Output feedback design for saturated linear plants using deadzone loops ⋆

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Abstract

In this paper, we present a LMI-based synthesis approach on output feedback design for input saturated linear systems by using deadzone loops. Algorithms are developed for minimizing the upper bound on the regional $\mathcal{L}_2$ gain for exogenous inputs with $\mathcal{L}_2$ norm bounded by a given value, and for minimizing this upper bound with a guaranteed reachable set or domain of attraction. The proposed synthesis approach will always lead to regionally stabilizing controllers if the plant is exponentially unstable, to semi-global results if the plant is non-exponentially unstable, and to global results if the plant is already exponentially stable, where the only requirement on the linear plant is detectability and stabilizability. The effectiveness of the proposed techniques is illustrated with one example.

Key words: output feedback control, input saturation, $\mathcal{L}_2$ gain, reachable set, domain of attraction, LMIs

1 Introduction

The behavior of linear, time-invariant (LTI) systems subject to actuator saturation has been extensively studied for several decades. More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks (see Postlethwaite et al. (2007) for a nice overview of application cases requiring a formal treament of the saturation constraints). Most of the research efforts geared toward constructive linear or nonlinear control for saturated plants can be divided into two main strands. In the first one, called anti-windup design, a pre-designed controller is given, so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). Given the pre-designed controller, anti-windup design addresses the controller augmentation problem aimed at maintaining the pre-designed controller behavior before saturation and introducing suitable modifications after saturation so that global (or regional) asymptotic stability is guaranteed (local asymptotic stability already holds by the properties of the saturation nonlinearity). Anti-windup research has been largely discussed and many constructive design algorithms have been formally proved to induce suitable stability and performance properties. Many of these constructive approaches (for example, see Sofrony et al. (2007); Turner et al. (2007); Roos and Biannic (2008); Cao et al. (2002); Crawshaw and Vinnicombe (2000); Gomes da Silva Jr and Tarbouriech (1998, 2005); Grimm et al. (2003, 2004a,b); Marcopoli and Phillips (1996); Mulder et al. (2001); Zaccarian and Teel (2002)) rely on convex optimization techniques and provide Linear Matrix Inequalities (LMIs) (see Boyd et al. (1994)) for the anti-windup compensator design. The second research strand, can be called "direct design", to resemble the fact that saturation is directly accounted for in the controller design and that no specification or constraint is imposed on the behavior of

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the closed-loop for small signals. Direct designs for saturated systems range from the well-known Model Predictive Control (MPC) techniques by Mayne et al. (2000), (especially suitable for discrete-time systems) to sophisticated nonlinear control laws which are able to guarantee global asymptotic stability for all linear saturated and globally stabilizable plants, for example, see the scheduled Riccati approach in Megretski (1996) and the nested saturations of Sussmann et al. (1994); Teel (1992). Several LMI-based methods on direct controller design for linear plants with input saturation have also been proposed, for example, see Gomes da Silva Jr and Tarbouriech (2001); Kendi et al. (1998); Mulder et al. (2009); Scherer et al. (1997). It is not our scope to mention here all the extensive literature on direct design for saturated systems, but it is worth mentioning that several constructive methods are available that differ in simplicity, effectiveness and formality. For example, the work in Cao et al. (2003) addresses LMI-based designs for systems with sensor saturation using similar tools to the ones that we adopt here (namely those from Gahinet and Apkarian (1994)). However the conditions that they derive in Cao et al. (2003) are very different from ours because the issues with sensors saturation (typically associated with weak detectability of the plant states) are dual to those concerning actuators saturation (namely, weak controllability). Another relevant reference is that in Garcia et al. (2007), where the authors address both actuators and sensors saturation. The problem addressed there is far more complicated than ours and, due to this fact, the solution derived in Garcia et al. (2007) is nonconvex (whereas ours is) and the conditions in Garcia et al. (2007) do not have a system theoretic interpretation. Our results, in turns, are convex and admit nice system theoretic interpretation because we address the input saturation problem only (namely, a simpler one as compared to Garcia et al. (2007)).

In this paper, we propose a synthesis method for the construction of output feedback controllers with an internal deadzone loop. This type of structure corresponds to the typical framework used since the 1980's for the design of control systems for saturated plants. See for example the work in Tyan and Bernstein (1995); Burgat and Tarbouriech (1998); Wu et al. (2000); Haddad and Kapila (1996); Takaba (1999); Gomes da Silva Jr and Tarbouriech (2001) and other references in Bernstein and Michel (1995). In our approach we will use the same regional analysis tools used in our recent papers Hu et al. (2006, 2008), and recast the underlying optimization problem for the selection of all the controller matrices (whereas in Hu et al. (2008) only the anti-windup component was selected and the underlying linear controller matrices were fixed). This approach parallels the approach proposed in Mulder et al. (2009) where classical sector conditions were used and extra assumptions on the direct input-output link of the plant were enforced. A similar assumption was also made in the recent paper Gomes da Silva Jr et al. (2005) which uses similar tools to ours to address both magnitude and rate saturation problems in a compensation scheme with some restrictions on the plant matrices as compared to ours.

Our contribution here is to transform the regional analysis result in Hu et al. (2006) into a regional output feedback synthesis technique for plants with input magnitude saturation, with guaranteed regional $\mathcal{L}_2$ gain and reachable set for a class of norm bounded disturbance inputs, as well as guaranteed domain of attraction. To this aim, we apply the elimination lemma following similar computations to those in Gahinet and Apkarian (1994) for $\mathcal{H}_\infty$ controller synthesis. Due to the special structure of our design goal, we are able to derive interesting feasibility results with precise system theoretic interpretations. In particular, the overall synthesis is cast as an optimization over LMIs, and under a detectability and stabilizability condition on the plant, the proposed design procedure always leads to regionally stabilizing controllers if the plant is exponentially unstable, to semi-global results if the plant is non-exponentially unstable, and to global results if the plant is already exponentially stable. We emphasize that these general feasibility properties of the proposed technique and the convexity of the underlying design procedure can be achieved by virtue of the special controller structure, which incorporates the deadzone loops, and this motivates and justifies the special structure considered here. Moreover, an interesting advantage of the approach proposed here is that due to the type of transformation that we use, it is possible to derive system theoretic interpretations of the feasibility conditions for the controller design (such as stabilizability and detectability of the plant). These results are novel and were not observed in any previous work. A preliminary version of the results in this paper has been presented in Dai et al. (2006) and some example studies using this same approach have been reported in Dai et al. (2007).

This paper is organized as follows: In Section 2 we formulate three problems that will be addressed in the paper; in Section 3 we state the LMI-based main conditions for output feedback controller synthesis and the procedure for the controller construction; in Section 4 we characterize (necessary and) sufficient conditions for the feasibility of the proposed synthesis; in Section 5 we illustrate the proposed constructions on one example.

Notation For compact presentation of matrices, given a square matrix X we denote $\mathrm{He}X := X + X^T$. For $P = P^T > 0$, we denote $\mathcal{E}(P) := \{x : \ x^T P x \leq 1\}$.

2 Problem statement

Consider a linear saturated plant,

$$\mathcal{P} \quad \begin{cases} \dot{x}_p = A_p x_p + B_{pu} u + B_{pw} w \\ y = C_{py} x_p + D_{p,yu} u + D_{p,yw} w \\ z = C_{pz} x_p + D_{p,zu} u + D_{p,zw} w \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ is the measurement output and $z \in \mathbb{R}^{n_z}$ is the performance output. Assume that (A_p, B_{pu}) is stabilizable and (C_{py}, A_p) is detectable.

The goal of this paper is the synthesis of a plant-order linear output feedback controller with internal loops:

$$\mathcal{C} \begin{cases} \dot{x}_c = A_c x_c + B_c y + E_1 \mathrm{dz}(y_c) \\ y_c = C_c x_c + D_c y + E_2 \mathrm{dz}(y_c) \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ (with $n_c = n_p$) is the controller state, $y_c \in \mathbb{R}^{n_u}$ is the controller output and $\mathrm{dz}(\cdot)$ is the deadzone function defined below. The relationship between the plant input u and the controller output y_c is described as

$$u = \mathrm{sat}(y_c), \tag{3}$$

where $\mathrm{sat}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is the symmetric saturation function * having saturation levels $\bar{u}_1, \ldots, \bar{u}_{n_u} > 0$ with its i-th component depending only on the i-th input component y_{ci} as follows: $u_i := \mathrm{sign}(y_{ci}) \min\{|y_{ci}|, \bar{u}_i\}, i = 1, ..., n_u$.

The resulting controller is a linear output feedback controller with internal deadzone loops via the nonlinear term $\mathrm{dz}(y_c)$, where $\mathrm{dz}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is the deadzone function defined as $\mathrm{dz}(y_c) := y_c - \mathrm{sat}(y_c)$ for all $y_c \in \mathbb{R}^{n_u}$. Furthermore, the resulting nonlinear closed-loop (1), (2), (3) is depicted in Fig. 1 and will be denoted *output feedback system with deadzone loops* henceforth.

The same output feedback structure was considered in Mulder et al. (2009), where convex synthesis methods for global (rather than regional, as we consider here) stability and performance were developed.

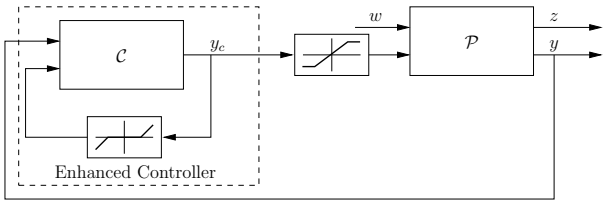

Fig. 1. The output feedback system with deadzone loops.

It is well known that linear saturated plants are characterized by weak stabilizability conditions. In particular, since the controller authority becomes almost zero for arbitrarily large signals, the global exponential stability can only be guaranteed if the plant is already exponentially stable. On the other hand, local and regional results are always achievable and semiglobal ones are achievable with non-exponentially unstable plants.

The following three regional properties for the closed-loop system (1), (2), (3) will be addressed in this paper.

Property 1 (Regional stability) *Given a set $\mathcal{S}_p \subset \mathbb{R}^{n_p}$. The origin of the closed-loop system is exponentially stable with domain of attraction including $\mathcal{S}_p \times \mathcal{S}_c$, where $\mathcal{S}_c \subset \mathbb{R}^{n_c}$ is a suitable set including $\{0\}$. (Since $x_c(0)$ can be set to 0, we may choose $\mathcal{S}_c = \{0\}$.)*

* Note that all the results in this paper also hold for non-symmetric saturations, for example, see Mulder et al. (2009), even with time-varying saturation limits, as long as a lower bound on both saturation limits is known.

Property 2 (Reachable set) *Given a set $\mathcal{R}_p \subset \mathbb{R}^{n_p}$ and a number $s > 0$. Every solution of the closed-loop system under $\|w\|_2 \leq s$ and initial condition $(x_p(0), x_c(0)) = (0, 0)$ satisfies $x_p(t) \in \mathcal{R}_p$ for all $t \geq 0$. The set $\mathcal{R}_p$ is called a bound on the reachable set for the plant state x_p.*

Property 3 (Regional $\mathcal{L}_2$ gain) *Given two numbers $s, \gamma > 0$. Every performance output $z(t)$ of the closed-loop system under $\|w\|_2 \leq s$ and initial condition $(x_p(0), x_c(0)) = (0, 0)$ satisfies $\|z\|_2 < \gamma\|w\|_2$. The number γ is called a bound on the regional $\mathcal{L}_2$ gain.*

In terms of the above properties, it is desirable to design output feedback laws to ensure the stability region include $\mathcal{S}_p \times \mathcal{S}_c$ with a large $\mathcal{S}_p$ (in the absence of disturbance), reachable set bounded by small $\mathcal{R}_p$ so that the plant state will remain (sufficiently) close to the origin, and regional $\mathcal{L}_2$ gain bounded by small γ so that the output energy is below a desirable value. Since there may be conflicts among these desirable properties, we formulate the following three design problems to balance different requirements.

Problem 1 *Consider the linear plant (1), a bound s on $\|w\|_2$, a set $\mathcal{R}_p \subset \mathbb{R}^{n_p}$ and a number $\gamma > 0$. Design an output feedback controller (2) which maximizes $\mathcal{S}_p$ so that $\mathcal{S}_p \times \{0\}$ is in the stability region of the closed-loop system, while guaranteeing $\mathcal{R}_p$ a bound for the reachable set and γ a bound for the regional $\mathcal{L}_2$ gain.*

Problem 2 *Consider the linear plant (1), a bound s on $\|w\|_2$, a set $\mathcal{S}_p \subset \mathbb{R}^{n_p}$ and a number $\gamma > 0$. Design an output feedback controller (2) which minimizes the bound $\mathcal{R}_p$ on reachable set for the closed-loop system, while guaranteeing $\mathcal{S}_p \times \{0\}$ in the stability region and a bound γ on the regional $\mathcal{L}_2$ gain.*

Problem 3 *Consider the linear plant (1), a bound s on $\|w\|_2$, and two sets $\mathcal{S}_p, \mathcal{R}_p \subset \mathbb{R}^{n_p}$. Design an output feedback controller (2) which minimizes the bound γ on regional $\mathcal{L}_2$ gain of the closed-loop system, while guaranteeing $\mathcal{R}_p$ a bound on reachable set and $\mathcal{S}_p \times \{0\}$ in the stability region.*

3 LMI-based design

This section provides a set of feasibility conditions for solving Problems 1 to 3 and a procedure to construct the output feedback controller (2).

3.1 Main feasibility theorem

Theorem 1 *Given $s > 0$. Consider the linear plant (1) with $\|w\|_2 \leq s$. Let $[N_1 \;\; N_2]^T$ span the null space of $[C_{py} \;\; D_{p,yw}]$. If the following LMIs in the variables $Q_{11}, P_{11} \in \mathbb{R}^{n_p \times n_p}$, $Q_{11} = Q_{11}^T > 0$, $P_{11} = P_{11}^T > 0$, $Y_p \in \mathbb{R}^{n_u \times n_p}$, $\gamma^2 > 0$, $\zeta \leq \frac{1}{s^2}$ are feasible:*

$$\mathrm{He}\begin{bmatrix} A_pQ_{11} + B_{pu}Y_p & B_{pw} & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_{pz}Q_{11} + D_{p,zu}Y_p & D_{p,zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0 \tag{4a}$$

$$\mathrm{He}\begin{bmatrix} N_1P_{11}A_pN_1^T + N_1P_{11}B_{pw}N_2^T - \frac{1}{2}N_2N_2^T & 0 \\ C_{pz}N_1^T + D_{p,zw}N_2^T & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0 \tag{4b}$$

$$\begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} > 0 \tag{4c}$$

$$\begin{bmatrix} \zeta\bar{u}_i^2 & Y_{pi} \\ Y_{pi}^T & Q_{11} \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u, \tag{4d}$$

(where Y_{pi} denotes the ith row of Y_p), then there exists an output feedback controller in the form of (2) and of order n_p, which guarantees the following three properties for the closed-loop system:

(1) the regional $\mathcal{L}_2$ gain bounded by γ;
(2) $\mathcal{E}(\zeta Q_{11}^{-1}) \times \{0\}$ inside the domain of attraction;
(3) the reachable set of the plant bounded by $\mathcal{E}((s^2 Q_{11})^{-1})$.

Moreover, given any feasible solution to the LMI constraints (4), a state-space representation of a controller guaranteeing these properties can be determined based on the matrices (Q_{11}, P_{11}) via the Procedure 1 reported next.

Remark 1 The LMI conditions in (4) have a system theoretic interpretation:

1) The first condition (4a) corresponds to a strengthened stabilizability condition for the plant (1). Indeed, substituting $Y_p = K_p Q_{11}$, (4a) corresponds to the LMI formulation of the bounded real lemma (see, e.g., Grimm et al. (2003)) characterizing the $\mathcal{L}_2$ gain from w to z for the plant controlled by a state feedback law $u = K_p x_p$. Therefore, (4a) constrains γ to be not smaller than the $\mathcal{L}_2$ gain of the plant stabilized by static state feedback. Note however that the corresponding state feedback gain K_p is further constrained by (4d), so that the open-loop plant $\mathcal{L}_2$ gain may be only reduced to a certain extent when larger values of s make the constraint (4d) tighter. As s approaches $+\infty$, Y_p will approach 0 and the constraint on γ enforced by (4a) will approach the global constraint corresponding to the $\mathcal{L}_2$ gain of the open-loop plant.
2) Applying (Grimm et al., 2004a, Lemma 2), it follows that the second condition (4b) is satisfied if and only if there exist $K_1 \in \mathbb{R}^{n_p \times n_y}$, $K_2 \in \mathbb{R}^{n_y \times n_y}$, $K_3 \in \mathbb{R}^{n_z \times n_y}$ such that

$$\mathrm{He}\begin{bmatrix} P_{11}A_p + K_1 C_{py} & P_{11}B_{pw} + K_1 D_{p,yw} & 0 \\ K_2 C_{py} & K_2 D_{p,yw} - \frac{I}{2} & 0 \\ C_{pz} + K_3 C_{py} & D_{p,zw} + K_3 D_{p,yw} & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0 \tag{5}$$

where (5) corresponds to a strengthened detectability condition for the plant. Indeed, if we had $K_2 = 0$ in the condition above, then the corresponding equation would mean that (if we take $L = P_{11}^{-1} K_1$) there exist L and K_3 such that the observer

$$\begin{aligned} \dot{\tilde{x}} &= A_p \tilde{x} + L(y - \tilde{y}) \\ \tilde{y} &= C_{yp}\tilde{x} \\ \tilde{z} &= C_{pz}\tilde{x} - K_3(y - \tilde{y}), \end{aligned} \tag{6}$$

for the system with unknown input

$$\begin{aligned} \dot{x} &= A_p x + B_{pw} w \\ y &= C_{yp} x + D_{p,yw} w \\ z &= C_{pz} x + D_{p,zw} w \end{aligned} \tag{7}$$

guarantees gain γ from w to the output observation error $(z - \tilde{z})$. The fact above can be checked writing down the error dynamics $e = x - \tilde{x}$ and imposing that there exists a disturbance attenuation Lyapunov function $V = e^T P_{11} e$. In particular, the LMI (4b) corresponds to imposing that:

$$\begin{aligned} &2e^T P_{11}(A_p + LC_{py})e + 2e^T P_{11}(B_{p,w} + LD_{p,yw})w \\ &+ \frac{1}{\gamma^2} z_e^T z_e - w^T w < 0, \end{aligned}$$

where $z_e = z - \tilde{z}$. ∘

3.2 Controller construction

In what follows, we provide a constructive algorithm for determining the matrices of the controller whose existence is established in Theorem 1.

Procedure 1 (Output feedback construction)

Step 1. *Solve the feasibility LMIs.* Find a solution $(Q_{11}, P_{11}, Y_p, \zeta)$ to the feasibility LMI conditions (4).

Step 2. *Construct the matrix Q.* (see also Grimm et al. (2004a); Hu et al. (2008).) Define the matrices $Q_{11} \in \mathbb{R}^{n_p \times n_p}$ and $Q_{12} \in \mathbb{R}^{n_p \times n_c}$, with $n_c = n_p$ as a solution to the following equation:

$$Q_{11}\, P_{11}\, Q_{11} - Q_{11} = Q_{12}\, Q_{12}^T. \tag{8}$$

Since Q_{11} and P_{11} are invertible and $Q_{11}^{-1} < P_{11}$ by the feasibility conditions, then $Q_{11}\, P_{11}\, Q_{11} - Q_{11}$ is positive definite. Hence there always exists a matrix Q_{12} satisfying equation (8). † Define the matrix $Q_{22} \in \mathbb{R}^{n_c \times n_c}$ as

$$Q_{22} := I + Q_{12}^T\, Q_{11}^{-1}\, Q_{12}. \tag{9}$$

Finally, define the matrix $Q \in \mathbb{R}^{n \times n}$ $(n = n_p + n_c)$ as

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}. \tag{10}$$

Step 3. *Controller synthesis LMI.* Construct the matrices $\Psi \in \mathbb{R}^{\bar{n} \times \bar{n}}$, $H \in \mathbb{R}^{(n_c+n_u) \times \bar{n}}$, $G \in \mathbb{R}^{(n_c+n_u+n_y) \times \bar{n}}$, and $T \in \mathbb{R}^{\bar{n} \times \bar{n}}$ $(\bar{n} = n_p + n_c + n_u + n_w + n_z)$ as follows:

$$\Psi = \begin{bmatrix} A_p Q_{11} & A_p Q_{12} & -B_{pu} U & B_{pw} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -Y_p & -Y_c & -U & 0 & 0 \\ 0 & 0 & 0 & -\frac{I}{2} & 0 \\ C_{pz} Q_{11} & C_{pz} Q_{12} & -D_{p,zu} U & D_{p,zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} \tag{11}$$

$$H = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ B_{pu}^T & 0 & I & 0 & D_{p,zu}^T \end{bmatrix} \tag{12}$$

$$G = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_{py} & 0 & 0 & D_{p,yw} & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} \tag{13}$$

$$T = \mathrm{diag}\{Q, I, I, I\}, \tag{14}$$

where $Y_c \in \mathbb{R}^{n_u \times n_c}$ is defined as $Y_c := Y_p (Q_{11})^{-1} Q_{12}$. Then, define the unknown variable $\bar{\mathbf{\Omega}}_U$ as:

$$\bar{\mathbf{\Omega}}_U := \begin{bmatrix} \bar{A}_c & \bar{B}_c & \bar{E}_1 U \\ \bar{C}_c & \bar{D}_c & \bar{E}_2 U \end{bmatrix}. \tag{15}$$

Finally, solve the output feedback controller LMI:

$$\Phi = \mathrm{He}(\Psi + \Psi_1) = \mathrm{He}(\Psi + H^T \bar{\mathbf{\Omega}}_U G T) < 0 \tag{16}$$

in the unknowns $\bar{\mathbf{\Omega}}_U \in \mathbb{R}^{(n_c+n_u) \times (n_c+n_u+n_y)}$, and $U \in \mathbb{R}^{n_u \times n_u}, U > 0$ diagonal.

Step 4. *Computation of the controller matrices.* From the matrices U and $\bar{\mathbf{\Omega}}_U$ in Step 3, compute the matrix $\bar{\mathbf{\Omega}}$ as

$$\bar{\mathbf{\Omega}} := \begin{bmatrix} \bar{A}_c & \bar{B}_c & \bar{E}_1 \\ \bar{C}_c & \bar{D}_c & \bar{E}_2 \end{bmatrix} = \bar{\mathbf{\Omega}}_U\, \mathrm{diag}(I, I, U^{-1}). \tag{17}$$

† Note that equation (8) always admits infinite solutions, parameterizing an infinite number of compensators inducing the same closed-loop performance. Understanding how to exploit this degree of freedom for the selection of a most desirable compensator is subject of future research.

Finally, the controller parameters in $\mathbf{\Omega}$ can be determined applying the following transformation:

$$\begin{aligned}\mathbf{\Omega} &:= \begin{bmatrix} A_c & B_c & E_1 \\ C_c & D_c & E_2 \end{bmatrix} \\ &= \bar{\mathbf{\Omega}} \begin{bmatrix} I & 0 & 0 \\ -\bar{X} D_{p,yu} \bar{C}_c & \bar{X} & \bar{X} D_{p,yu}(I - \bar{E}_2) \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{18}$$

where $\bar{X} := (I + D_{p,yu}\bar{D}_c)^{-1}$ and the matrix in brackets is always nonsingular.

Remark 2 A few comments on some peculiarities of Procedure 1 are in order. Note that equation (8) at step 2 admits infinitely many solutions: given any solution Q_{12} (e.g., computed using `sqrtm()` as in our example study), for any square matrix E such that $EE^T = I$, $\bar{Q}_{12} := Q_{12}E$ is an alternative solution. In principle, each solution could lead to different families of controllers, however by Theorem 1 it is guaranteed that all of them will lead to the same optimal guaranteed performance level. Moreover, numerical experience (see, e.g., the example in Section 5) indicates that the controllers obtained from different solutions induce indistinguishable closed-loop responses. Note also that the controller matrix A_c resulting from this construction is not necessarily Hurwitz, however one could augment the LMI (16) with extra conditions with the goal of making, e.g., $A_c^T P_c + P_c A_c$ negative definite for some $P_c = P_c^T > 0$. This fix, however, wouldn't be associated with guarantees on the existence of a solution. Finally, it should be pointed out that numerical problems can be experienced when solving (16) for large dimensions of A_p. In those cases a useful approach is to rely on the explicit formulas for the selection of the controller matrices in (16) (see, e.g., Gahinet (1996)).

4 Feasibility and solution to the output feedback synthesis problems

In this section, the feasibility conditions established in Theorem 1 will be used to provide conditions for the solvability of Problems 1 to 3.

4.1 Feasibility of and solution to Problem 1

The objective of Problem 1 is to maximize the stability region with guaranteed regional $\mathcal{L}_2$ gain and reachable set. Using Theorem 1, we need to maximize the size of the ellipsoid $\mathcal{E}(\zeta Q_{11}^{-1})$ (an estimate of the stability region) by imposing a given bound γ on the regional $\mathcal{L}_2$ gain and a given bound $\mathcal{R}_p$ on $\mathcal{E}((s^2 Q_{11})^{-1}$ which includes the reachable set. We state the corollary below for a generic measure $\alpha_R(\cdot)$ of the size of the ellipsoid $\mathcal{E}(\zeta Q_{11}^{-1})$. This is typically done with respect to some shape reference set of the desired stability region.

Corollary 1 *Given s, $\mathcal{R}_p$ and γ, a solution to Problem 1 is obtained by applying Procedure 1 to the optimal solution of the following problem (whenever feasible):*

$$\begin{aligned} &\sup_{Q_{11}, P_{11}, Y_p, \zeta} \alpha_R(\mathcal{E}(\zeta Q_{11}^{-1})), \textit{ subject to} \\ &(4a), (4b), (4c), (4d), && (19a) \\ &\mathcal{E}(Q_{11}^{-1}/s^2) \subset \mathcal{R}_p. && (19b) \end{aligned}$$

When $\alpha_R(\mathcal{E}(\zeta Q_{11}^{-1})) = \det(Q_{11}/\zeta)$, the objective is to maximize the volume of $\mathcal{E}(\zeta Q_{11}^{-1})$. Alternatively, given a shape reference set $\mathcal{X}_R$, α_R can be chosen as the maximal α so that $\alpha\mathcal{X}_R \subset \mathcal{E}(\zeta Q_{11}^{-1})$. When $\mathcal{X}_R$ is an ellipsoid $\{x_p : x_p^T (S_p)^{-1} x_p \leq 1\}$, this translates to $\alpha^2 S_p \leq Q_{11}/\zeta$. Thus the problem (19) can be cast as

$$\begin{aligned} &\inf_{Q_{11}, P_{11}, Y_p, \zeta, \alpha} 1/\alpha^2, \text{ subject to} \\ &(4a), (4b), (4c), (4d), (19b) && (20a) \\ &\zeta S_p \leq (1/\alpha^2) Q_{11}, && (20b) \end{aligned}$$

which is a generalized eigenvalue problem.

Remark 3 Based on Corollary 1, reduced LMI conditions can be written to only maximize the estimate of the domain of attraction without considering the effect of disturbances ($w = 0$):

$$\sup_{Q_{11},P_{11},Y_p} \alpha_R(\mathcal{E}(Q_{11}^{-1})), \text{ subject to}$$
$$\mathrm{He}[A_pQ_{11} + B_{pu}Y_p] < 0 \tag{21a}$$
$$\mathrm{He}\big[N_1P_{11}A_pN_1^T\big] < 0 \tag{21b}$$
$$(4c), (4d) \tag{21c}$$

where we can fix $\zeta = 1$. Note however that in this simplified case, the size of the matrices at step 3 of Procedure 1 should be reduced. In particular, the last two block rows and columns of Ψ should be removed, and the last two columns of H and G should be removed too.

From (21) it is straightforward to conclude that if the plant is exponentially stable, then global exponential stability and finite $\mathcal{L}_2$ gain can be achieved by the proposed output feedback controller ($Y_p = 0$ is sufficient). If the plant is not exponentially unstable, semiglobal results are obtainable. Regional results can always be obtained in the general case and the size of the maximal feasible domain of attraction depends on the particular problem.

4.2 Feasibility of and solution to Problem 2

The objective of Problem 2 is to minimize the size of the reachable set with guaranteed stability region and regional $\mathcal{L}_2$ gain. By Theorem 1, a bound for the reachable set is given as the ellipsoid $\mathcal{E}((s^2Q_{11})^{-1})$. Since an ellipsoid may have arbitrarily small volume but still contain points far away from the origin, (e.g., when some eigenvalues of Q_{11} are very small and some are very large), the volume may not be a proper measure. Instead, we would like to use a measure that indicates the"outer" size of the reachable set. Given a shape reference set $\mathcal{X}_r$, the outer size of $\mathcal{E}((s^2Q_{11})^{-1})$ is defined as $\alpha_r := \min\{\alpha : \mathcal{E}((s^2Q_{11})^{-1}) \subset \alpha\mathcal{X}_r\}$.

Corollary 2 *Given $s, \gamma > 0$ and $\mathcal{S}_p \subset \mathbb{R}^{n_p}$, a solution to Problem 2 is obtained by applying Procedure 1 to the optimal solution to the following problem (whenever feasible):*

$$\min_{Q_{11},P_{11},Y_p,\zeta} \alpha_r(\mathcal{E}((s^2Q_{11})^{-1}), \textit{ subject to}$$
$$(4a), (4b), (4c), (4d), \tag{22a}$$
$$\mathcal{S}_p \subset \mathcal{E}(\zeta Q_{11}^{-1}) \tag{22b}$$

When the shape reference set $\mathcal{X}_r$ is an ellipsoid, $\mathcal{X}_r = \{x_p : x_p^TR_p^{-1}x_p \le 1\}$, $R_p = R_p^T > 0$, $\alpha_r(\mathcal{E}((s^2Q_{11})^{-1}) = \min\{\alpha : s^2Q_{11} \le R_p\alpha^2\}$. Thus the optimization problem (22) becomes

$$\min_{Q_{11},P_{11},Y_p,\zeta} \alpha^2, \text{ subject to}$$
$$(4a), (4b), (4c), (4d), (22b) \tag{23a}$$
$$s^2Q_{11} \le \alpha^2R_p \tag{23b}$$

Similarly, $\mathcal{X}_r$ can be selected as an unbounded set:

$$\mathcal{X}_r = \{x_p : \ |Cx_p| \le 1\},$$

where $C \in \mathbb{R}^{1\times n_p}$ is a given row vector. Then $\mathcal{E}((s^2Q_{11})^{-1}) \subset \alpha\mathcal{X}_r$ if and only if $s^2CQ_{11}C^T \le \alpha^2$. If both (23a) and $s^2CQ_{11}C^T \le \alpha^2$ are feasible, then $|Cx_p(t)| \le \alpha$ for all t under $\|w\|_2 \le s$. Thus, if our objective is to minimize the size of a certain output Cx_p, we just need to replace (23b) with $s^2CQ_{11}C^T \le \alpha^2$.

Remark 4 If there is no requirement for a guaranteed $\mathcal{L}_2$ gain, the problem of minimizing the reachable set can be simplified by setting $\gamma = \infty$, which is equivalent to removing all the block rows and columns involving γ^2 in (4a) and (4b). Similar to the discussion in Remark 3, when following Procedure 1 for this simplified case, all the block rows and columns involving γ^2 should be removed from Φ in (16).

4.3 Feasibility of and solution to Problem 3

Similarly to the previous two sections for design Problems 1 and 2, here we use Theorem 1 to give a solution to Problem 3.

Corollary 3 *Given $s > 0$, $\mathcal{R}_p, \mathcal{S}_p \subset \mathbb{R}^{n_p}$, a solution to Problem 3 is obtained by applying Procedure 1 to the optimal solution to the following problem (whenever feasible):*

$$\min_{Q_{11}, P_{11}, Y_p, \zeta, \gamma^2} \gamma^2, \text{ subject to}$$

$$(4a), (4b), (4c), (4d), \tag{24a}$$

$$\mathcal{E}((s^2 Q_{11})^{-1}) \subset \mathcal{R}_p \tag{24b}$$

$$\mathcal{S}_p \subset \mathcal{E}(\zeta Q_{11}^{-1}) \tag{24c}$$

When both $\mathcal{S}_p$ and $\mathcal{R}_p$ are ellipsoids, i.e., $\mathcal{S}_p = \mathcal{E}(S_p^{-1})$ and $\mathcal{R}_p = \mathcal{E}(R_p^{-1})$, $S_p = S_p^T > 0$ and $R_p = R_p^T > 0$, (24b) and (24c) can be written as $\zeta S_p \leq Q_{11} \leq R_p/s^2$.

Alternative shapes for the guaranteed reachable set $\mathcal{R}_p$ and for the guaranteed stability region $\mathcal{S}_p$ can be selected as those in the previous two subsections.

5 A Numerical Example

Consider the inverted pendulum in Dai et al. (2007). The linearized model in appropriately scaled variables is given by:

$$\mathcal{P} \begin{cases} \tau_1 \dot{x}_1 &= -x_2 + d; \\ \tau_2 \dot{x}_2 &= -x_1 + x_3; \\ \tau_3 \dot{x}_3 &= x_2 + u; \\ y/z &= x_3 \end{cases} \tag{25}$$

with parameters $\tau_1 = \tau_2 = \tau_3 = 1.0s$. In (25), the state variables correspond to the horizontal speed of the pendulum center of gravity (cg-x_1), the horizontal distance between the cg of the pendulum and that of the slider (x_2) and the horizontal speed of the slider (x_3). The disturbance d represents an unknown horizontal force F_z exerted on the pendulum cg, which has bounded magnitude. The input u represents the force F_u exerted by the actuator, which is constrained in magnitude (stroke) between ± 1. The transfer function has one pole at the origin, two poles symmetrically on the real axis at $\pm\sqrt{2}$. Thus the open-loop system is exponentially unstable.

For simulation purposes, the disturbance is initially selected as a constant force of $0.485N$ happening at $5.1s$ with duration $0.9s$. By applying Procedure 1, setting $s^2 = 0.01$, the following controller matrices in $\mathbf{\Omega}$ are obtained, which guarantee a performance level of $\gamma = 3.28$:

$$\left[\begin{array}{ccc|c|c} -273.237 & 273.950 & 97.058 & -147.364 & -0.000 \\ 277.059 & -280.465 & -82.696 & 50.975 & 0.000 \\ 89.676 & -95.096 & 87.741 & -716.184 & -0.000 \\ \hline 2.271 & -3.425 & 17.933 & -118.550 & 0.99 \end{array}\right] \tag{26}$$

having condition number 824.875. The corresponding response of the closed-loop system, labeled "Controller I," is represented by the thin solid curves in Figure 2. In the upper plot, the thin solid curve shows the horizontal speed of the slider, and the dash-dotted curve shows the disturbance signal. The lower plot shows the saturated control input. From the figure it appears that the output feedback controller in (26) succeeds in rejecting the disturbance while exhibiting desirable performance.

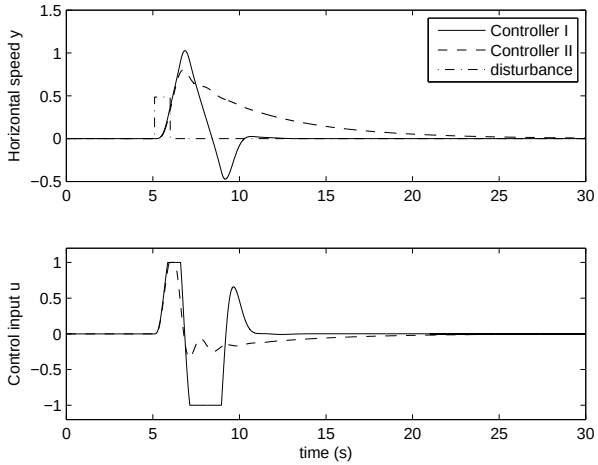

Fig. 2. Response of the inverted pendulum.

With reference to Remark 2 on the selection of Q_{12}, we use here Matlab's function `sqrtm()` to get a solution of equation (8). Alternative controllers have been computed using different scaling matrices E, as specified in the remark. These controllers differ in terms of the matrices, but lead to the same performance level and to indistinguishable responses (all overlapping the solid curves in Figure 2).

If we increase the value of s in Procedure 1 to $s^2 = 0.16$, the minimized $\mathcal{L}_2$ gain grows to 10.61 and the output feedback controller matrices in $\mathbf{\Omega}$ become:

$$\left[\begin{array}{ccc|c|c} -354.664 & 458.275 & 271.819 & 208.455 & 0.448 \\ 456.711 & -543.778 & -401.107 & 949.488 & 2.513 \\ 274.226 & -399.716 & -166.735 & -1253.9 & -3.489 \\ \hline -0.652 & -3.141 & 4.165 & -93.412 & 0.777 \end{array}\right] \tag{27}$$

with condition number 1529.3. The response of the system, labeled "Controller II," to the same disturbance used above is represented by the dashed curves in Fig. 2. This response appears less desirable than the one induced by controller I in (26) (solid curves in the same figure). On the other hand, the advantage of controller II versus controller I is that the former tolerates a larger class of disturbances. This can be verified by simulations (omitted here due to space constraints) showing that under the same disturbance duration of $0.9s$, controller II is able to reject disturbances up to a magnitude of $0.8N$ while controller I generates diverging responses already for disturbances of larger size than $0.5N$. This comparative property of the responses induced by the two controllers is in accordance with the results of Theorem 1 stating that controllers designed with larger values of s can tolerate disturbances with larger energy but lead to worse regional $\mathcal{L}_2$ gain performance.

6 Conclusions

In this paper we proposed a novel synthesis approach for the construction of an output feedback controller with internal deadzone loops. A systematic approach for the synthesis of the proposed controller is cast as a convex optimization problem over LMIs and a variety of regional stability and performance goals can be achieved. Under a detectability and stabilizability condition on the plant, the proposed design procedure always leads to regionally stabilizing controllers if the plant is exponentially unstable, to semi-global results if the plant is non-exponentially unstable, and to global results if the plant is already exponentially stable.

Appendix

Proof of Theorem 1.

After tedious computations, the dynamics of the closed-loop system (1), (2), (3) can be written as

$$\begin{aligned} \dot{x} &= Ax + B_q \mathrm{dz}(y_c) + B_w w \\ y_c &= C_y x + D_{yq} \mathrm{dz}(y_c) + D_{yw} w \\ z &= C_z x + D_{zq} \mathrm{dz}(y_c) + D_{zw} w, \end{aligned} \tag{28}$$

where

$$\left[\begin{array}{c|c|c} A & B_q & B_w \\ \hline C_y & D_{yq} & D_{yw} \\ \hline C_z & D_{zq} & D_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} A_p & 0 & -B_{pu} & B_{pw} \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline C_{pz} & 0 & -D_{p,zu} & D_{p,zw} \end{array}\right] + \left[\begin{array}{cc} 0 & B_{pu} \\ I & 0 \\ \hline 0 & I \\ \hline 0 & D_{p,zu} \end{array}\right] \bar{\mathbf{\Omega}} \left[\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ C_{py} & 0 & 0 & D_{p,yw} \\ 0 & 0 & I & 0 \end{array}\right] \tag{29}$$

and $\bar{\Omega}$ is defined in (15), (17) and related to controller matrices in $\mathbf{\Omega}$ via the invertible transformation in (18). For system (28), the following lemma is a slight extension of (Hu et al., 2006, Theorem 2, item (3)) with the new variable $\zeta > 0$ to reduce the conservativeness of the estimate of the stability region.

Lemma 1 *Given $s > 0$. Suppose there exist $Q = Q^T > 0$, $U > 0$ diagonal, $\gamma^2 > 0$, $\zeta \leq \frac{1}{s^2}$, Y such that*

$$\Phi := \mathrm{He} \begin{bmatrix} AQ & B_q U & B_w & 0 \\ C_y Q - Y & -U + D_{yq} U & D_{yw} & 0 \\ 0 & 0 & -\frac{I}{2} & 0 \\ C_z Q & D_{zq} U & D_{zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0 \tag{30a}$$

$$\begin{bmatrix} \zeta \bar{u}_i^2 & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u, \tag{30b}$$

then the closed-loop system (28) is well posed and its origin is exponentially stable with stability region containing the set $\mathcal{E}(\zeta Q^{-1})$. Moreover, if $x(0) = 0$ and $\|w\|_2 \leq s$, then $x(t) \in \mathcal{E}((s^2 Q)^{-1})$ for all $t \geq 0$ and $\|z\|_2 \leq \gamma \|w\|_2$.

The rest of the proof shows that the conditions in (4) are feasible if and only if conditions (30) are satisfied by the output feedback controller arising from Procedure 1. Partition Q in (30) according to the dimensions of x_p and x_c as $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$. Also partition $Y = \begin{bmatrix} Y_p & Y_c \end{bmatrix}$ accordingly. Then after many computations, the matrix Φ in (30a) is shown to coincide with the matrix Φ in (16). Moreover, nonsingularity of the matrix $I + D_{p,yu} \bar{D}_c$ at step 4 of Procedure 1 follows from the well posedness of the feedback interconnection, established in Lemma 1.

Using the projection lemma, as in Grimm et al. (2003) (see also Gahinet and Apkarian (1994)), there exist $\bar{\mathbf{\Omega}}_U$ such that $\Phi = \mathrm{He}(\Psi + \Psi_1) < 0$ if and only if

$$\mathrm{He}\ \left(W_H^T \Psi W_H\right) < 0, \tag{31a}$$

$$\mathrm{He}\ \left(W_G^T T^{-1} \Psi T^{-1} W_G\right) < 0 \tag{31b}$$

where W_H generates (namely, its column form a base of) the null space of H and W_G generates the null space of G (note that many selections of W_H and W_G may be used, still the corresponding conditions in (31) will all be

equivalent). A possible selection for W_H is

$$W_H = \begin{bmatrix} I & 0 & -B_{pu} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & -D_{p,zu} & 0 & I \end{bmatrix}^T .$$

Supposing that $[N_1\ N_2]^T$ generates the null space of $[C_{py}\ D_{p,yw}]$, a possible selection for W_G is

$$W_G = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}^T .$$

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = Q^{-1}$. Then, after some tedious computations, equation (31a) simplifies to (4a) and equation (31b) simplifies to (4b).

Note that (4a) and (4b) are stated in terms of P_{11} and Q_{11}, which are constrained by

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = I. \tag{32}$$

Since the controller is of the same order as the plant, then P_{11} and Q_{11} have the same size and by (Grimm et al., 2003, Lemma 6) it is sufficient for (32) that $P_{11} > {Q_{11}}^{-1}$. This corresponds to condition (4c) in Theorem 1. As for conditions (30b) in Lemma 1, since Y_{pi} satisfy (4d), then it is sufficient ‡ to select $Y_i = \begin{bmatrix} Y_{pi}Q_{11}^{-1} & 0 \end{bmatrix} Q$. To see this, note that

$$\begin{aligned} Y_i Q^{-1} Y_i^T &= \begin{bmatrix} Y_{pi}Q_{11}^{-1} & 0 \end{bmatrix} Q \begin{bmatrix} Y_{pi}Q_{11}^{-1} & 0 \end{bmatrix}^T \\ &= Y_{pi}Q_{11}^{-1}Y_{pi}^T \leq \zeta \bar{u}_i^2. \end{aligned}$$

Finally, by Schur complement, the last inequality easily transforms into (30b). □

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‡ Note that this corresponds to choosing the feasible selection $Y = [Y_p\ Y_p Q_{11}^{-1} Q_{12}]$.

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