Static anti-windup design for discrete-time large-scale cross-directional saturated linear control systems

I. Queinnec, S. Tarbouriech, S. Gayadeen, L. Zaccarian

Abstract

The paper proposes some design strategies of static anti-windup control schemes for systems describing cross-directional processes. These are processes in which the variations of a variable in a profile orthogonal to the direction of propagation of the variable are controlled. Actually, the anti-windup synthesis approach takes advantages of the particular structure of such systems for which the interaction map between sensors and actuators along the propagation direction is expressed in terms of reduced singular value decompositions. The approach is developed within the framework of a synchrotron machine, in which electrons are accelerated in a closed circular path and bent by strong electromagnetic fields. Different results are proposed with increasing computational complexity to evaluate the potential interest of low dimension problems to solve the anti-windup design problem.

I. INTRODUCTION

Synchrotron light is the electromagnetic radiation emitted by charged particles (electrons) that move at high speeds and change direction [6], [13], [5]. The acceleration of the electrons is performed in a circumference storage ring in which they are confined by magnetic fields. The ring consists in a succession of identical cells involving straight sections and bending magnets to curve the electron beam. Such electron beam is subjected to disturbance and a control strategy is required to maintain the electron beam position around the storage ring by acting on magnet power. Moreover, the effect on the beam location of a change in a single corrector magnet extends around the ring, resulting in strong interactions among the responses of different magnets. Such beam stabilization systems may be considered to be analogous to cross-directional processes, for which it is required to control variations of a measured variable in a profile that is orthogonal to the direction of propagation of the variable. Typical examples that belong to this class are issued from rolling processes involved in paper machines, plastic film extrusion or metal forming [11], [17], [3], [4]. A special issue of the IEE Proc - Control Theory Applications was devoted to such industrial web processes in 2002 [2]. Back to our case study, such a cross-directional control strategy has then been applied to control profiles in the direction orthogonal to the direction of propagation of the beam [13].

On the other hand, it has been underlined that saturation of magnetic elements used as actuators may strongly deteriorate the closed-loop behavior of the system [8], [1], [15], involving unfavorable disturbances within wide ranges of frequencies. Anti-windup strategies [18], [16], may then represent an appropriate framework to mitigate those saturation effects. Very few works have been reported in the literature addressing this issue in the context of cross-directional control [14], [12], [6], and also model-predictive control strategies have been barely exploited [10]. In this paper, we propose to implement a static linear anti-windup scheme, acting on the controller state and output equations, that statically reintroduces the difference between the computed control action and its saturated version in the control scheme for performance improvement. Moreover, using the peculiar structure of the problem, mapping the influence of each actuator to each sensor, we propose several solutions to the anti-windup design problem which allow to obtain a trade-off between the computational complexity and the potential for effective disturbance rejection.

The rest of the paper is structured as follows. The problem under consideration is stated in Section II, where the overall structure of the system involving the anti-windup action is presented. Three solutions are proposed in Section III with increasing computational complexity, whose proofs are omitted for space constraints. Optimization issues are reported in Section IV and the application to an example inspired by a synchrotron model is proposed in Section V. Finally, some concluding remarks are drawn in Section VI.

Notation. Given matrices $A$ and $B$, $A \otimes B$ denotes their Kronecker product. Given a square real matrix $X$, we denote $\text{He}X = X + X^T$, where $X^T$ denotes the transpose of $X$. Given a sequence $k \mapsto v(k), k \in \mathbb{N}$, the $\ell_2$ norm of $v$ corresponds to $\|v\|_2 := \sqrt{\sum_{k=0}^{+\infty} |v(k)|^2}$, where the vector norm used for $|v(k)|$ is the standard Euclidean norm.

II. PROBLEM STATEMENT

Inspired by the works in [13], [6] we consider a linear discrete-time plant comprising $N$ dynamic actuators, each of them having a scalar transfer function $p(z)$ and $N$ sensors characterized by static linear transfer functions. We assume that sensors

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and actuators are spatially interconnected by instantaneous relations, in such a way that the overall input/output dynamics can be described by the vector relation

\[ y = p(z)Bu + d = B\text{diag}\{p(z), p(z), \ldots, p(z)\}u + d, \quad (1) \]

where \( B \in \mathbb{R}^{N \times N} \) is a suitable square matrix and \( d \) is a suitable performance input. Following [13], [6], it is reasonable to control plant (1) by designing a modular controller that works in the “modal space”, which corresponds to the directions obtained by a singular value decomposition of matrix \( B \):

\[ B = \Phi \Sigma \Psi^T, \quad (2) \]

where \( \Phi \) and \( \Psi \) are square orthogonal matrices of suitable dimensions (namely they satisfy \( \Phi^T = \Phi^{-1} \) and \( \Psi^T = \Psi^{-1} \)). Then using linearity, the controller for plant (1) can be designed in the modal space by selecting the desired plant input \( y_c \) as

\[ y_c = c(z)B^{-1}y = \Psi \Sigma^{-1}\text{diag}\{c(z), c(z), \ldots, c(z)\}\Phi^Ty, \quad (3) \]

where the scalar transfer function \( c(z) \) in (3) is selected in such a way that its feedback interconnection with \( p(z) \) induces suitable exponential stability and performance properties from the disturbance to the output. As already noted in [13], [6], the linear feedback interconnection \( u = y_c \) between (1) and (3) induces a desirable closed-loop behavior well characterized in the modal space.

In this paper we are interested in the situation where each transfer function \( p(z) \) at the actuator nodes is subject to magnitude saturation, which can be represented by

\[ u_i = \text{sat}(y_{ci}), \quad \text{sat}(s) = \begin{cases} -\bar{u}, & \text{if } s \leq -\bar{u} \\ \bar{u}, & \text{if } s \geq \bar{u} \\ s, & \text{otherwise}, \end{cases} \quad (4) \]

where we fix a uniform bound \( \bar{u} \) for all the actuators, due to the uniform nature of the addressed problem. Henceforth, with a slight abuse of notation, we use interchangeably the symbol \( \text{sat} \) for both the scalar and the (decentralized) vector saturation function. The presence of saturation makes it impossible to preserve the transfer function description of the closed
\[ x_{cl}^+ = (A_{cl} \otimes I_N)x_{cl} + (B_{cl,u} \otimes I_N)\Sigma \Psi^T dz(\Psi \Sigma^{-1} \nu) + (B_{cl,v} \otimes I_N)v + (B_{cl,d} \otimes I_N)\Phi^T d \]
\[ \nu = (C_{cl,u} \otimes I_N)x_{cl} + (D_{cl,u,v} \otimes I_N)\Sigma \Psi^T dz(\Psi \Sigma^{-1} \nu) + (D_{cl,v,u} \otimes I_N)v + (D_{cl,v,d} \otimes I_N)\Phi^T d \]
\[ y = \Phi ((C_{cl,y} \otimes I_N)x_{cl} + (D_{cl,y,u} \otimes I_N)\Sigma \Psi^T dz(\Psi \Sigma^{-1} \nu) + (D_{cl,y,v} \otimes I_N)v + (D_{cl,y,d} \otimes I_N)\Phi^T d) + d. \]

loop and a suitable state-space representation of the system must be considered, making a clear distinction among signals leaving in the actuator’s space (where the saturation nonlinearity is acting) and in the sensor’s space (where the disturbance is acting). In particular, we consider the following state-space representation for each one of the scalar transfer functions \( p(z) \) and \( c(z) \):

\[ p(z) : \begin{cases} x_{pi}^+ = A_p x_{pi} + B_p u_{pi} \\ \zeta_i = C_p x_{pi} + D_p u_{pi}, \end{cases} \]
\[ c(z) : \begin{cases} x_{ci}^+ = A_c x_{ci} + B_c u_{ci} + v_{i1} \\ v_i = C_c x_{ci} + D_c u_{ci} + v_{i2}, \end{cases} \]

where \( x_{pi} \in \mathbb{R}^{n_p}, x_{ci} \in \mathbb{R}^{n_c}, \) and \( v_{i1}, v_{i2} \) are input signals available to perform a suitable correction (anti-windup action) for mitigating the negative effects of saturation at the controller level.

With these equations in place, the overall control system corresponds to the block diagram represented in Figure 1, where we represent the saturation function with its equivalent form \( sat(u) = u - dz(\psi(v)) \), with \( dz \) denoting the deadzone function. The equations describing the closed loop with saturation correspond to:

\[ x_p^+ = (A_p \otimes I_N)x_p + (B_p \otimes I_N)\Sigma \Psi^T \text{sat}(\Psi \Sigma^{-1} \nu) \]
\[ x_c^+ = (A_c \otimes I_N)x_c + (B_c \otimes I_N)\Phi^T y + v_1 \]
\[ \nu = (C_c \otimes I_N)x_c + (D_c \otimes I_N)\Phi^T y + v_2 \]
\[ y = \Phi ((C_p \otimes I_N)x_p + (D_p \otimes I_N)\Sigma \Psi^T \text{sat}(\Psi \Sigma^{-1} \nu)) + d \]

where \( x_p \) is the vector stacking all the components \( x_{pi} \) and similarly for \( x_c, \nu, v_1, \) and \( v_2 \).

Due to the peculiar structure of system (8), and using linearity, we may perform the convenient transformation of the closed loop represented in Figure 2, and corresponding to the equations (7) at the top of the page, where we use \( x_{cl} = (x_p, x_c) \) and all the matrices are uniquely defined based on the data in (5) and (6) if the linear closed-loop between \( p(z) \) and \( c(z) \) is (linearly) well posed. For the sake of completeness, we report them below for the simplified case where \( D_p = 0 \) which is the case for our application study of Section V:

\[
\begin{bmatrix}
A_{cl} \\
C_{cl,u} \\
C_{cl,y}
\end{bmatrix} =
\begin{bmatrix}
A_p + B_p D_c C_p & B_p C_c & A_c \\
B_c C_p & D_c C_p & C_c \\
C_p & 0 & 0
\end{bmatrix}
\begin{bmatrix}
B_{cl,u} \\
B_{cl,v} \\
B_{cl,d}
\end{bmatrix}
\begin{bmatrix}
D_{cl,u,v} & D_{cl,v,u} & D_{cl,v,d} \\
D_{cl,u,v} & D_{cl,v} & D_{cl,u,d} \\
D_{cl,y,u} & D_{cl,y,v} & D_{cl,y,d}
\end{bmatrix}
\begin{bmatrix}
-B_p & 0 & B_p D_c \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
B_{cl,d} \\
D_{cl,v,d} \\
D_{cl,y,d}
\end{bmatrix}
\]

With these equations in place, we recognize that the closed-loop dynamics (7) is linear before the action of the perturbation caused by the nonlinear term \( \Psi^T dz(\psi(v)) \). Due to this reason, and following an anti-windup type of paradigm, the problem that we address in this paper is that of designing a matrix gain \( D_{aw} \) in such a way that the selection

\[ v = D_{aw} \Sigma \Psi^T dz(\Psi \Sigma^{-1} \nu), \]

ensures suitable stability and disturbance rejection from input \( d \) to output \( y \) for dynamics (7), (9). In particular, one aspect that we will take into account in this paper is related to the curse of dimensionality, so in the next section we will propose three solutions having increasing computational complexity and increasing potential for effective disturbance rejection.

III. Problem solution

A first result that we can immediately provide comes from exploiting the orthogonality properties of matrices \( \Phi \) and \( \Psi \) and the special structure of the matrices appearing in (7). Then the anti-windup gain can be conveniently designed by first solving a small-dimensional LMI optimization, and then lifting the solution to the larger space, as indicated in the next theorem.
\[
\begin{bmatrix}
A_{cl} & B_{cl,u} & B_{cl,v} & B_{cl,d} \\
C_{cl,u} & D_{cl,vu} & D_{cl,vd} & D_{cl,yu} \\
C_{cl,v} & D_{cl,yv} & D_{cl,yd} & D_{cl,yd}
\end{bmatrix}
\begin{bmatrix}
A_{cl} \otimes I_N & B_{cl,u} \otimes I_N & B_{cl,v} \otimes I_N & B_{cl,d} \otimes I_N \\
C_{cl,u} \otimes I_N & D_{cl,vu} \otimes I_N & D_{cl,vd} \otimes I_N & D_{cl,yu} \otimes I_N \\
C_{cl,v} \otimes I_N & D_{cl,yv} \otimes I_N & D_{cl,yd} \otimes I_N & D_{cl,yd} + I_N
\end{bmatrix}
\]

(12)

**Theorem 1:** Fix a bound \(s\) on the maximum size of the disturbance, in terms of its \(\ell_2\) norm. Consider any feasible selection of variables \(Q_0 \in \mathbb{R}^{n_+ \times n_+}, u_0 \in \mathbb{R}, X_0 \in \mathbb{R}^{(n_+ + 1) \times 1}, Y_0 \in \mathbb{R}^{1 \times (n_+ + n_z)}, \gamma^2 \in \mathbb{R}\) for the following LMI constraints:

\[
Q_0 = Q_0^T > 0, u_0 > 0
\]

(10a)

\[
\begin{bmatrix}
A_{cl}^2 Q_0 A_{cl} - Q_0 & u_0 B_{cl,u} + B_{cl,v} X_0 & B_{cl,d} & 0 \\
C_{cl,v} Q_0 + Y_0 & u_0 (D_{cl,vu} - I) + D_{cl,vd} X_0 & D_{cl,vd} & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
C_{cl,y} Q_0 & u_0 D_{cl,yu} + D_{cl,yv} X_0 & 1 + D_{cl,yd} - \frac{\sigma^2}{\sigma_m^2} \gamma^2 & < 0
\end{bmatrix}
\]

(10b)

where \(\sigma_m, \sigma_M\) denote, respectively, the minimum and maximum diagonal entries of \(\Sigma\). Then the selection:

\[
D_{aw} = u_0^{-1} X_0 \otimes I_N
\]

(11)

is such that any solution to the anti-windup closed-loop system (7), (9), (11) starting from zero initial conditions and with a disturbance input \(d\) satisfying \(\|d\|_2 \leq s\), satisfies the \(\ell_2\) performance bound \(\|y\|_2 \leq \gamma \|d\|_2\).

The construction in Theorem 1 is appealing because it requires solving a small-dimensional problem involving the size of each plant-controller pair \(p(z), c(z)\) rather than involving all the \(N\) actuator/sensor pairs distributed along the plant. Nevertheless, the arising conditions may be overly conservative and may provide an anti-windup construction that is not as good as one could get by way of a fully centralized design. To begin with, one may even experience infeasibility of constraints (10) and then be willing to solve a more complex, centralized, design problem.

Due to this reason, we propose below two other solutions that enlarge the feasibility set of the corresponding constraints. To the end of providing a compact representation of the proposed solutions, we define the matrices reported at the top of the page in (12), representing the entries in the centralized dynamics (7) with the scaling caused by \(\Sigma\) but without the input/output rotations caused by matrix \(\Phi\). With the definitions in (12), we can now formulate our second anti-windup synthesis result, corresponding to the next theorem.

**Theorem 2:** Fix a bound \(s\) on the maximum size of the disturbance, in terms of its \(\ell_2\) norm. Consider any feasible selection of variables \(Q \in \mathbb{R}^{N(n_+ + n_z) \times N(n_+ + n_z)}, U \in \mathbb{R}^{N \times N}, diagonal, X \in \mathbb{R}^{N(n_+ + 1) \times N}, Y \in \mathbb{R}^{N \times N(n_+ + n_z)}, \gamma^2 \in \mathbb{R}\) for the following LMI constraints:

\[
Q = Q^T > 0, U > 0 \text{ diagonal}
\]

(13a)

\[
\begin{bmatrix}
A_{cl}^2 Q_0 A_{cl} - Q_0 & B_{cl,u} \bar{U} + B_{cl,v} X & B_{cl,d} & 0 \\
C_{cl,v} Q_0 + Y (D_{cl,vu} - I) \bar{U} + D_{cl,vd} X & D_{cl,vd} & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
C_{cl,y} Q_0 & D_{cl,yu} \bar{U} + D_{cl,yv} X & D_{cl,yd} - \frac{\gamma^2}{2} I & < 0
\end{bmatrix}
\]

(13b)

where we used the (linear) relation \(\bar{U} := \Sigma \Psi^T U \Psi \Sigma\).
Then the selection:

$$D_{aw} = X U^{-1} = X \Sigma^{-1} \Psi^T U^{-1} \Psi \Sigma^{-1}$$ (14)

is such that any solution to the anti-windup closed-loop system (7), (9), (14) starting from zero initial conditions and with a disturbance input \(d\) satisfying \(\|d\| \leq s\), satisfies the \(\ell_2\) performance bound \(\|y\| \leq \gamma \|d\|\).

It is emphasized that the construction of Theorem 2 involves significantly heavier computations, due to the presence of full-size variables in the constraints (13). Indeed, the construction provides a centralized design, while the previous one of Theorem 1 was decentralized. The advantage of this second formulation is established in the next section where comparative analysis of the optimal gains is carried out. The range of different solutions proposed in this paper is completed by the following result that proposes an even more computationally intensive solution that has greater potential as compared to that of Theorem 2.

**Theorem 3:** Fix a bound \(s\) on the maximum size of the disturbance, in terms of its \(\ell_2\) norm. Consider any feasible selection of variables \(Q \in \mathbb{R}^{N(n_p + n_c) \times N(n_p + n_c)}, U \in \mathbb{R}^{N \times N}\), diagonal, \(X \in \mathbb{R}^{N \times N(n_p + n_c)}, Y \in \mathbb{R}^{N \times N(n_p + n_c)}, \gamma^2 \in \mathbb{R}\) for the LMIs (13a), (13b), together with:

$$
\begin{bmatrix}
Q & Y^T \\
Y & \Sigma^{-1}
\end{bmatrix} \geq 0, \quad \forall i = 1, \ldots, N,
$$

(15)

where \(Y[i]\) denotes the \(i\)-th row of matrix \(Y\).

Then the selection (14) is such that any solution to the anti-windup closed-loop system (7), (9), (14) starting from zero initial conditions and with a disturbance input \(d\) satisfying \(\|d\| \leq s\), satisfies the \(\ell_2\) performance bound \(\|y\| \leq \gamma \|d\|\).

The proofs of the theorems above have been omitted in this version due to lack of space. They use the following lemma, which is a generalization of the well-known generalized sector property proposed in [7], [9].

**Lemma 3.1:** Given any orthogonal matrix \(\Psi\) and any diagonal positive definite matrices \(W\) and \(\Sigma\), denote \(\bar{W} := \Sigma^{-1} \Psi^T W \Psi \Sigma^{-1}\) and \(dz_{\Psi}(\nu) := \Sigma \Psi^T dz_{\Psi}(\Psi \Sigma^{-1} \nu)\). Then the following relation holds for any \(u, w \in \mathbb{R}^N\):

$$dz_{\Psi}(w) = 0 \Rightarrow dz_{\Psi}(\nu)^T \bar{W}(\nu - dz_{\Psi}(\nu) + w) \geq 0.$$ (16)

### IV. Optimal Performance Levels

The three anti-windup solutions presented in the previous section and illustrated in Theorems 1, 2 and 3, provide three different levels of trade-off between computational burden and size of the feasibility set. When performing anti-windup design, we are interested in minimizing the worst case \(\ell_2\) bound corresponding to (the square root of) the value of \(\gamma^2\) in each one of the proposed constructions. In particular, once an upper bound \(s > 0\) is fixed for the worst case \(\ell_2\) norm of the disturbance input, three alternative optimized anti-windup designs can be performed by solving one of the following convex LMI-optimization problems:

$$
\gamma_1^2 = \min_{Q, u, x, y, \gamma^2} \gamma^2, \quad \text{subject to (10)},
$$

(17)

$$
\gamma_2^2 = \min_{Q, U, X, Y, \gamma^2} \gamma^2, \quad \text{subject to (13)},
$$

(18)

$$
\gamma_3^2 = \min_{Q, U, X, Y, \gamma^2} \gamma^2, \quad \text{subject to (13a), (13b), (15)}.
$$

(19)

Then all three optimization problems above are interesting because they are of increasing computational complexity but are guaranteed to provide nonincreasing performance guarantees. In particular, the following proposition establishes the interest in adopting increasingly complex anti-windup constructions to the end of obtaining the smallest possible value for the guaranteed \(\ell_2\) gain.

**Proposition 1:** The optimal values of the three optimization problems in (17), (18), and (19) satisfy:

$$\gamma_1^2 \geq \gamma_2^2 \geq \gamma_3^2.$$ (20)

From the point of view of the computational burden, Table I, where we use \(n_t = n_p + n_c\), well illustrates the increasing computational resources necessary to solve the optimization problems.

### V. Application to a Synchrotron Model

In this section we test our synthesis algorithm on numerical data whose values are issued from [6]. The open-loop dynamic response of the system is dominated by the first order response of the power supply units for the corrector magnet and the delay in the sensor data acquisition and processing, so that [6, eq. (38)]:

$$p(z) = \frac{0.3558}{z^8 - 0.6442z^7},$$
which corresponds to a bandwidth of the actuator response of $700 \text{ rad.s}^{-1}$, a sample interval of $100 \mu s$ and a total delay of 7 samples. The controller is defined as:

$$c(z) = \frac{q(z)}{1 - p(z)q(z)}$$

where $q(z)$ is an IMC filter dynamics selected as $[6, \text{eq. (39)}]$:

$$q(z) = 0.4 \frac{0.3741z - 0.241}{z - 0.8669}.$$  

The bound on the magnitude saturation and the maximum size of the disturbance are set as

$$\overline{\gamma} = 1, \quad s = 10$$

The static map from the $N$ actuators to the $N$ sensors positions is extracted from the steady state response matrix in the vertical plane of a real machine from Diamond Light Source, for various values of $N$. Let us consider a case with $N = 4$ for which one selects

$$B = \begin{bmatrix}
0.5984 & 0.9022 & -0.9192 & 1.0138 \\
1.5123 & 0.7611 & -0.7681 & 0.7771 \\
-1.4066 & -0.7489 & 0.6329 & -0.6318 \\
0.6533 & 0.2355 & -0.1403 & 0.5424
\end{bmatrix},$$

Solutions of optimization problems (17), (18) and (19), are given by:

- **Thm 1 (optimization (17)):** $\gamma_1 = 219.1116$ and

$$D_{aw1} = \begin{bmatrix}
0.0601 \\
-0.0362 \\
-0.0021 \\
0.0002 \\
-0.0001 \\
-0.0004 \\
-0.0035 \\
0.0216 \\
-0.0571 \\
0.5673
\end{bmatrix} \otimes I_N.$$

- **Thm 2 (optimization (18)):** $\gamma_2 = 4.4737$ and

$$D_{aw2} = \begin{bmatrix}
D_{aw21} & \bullet & \bullet \\
\bullet & D_{aw22} & \bullet \\
\bullet & \bullet & D_{aw23} \\
\bullet & \bullet & \bullet & D_{aw24}
\end{bmatrix},$$

where $\bullet$ denotes negligible elements (close to 0) outside the block-diagonal structure and
\[
\begin{bmatrix}
D_{aw,21} & D_{aw,22} & D_{aw,23} & D_{aw,24} \\
2.2954 & 42.6375 & 288.5621 & 5440.724 \\
-1.4793 & -27.4789 & -185.9719 & -3506.426 \\
-0.0002 & -0.0035 & -0.0234 & -0.4416 \\
-0.0002 & -0.0033 & -0.0223 & -0.4199 \\
-0.0002 & -0.0034 & -0.0228 & -0.4297 \\
-0.0002 & -0.0034 & -0.0232 & -0.4372 \\
-0.0001 & -0.0026 & -0.0179 & -0.3368 \\
-0.0006 & -0.0104 & -0.0706 & -1.3318 \\
0.0032 & 0.0590 & 0.3991 & 7.5238 \\
\end{bmatrix}
\]

- Thm 3 (optimization (19)): \( \gamma_3 = 4.4738 \) and \( D_{aw,3} \) presents the same “almost block-diagonal” structure as \( D_{aw,2} \) with

\[
\begin{bmatrix}
D_{aw,31} & D_{aw,32} & D_{aw,33} & D_{aw,34} \\
1.9549 & 36.4470 & 245.7633 & 4622.627 \\
-1.2611 & -23.5130 & -158.5460 & -2982.168 \\
-0.0007 & -0.0133 & -0.0936 & -1.7588 \\
-0.0005 & -0.0103 & -0.0677 & -1.2442 \\
-0.0005 & -0.0097 & -0.0662 & -1.2969 \\
-0.0006 & -0.0116 & -0.0801 & -1.4569 \\
-0.0003 & -0.0055 & -0.0365 & -0.7391 \\
-0.0021 & -0.0422 & -0.2804 & -5.1625 \\
0.0107 & 0.2061 & 1.3702 & 25.5798 \\
-12.1675 & -244.4823 & -1654.357 & -31135.45
\end{bmatrix}
\]

One can comment that Theorem 2 does not add much conservatism with respect to Theorem 3, and is associated with a significantly reduced computation time (thanks to a reduced number of lines). This is illustrated in Table V, which compares the computational times of the three optimization problems for increasing values of \( N \). All numerical evaluations have been computed on a standard Macbook Air with a Matlab® function using lmi lab as parser and optimizer. Realistic case studies of actuators/sensors would be difficult to handle with this configuration (except for Thm 1, of course) as they may involve much more than 100 elements, but all the ingredients for solving such a large-scale anti-windup problem are now available.

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**Table II**

Comparative synthesis computational time among the three anti-windup constructions (17), (18), (19), for various values of actuators/sensors configurations.

On the other hand, the bound \( \gamma_1 \) obtained from Theorem 1 is significantly larger than the one obtained from Theorems 2 and 3. However, this bound is partly conservative due to the structure of the optimization variables. This fact can be appreciated by using Theorem 3 as a performance analysis tool for various fixed selections of \( D_{aw} \). Then, Table III shows that the performance analysis with the anti-windup gain \( D_{aw,1} \) confirms that this simplified strategy is sufficient to implement an anti-windup action with a strongly reduced time of computation of the anti-windup gain and a slightly reduced performance. Note also that although Proposition 1 guarantees by construction that \( \gamma_3 < \gamma_2 \), this is not necessarily true in practice due to numerical issues arising from the increased computational burden.
VI. Conclusion

In this paper we proposed three static linear anti-windup construction for discrete-time linear cross-directional control systems. The work is inspired by a relevant application related to synchrotron beam control. The proposed constructions have been characterized in terms of computational complexity and guaranteed performance level. In particular, the proposed designs range from the least conservative and most computationally complex one to the most conservative and least computationally complex one. Numerical results confirmed the stated results, when tested on plant-controller data taken from a synchrotron model. Future work involves possibly reducing the conservatism of our first construction, testing the proposed techniques on a more realistic model with a larger number of actuators/sensors, and applying our construction to alternative examples of cross-directional processes.

References