60) in Theorems 6 to 8 appear to be less tractable because of the bilinear term $P_jB_qT_iH$ in the first blocks of the matrices. Also, the same H for all P_j's seems to offer fewer degrees of freedom as compared with different Y_j's for different Q_j's in Theorems 3 to 5. However, numerical examples show that Theorems 6 to 8 may produce better results in some cases.

5 Examples

Example 1 Consider system (1) with the following parameters:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]. \tag{62}$$

The well-posedness of the system is easily verified through Claim 2. We use the four methods in Theorems 1, 2, 4 and 7 to estimate the nonlinear $\mathcal{L}_2$ gain. The resulting estimates are plotted in Fig. 2, where the dotted curve is from applying quadratics via NDI (Theorem 2), the dash-dotted one is from applying quadratics via PDI (Theorem 1), the dashed one is from applying max quadratics (with $J = 2$) via PDI (Theorem 7) and the solid one is from applying convex hull quadratics ($J = 2$) via PDI (Theorem 4). Each of the four curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value will be an

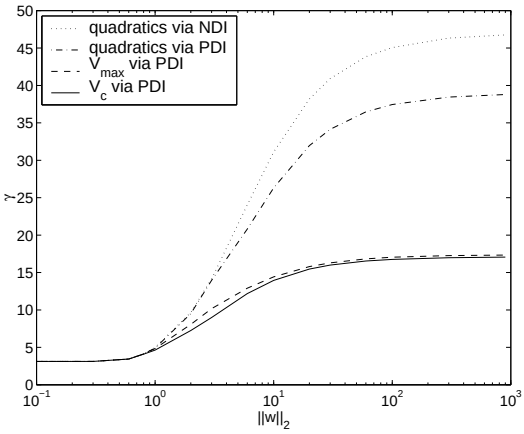

Figure 2: Different estimates of the nonlinear $\mathcal{L}_2$ gain: Case 1.

estimate of the global $\mathcal{L}_2$ gain. As expected, applying quadratics via PDI always leads to better results than applying quadratics via NDI, and applying one of the two non-quadratics always leads to better results than applying quadratics. However, the relationship between the results from applying the two non-quadratic functions is not definite. The situation exhibited in Fig. 2 can be reversed if we change the parameters of the system. In what follows, we present several scenarios through some adjustments of the parameters.

Case 2: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via NDI is unbounded (or, global stability is not confirmed), while that by using quadratics via PDI is 170.1473. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are respectively 20.7833 and 19.3307.

Case 3: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via either NDI or PDI is unbounded. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are respectively 42.3354 and 31.6731.

The above two situations also show how the stability and performance results by the same method can be affected by the parameter D_{yq} which describes the algebraic loop. As discussed in [39], this parameter

is one of the two key design parameters in static anti-windup synthesis and can have a dramatic impact on anti-windup performance.

Due to space limitation, we will not present computational results about the estimation of the domain of attraction or the estimation of the reachable set. Interested readers are referred to [27] for some numerical results. ¿From the different situations exhibited through the $\mathcal{L}_2$ gain, it is not hard to infer that the difference among the estimations by using quadratics/non-quadratics via NDI/PDI can be made arbitrarily large through adjusting the four elements of D_{yq}. For instance, Case 2 suggests that the estimate of the domain of attraction by using quadratics via NDI is bounded while that by using quadratics via PDI is the whole state space. Case 3 suggests that the domain of attraction estimated by non-quadratic functions is the whole state space while that by quadratics (via PDI or NDI) is bounded. On the other hand, the estimate of the reachable set by non-quadratics can be bounded while that by quadratics is not.

We should remark that for this particular example, the algorithm for applying convex hull quadratics converges very well for all the values of s that we considered in our numerical computation, even under different parameter changes. The algorithm for applying max quadratics generally converges well but for some values of s it showed some difficulties where we needed to stop the algorithm and restart it from different initial values of λ_{ijk} which are randomly generated. In any case, improvement is expected from the non-quadratic functions.

Example 2 We adopt Example 2 from [16]. The plant is a cart-spring-pendulum system with one control input, one disturbance input, four states and one measurement output. The plant and controller parameters can be found in [16]. For this example, the closed-loop system without anti-windup compensation is not globally stable. Also, there exists no static anti-windup compensation to make the global $\mathcal{L}_2$ gain bounded. With dynamic anti-windup augmentation, an upper bound for the achievable global $\mathcal{L}_2$ gain is found to be 181.1424 (by using quadratic Lyapunov functions). When this achievable gain is approached, some parameters of the anti-windup compensator will approach infinity. To make the parameters within a reasonable range, we have to allow a slightly larger global $\mathcal{L}_2$ gain. A particular dynamic anti-windup compensator is given as follows, with notation adopted from [16],

$$\Lambda_1 = \begin{bmatrix} -10.0484 & -8.6696 & 5.9466 & -34.8168 \\ 16.7846 & -0.0077 & -50.5254 & 33.3906 \\ 27.2580 & 12.9076 & -176.8422 & -20.1985 \\ 6.8086 & 9.5653 & -54.0989 & -35.0035 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.6253 \\ 0.2146 \\ 1.5342 \\ 0.4100 \end{bmatrix} \times 1000,$$

$$\Lambda_3 = \begin{bmatrix} 0.0157 & -0.0010 & -0.0148 & 0.0105 \\ 0.3209 & -0.1315 & 0.1458 & 0.6281 \\ 0.0972 & -0.0763 & 0.1102 & -0.0196 \\ 7.4719 & -5.0878 & 2.7569 & -1.0528 \\ -0.1152 & -0.0367 & 0.5992 & 0.0387 \end{bmatrix} \times 10000, \quad \Lambda_4 = \begin{bmatrix} 0.1467 \\ 0.3452 \\ -0.6949 \\ 2.4840 \\ -5.4618 \end{bmatrix} \times 10000.$$

When quadratic Lyapunov functions are used via the PDI, the estimated global $\mathcal{L}_2$ gain is 182.3080. When V_c (with $J = 2$) is used via the PDI, a slightly smaller estimate is given as 181.2326. For other values of bound on $\|w\|_2$, the improvement by using V_c is also small. However, if we change some parameters of the system, the difference between estimates by quadratics and nonquadratics can be arbitrarily large.

For this particular system, we have $D_{yq} = \Lambda_4(5)$. Hence the algebraic loop is directly affected by $\Lambda_4(5)$. Suppose that we change $\Lambda_4(5)$ from -54618 to -52618. Two estimates of the nonlinear $\mathcal{L}_2$ gain are plotted in Fig. 3, where the dashed curve corresponds to the estimate obtained by applying quadratic functions and the solid one to that obtained by applying V_c (with $J = 2$), both via PDI description. Also plotted as a dash-dotted curve is the estimate obtained by using V_c when $\Lambda_4(5) = -54618$. The above computational results suggest that nonquadratic functions may also have advantage for analyzing robust performance under parameter perturbations. This will motivate further research problems.

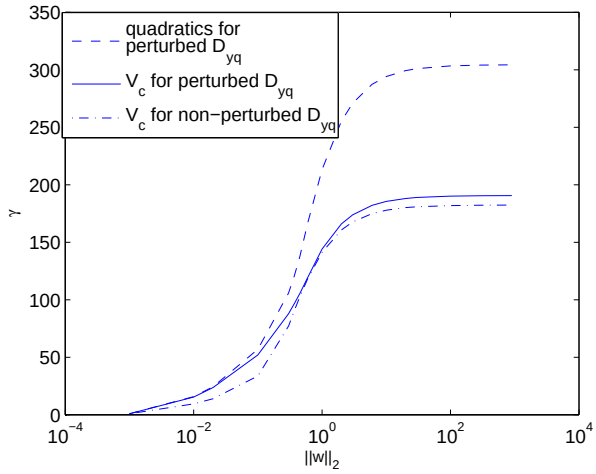

Figure 3: Different estimates of $\mathcal{L}_2$ gain under parameter perturbation.

The order of the closed-loop system for this example is 12, including the state of the plant, the controller and the dynamic anti-windup compensator. The BMI problem for V_c with $J = 2$ involves 189 variables (the two matrices Q_1 and Q_2 for V_c contain 156 variables). It takes about 2 hours to generate the solid curve (a connection of 18 points). The smoothness of the curve suggests the uniformity of the convergence to some optimal or suboptimal solutions, considering that the algorithm was run only once for each value of $\|w\|_2$ and the initial values of λ_{ijk}'s were chosen randomly.

6 Conclusions

For a general system with saturation or deadzone components, regional stability and performance analysis relies on an effective regional treatment of the algebraic loop and the deadzone function. This paper provides such a treatment which yields two forms of parameterized differential inclusions. Applying available tools based on quadratic Lyapunov functions to these differential inclusions, we obtained conditions for stability and performance in the form of LMIs. These conditions are easily tractable but could be conservative in view of the quadratic Lyapunov functions applied. Further improvement relies on using non-quadratic Lyapunov functions. We explored a pair of conjugate Lyapunov functions in this paper and reduced the conservatism of the conditions with a series of BMI conditions. Numerical experience shows that these BMI conditions can be effectively solved with the path following method. Although there is no guarantee that the global optimal solutions will be obtained, the great potential of these non-quadratic Lyapunov functions has been revealed by numerical examples. The effectiveness demonstrated through these examples motivates further investigation on these non-quadratic Lyapunov functions and the development of more efficient algorithms to handle them for more complicated situations. This paper's results lay foundations for the design of saturated controllers and for the design of anti-windup compensators. Preliminary results have been obtained in [29] for regional dynamic anti-windup design which is based on the analysis result by applying quadratic functions via NDI. The analysis results based on PDI and nonquadratic functions can be applied for design purposes by incorporating controller design parameters into the existing optimization problem. In this regard, main efforts will be devoted to making the optimization problems more tractable through careful algebraic manipulation and appropriate parameter transformations.

A Proofs of technical lemmas

Proof of Claim 1. The sufficiency was shown in [53]. [2] Here we show the necessity. Let ϕ_i be a saturation function. It is easy to verify that for each $\delta \in [0,1]$, and each $d \in [-1,1]$, there exist $a, b \in \mathbb{R}, a-b=d$ such that $\phi_i(a)-\phi_i(b)=\delta(a-b)$. We have the same property if ϕ_i is a deadzone function. Now suppose that $\det(I-D\Delta)=0$ for a certain $\Delta \in \mathrm{co}\mathcal{K}$. Then there exist $\delta_i \in [0,1], i \in I[1,m]$, and a nonzero vector $s \in \mathbb{R}^m, s_i \in [-1,1]$ such that

$$(I - D\mathrm{diag}[\delta_1, \delta_2, \cdots, \delta_m])s = 0. \tag{63}$$

Note that s can always be scaled to satisfy $s_i \in [-1,1]$. Let $a_i, b_i \in \mathbb{R}$ be chosen such that $a_i - b_i = s_i$ and $\phi_i(a_i)-\phi_i(b_i)=\delta_i(a_i-b_i)=\delta_i s_i$. Define $u_1 := [a_1 \;\; a_2 \;\; \cdots \;\; a_m]^T$, and $u_2 := [b_1 \;\; b_2 \;\; \cdots \;\; b_m]^T$ and let $v_1 = u_1 - D\phi(u_1), v_2 = u_2 - D\phi(u_2)$. Then $u_1 - u_2 = s \neq 0$ and

$$\phi(u_1) - \phi(u_2) = \mathrm{diag}[\delta_1, \delta_2, \cdots, \delta_m]s. \tag{64}$$

It follows from (63) and (64) that $v_1 - v_2 = s - D(\phi(u_1)-\phi(u_2)) = (I - D\mathrm{diag}[\delta_1, \delta_2, \cdots, \delta_m])s = 0$. This shows that there are two solutions u_1 and u_2 corresponding to the same $v_1 = v_2$. Therefore we conclude that well-posedness implies that $\det(I-D\Delta) \neq 0$ for all $\Delta \in \mathrm{co}\mathcal{K}$. □

Proof of Claim 2. We first show that

$$\mathrm{co}\{\det(I - D_{yq}\Delta) : \Delta \in \mathrm{co}\mathcal{K}\} = \mathrm{co}\{\det(I - D_{yq}K_i) : \; i \in I[1,2^m]\}. \tag{65}$$

Let the diagonal elements of Δ be $d_1, d_2, \cdots, d_m$. Then $\det(I - D_{yq}\Delta)$ is a multi-linear function of d_i's. This means that $\det(I - D_{yq}\Delta) = f_1(d_2, d_3, \cdots, d_m)d_1 + f_0(d_2, d_3, \cdots, d_m)$ for some multi-linear functions f_1 and f_0. Hence the maximum or the minimum of $\det(I - D_{yq}\Delta)$ is obtained at $d_1 = 1$ or $d_1 = -1$. Same can be said for $d_2, d_3, \cdots, d_m$. This verifies (65) and the first part of the claim.

The relation (5), repeated below,

$$\mathrm{co}\{(I - \Delta D_{yq})^{-1}\Delta : \; \Delta \in \mathrm{co}\mathcal{K}\} \subseteq \mathrm{co}\{(I - K_i D_{yq})^{-1}K_i : \; i \in I[1,2^m]\}, \tag{66}$$

can be shown with arguments similar to those in [2, page 57-58]. Recall that for a subset S of a vector space, $x_0 \in S$ is an extreme point of $\mathrm{co}\{S\}$ if and only if there exists a vector c such that $\langle c, x\rangle < \langle c, x_0\rangle$ for all $x \in S \setminus \{x_0\}$. Let $C \in \mathbb{R}^{m\times m}$ be an arbitrary matrix and consider both C and $(I - \Delta D_{yq})^{-1}\Delta$ as real vectors. The inner product of C and $(I - \Delta D_{yq})^{-1}\Delta$, i.e., $\mathrm{trace}(C^T(I - \Delta D_{yq})^{-1}\Delta)$, can be expressed as $(a_1 d_1 + a_0)/(b_1 d_1 + b_0)$, where a_1, a_0, b_1, b_0 are functions of $d_2, \cdots, d_m$. By the well-posedness condition, $b_1 d_1 + b_0 \neq 0$ for all $d_1 \in [-1,1]$. It can be easily verified that $(a_1 d_1 + a_0)/(b_1 d_1 + b_0)$ either increases or decreases over the interval. Hence the maximum or the minimum of $\mathrm{trace}(C^T(I - \Delta D_{yq})^{-1}\Delta)$ is obtained at $d_1 = 1$ or $d_1 = -1$. Same can be said for $d_2, d_3, \cdots, d_m$. This means that every extreme point of the set on the lefthand side of (66) belongs to the set on the righthand side. This completes the proof. □

Proof of Claim 3. We first consider the case where $M = I$ and assume that $2I - D_{yq} - D_{yq}^T = R^2$, where R is symmetric and nonsingular. We will show that

$$\mathrm{co}\{(I - K_i D_{yq})^{-1}K_i : \; i \in I[1,2^m]\} \subseteq \{R^{-2} + R^{-1}\Omega R^{-1} : \|\Omega\| \leq 1\}. \tag{67}$$

Since the set on the righthand side of (67) is convex, to prove (67) and that $(I - K_i D_{yq})^{-1}K_i$ is on the boundary of the set on the righthand side, it suffices to show that there exists Ω_i, $\|\Omega_i\| = 1$, such that

$$K_i(I - D_{yq}K_i)^{-1} = R^{-2} + R^{-1}\Omega_i R^{-1}. \tag{68}$$

[2] Note that in [53] a necessary assumption on the radial unboundedness of the function has inadvertently been omitted (compare also with [50]).

Let

$$\Omega_i = RK_i(I - D_{yq}K_i)^{-1}R - I.$$

Then clearly it satisfies (68). We need to prove that $\|\Omega_i\| = 1$.

Since K_i is a diagonal matrix with 0 or 1 at each diagonal, we have $K_i = K_i^2$. Hence

$$K_iR^2K_i = 2K_iI - K_iD_{yq}K_i - K_iD_{yq}^TK_i = K_i(I - D_{yq}K_i) + (I - D_{yq}K_i)^TK_i.$$

Multiplying on the left by $R(I - D_{yq}K_i)^{-T}$ (where $X^{-T} = (X^T)^{-1}$) and on the right by $(I - D_{yq}K_i)^{-1}R$ we have

$$R(I - D_{yq}K_i)^{-T}K_iR^2K_i(I - D_{yq}K_i)^{-1}R = R(I - D_{yq}K_i)^{-T}K_iR + RK_i(I - D_{yq}K_i)^{-1}R.$$

This leads to $(RK_i(I - D_{yq}K_i)^{-1}R - I)^T(RK_i(I - D_{yq}K_i)^{-1}R - I) = I$, i.e., $\Omega_i^T\Omega_i = I$. This not only proves (67) but also shows that $K_i(I - D_{yq}K_i)^{-1}$ is on the boundary of the set on the righthand side of (67).

Now consider an arbitrary diagonal positive matrix M. Then $K_i = MK_iM^{-1}$ and

$$K_i(I - D_{yq}K_i)^{-1} = MK_iM^{-1}(I - D_{yq}MK_iM^{-1})^{-1} = MK_i(I - M^{-1}D_{yq}MK_i)^{-1}M^{-1},$$

where we have used the fact that $X(I - YX)^{-1} = (I - XY)^{-1}X$. Applying (67) by replacing D_{yq} with $M^{-1}D_{yq}M$ we have

$$K_i(I - D_{yq}K_i)^{-1} \in \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} : \|\Omega\| \le 1\},$$

where $S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^TM^{-1}$. It is also straightforward to conclude that $K_i(I - D_{yq}K_i)^{-1}$ is on the boundary of the set at the righthand side. □

Proof of Lemma 3. Without loss of generality, consider $j = 1$. Note that

$$V_c(x) = \frac{1}{2}\min\left\{x^{\mathrm{T}}\left(Q_1 + \sum_{j=2}^{N}\gamma_k(Q_k - Q_1)\right)^{-1}x : \sum_{k=2}^{N}\gamma_k \le 1, \gamma_k \ge 0\right\}.$$

It is implied here that $\gamma_1 = 1 - \sum_{k=2}^{N}\gamma_k$. For a fixed x, define

$$\phi(\gamma_2, \gamma_3, \cdots, \gamma_N) := \frac{1}{2}x^{\mathrm{T}}\left(Q_1 + \sum_{k=2}^{N}\gamma_k(Q_k - Q_1)\right)^{-1}x.$$

Then by Schur complements, for any $c > 0$, the set

$$\left\{(\gamma_2, \gamma_3, \cdots, \gamma_N) : \phi(\gamma_2, \gamma_3, \cdots, \gamma_N) \le c, \sum_{k=2}^{N}\gamma_k \le 1, \gamma_k \ge 0\right\} \tag{69}$$

is convex. Hence the optimal $(\gamma_2, \gamma_3, \cdots, \gamma_N)$'s that minimize ϕ form a convex set.

If $x \in \bigcup_{\delta\in[0,1]}\delta E_1$, then $V_c(x) = \frac{1}{2}x^{\mathrm{T}}Q_1^{-1}x$, implying that the minimal value of ϕ is reached at $(\gamma_2, \gamma_3, \cdots, \gamma_N) = (0, 0, \cdots, 0)$. This means that at this point, $\partial\phi/\partial\gamma_k \ge 0$ for all $k \in I[2, N]$, i.e.,

$$x^{\mathrm{T}}Q_1^{-1}(Q_k - Q_1)Q_1^{-1}x \le 0 \quad \forall\, k \in [2, N]. \tag{70}$$

On the other hand, it is also clear that (70) implies that the minimal value of ϕ is reached at $(0, 0, \cdots, 0)$ by the convexity of the set in (69). Hence, (70) is equivalent to $V_c(x) = \frac{1}{2}x^{\mathrm{T}}Q_1^{-1}x$. In summary, we have

$$\bigcup_{\delta\in[0,1]}\delta E_1 = \{x \in L_{V_c} : x^{\mathrm{T}}Q_1^{-1}(Q_k - Q_1)Q_1^{-1}x \le 0, k \in I[1, N]\}.$$

□

Proof of Lemma 6. 1) Note that for any positive definite matrix P_j, we have

$$x^T P_j x - x_0^T P_j x_0 - 2x_0^T P_j(x - x_0) = (x - x_0)^T P_j (x - x_0). \tag{71}$$

Since $V_{\max}(x_0) = x_0^T P_j x_0$ for $j \in I[1, J_0]$, we obtain

$$V_{\max}(x) \geq x^T P_j x \geq V_{\max}(x_0) + 2x_0^T P_j(x - x_0) \;\; \forall j \in I[1, J_0], x \in \mathbb{R}^n.$$

Applying convex combination of the above inequalities,

$$V_{\max}(x) \geq V_{\max}(x_0) + c^T(x - x_0) \quad \forall\, c \in \mathrm{co}\{2P_j x_0 : j \in I[1, J_0]\}, x \in \mathbb{R}^n.$$

This shows $\mathrm{co}\{2P_j x_0 : j \in I[1, J_0]\} \subset \partial V_{\max}(x_0)$. To show the converse, we consider an arbitrary $c \notin \mathrm{co}\{2P_j x_0 : j \in I[1, J_0]\}$. Then there exist $\zeta \in \mathbb{R}^n$, $|\zeta|_2 = 1$, and $\varepsilon > 0$ such that

$$c^T(\alpha\zeta) > 2x_0^T P_j(\alpha\zeta) + \alpha\varepsilon \quad \forall \alpha > 0, \;\; j \in I[1, J_0]. \tag{72}$$

Let $x = x_0 + \alpha\zeta$, then from (71) and (72) we obtain

$$x^T P_j x - V_{\max}(x_0) < c^T(x - x_0) - \varepsilon\alpha + \alpha^2 \zeta^T P_j \zeta, \;\; \forall \alpha > 0, j \in I[1, J_0].$$

It is clear that there always exists a sufficiently small $\alpha > 0$ such that

$$x^T P_j x - V_{\max}(x_0) < c^T(x - x_0) \quad \forall j \in I[1, J_0].$$

Also note that when $x - x_0 = \alpha\zeta$ is sufficiently small, we still have $V_{\max}(x) = \max\{x^T P_j x : j \in I[1, J_0]\}$. In summary, there exists an $x \in \mathbb{R}^n$ such that

$$V_{\max}(x) - V_{\max}(x_0) < c^T(x - x_0).$$

This shows that $c \notin \partial V_{\max}(x_0)$ and confirms that $\partial V_{\max}(x_0) \subset \mathrm{co}\{2P_j x_0 : j \in I[1, J_0]\}$. Therefore, we conclude that $\partial V_{\max}(x_0) = \mathrm{co}\{2P_j x_0 : j \in I[1, J_0]\}$.

2) By (71), with $x = x_0 + t\zeta$, we have

$$x^T P_j x - x_0^T P_j x_0 = 2t x_0^T P_j \zeta + t^2 \zeta^T P_j \zeta. \tag{73}$$

Again, for sufficiently small t, we have $V_{\max}(x) = \max\{x^T P_j x : j \in I[1, J_0]\}$. Ignoring the second order term for sufficiently small t, we obtain

$$V_{\max}(x) = V_{\max}(x_0) + t \max\{2x_0^T P_j \zeta : \;\; j \in I[1, J_0]\}.$$

This leads to (53). □

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