Linear Output Regulation with Dynamic Optimization for Uncertain Linear Over-actuated Systems ⋆

Matteo Cocetti [c,a], Andrea Serrani [b], Luca Zaccarian [c,a]

[a] *Department of Industrial Engineering, 9 Sommarive, University of Trento, Trento, IT, 38123.*

[b] *Department of Electrical and Computer Engineering, 2015 Neil Ave, The Ohio State University, Columbus, OH, 43210.*

[c] *LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, FR, 31077.*

Abstract

This paper considers the linear output regulation problem for uncertain over-actuated plants. The general form of input redundancy considered in this work implies the existence of multiple control inputs and state trajectories compatible with a prescribed reference for the output. On-line selection, according to certain performance criteria, of the most suitable of these inputs-state trajectories leads to a linear output regulation problem with *dynamic redundancy allocation*. We present a solution that augments the well known *internal model control scheme* with two additional dynamical systems. The first one, named *annihilator*, parametrizes the inputs and the corresponding state trajectories that are invisible from the output. The second one, named *redundancy allocator*, dynamically selects the best solution according to a predefined performance criterion. We derive explicit solutions for the performance criterion equal to relaxed 1, 2, and ∞- norms of the plant input. This set-up is a particular case of the dynamic redundancy allocation problem named *dynamic input allocation*. The proposed solutions can be implemented in an error feedback form and are especially suitable for optimizing sparsity, power and amplitude of the control input. Finally, structural stability, robustness and existence of a unique steady-state are proven.

Key words: Linear output regulation; dynamic input allocation; optimization; uncertain systems.

1 Introduction

Intuitively speaking a system is over-actuated when the number of control inputs is larger than the number of regulated outputs. Over-actuation naturally arises every time there are multiple actuators performing the same action and this is often the case in many engineering applications. The presence of more actuators than strictly necessary could be desirable for many reasons, e.g., safety, fault-tolerant policies, performance or consumption optimization. Popular and well studied examples of over-actuated systems are high performance aircraft [2] and ships and underwater vehicles [14]. However many other application fields are increasing in number, see for instance [22]. A large number of actuators introduces a certain *degree of redundancy*, meaning that there exists an entire family of input functions and possibly of state trajectories that are compatible with a prescribed reference for the output [21,17,7,12], or more precisely the system does not have a unique right inverse. This redundancy is exploited to design a *dynamic redundancy allocator* whose primary objective is to dynamically select, among all the feasible inputs, the best one according to a certain cost function.

The setup considered in this work differs from previous studies, notably, [17] and [12], in several aspects. First, we explicitly take into consideration parametric uncertainties in the plant showing that the problem is structurally well posed. Second, we derive a closed form solution for cost functions involving different norms of the input, solving the so called *dynamic input optimization problem* for three significant cases that involve sparsity, power and amplitude of the plant control signal. The derived strategies only require the tracking error as input. Third, we precisely formulate the dynamic allocation problem within the framework of robust linear output regulation and we show that, in all cases, a dynamic allocator can be designed that provides global exponential stability of the closed-loop system (when the exoge-

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Email addresses: matteo.cocetti@unitn.it (Matteo Cocetti), serrani.1@osu.edu (Andrea Serrani), zaccarian@laas.fr (Luca Zaccarian).

nous signals are disconnected), uniform boundedness of all trajectories, and exponential convergence to a unique steady-state by way of a global contractivity property.

A preliminary version of this work have been presented in [5], but here we largely extend the ideas and we present results and proofs in a more general and clear setting. The improvements are essentially threefold. First, we extend the allocation strategy for generic strongly convex objective functions. Second, we provide explicit closed form solutions for three different cost functions of practical interest, i.e., relaxations of the 1, 2 and ∞ norm of the plant input, whereas in [5] only the 2 and ∞ norms were considered. Third, we remove the need for an additional tuning gain required in the solution presented in [5] and we extend the results from semiglobal to global. These latter properties, in turn, have been achieved via the introduction of mild regularity assumptions on the cost function, not required in [5].

The paper is organized as follows: in Section 2 we present the set-up and we formally define the *dynamic allocation problem*. In Section 3 we solve the problem in a fairly general setting. In Section 4 we specialize the findings of Section 3 into the *dynamic input allocation* set up and we provide explicit design for three specific choices of the cost function that are of practical interest, namely relaxed 1, 2, and ∞-norm of the control input. In 5 we present some simulations to show the effectiveness of the proposed techniques. Conclusions are offered in Section 6.

Notation: Let $\mathbb{R}^n$ denote the set of real vectors of dimension n; given a constant $c \in \mathbb{R}$ we write $\mathbb{R}_{\geq c}$ to denote the subset $[c, \infty) \subset \mathbb{R}$. Calligraphic symbols such as $\mathcal{M}$ denote sets, while the formal script font is used to denote real vector spaces locally isomorphic to Euclidean spaces, e.g., $\mathscr{X}$. For a vector $x \in \mathbb{R}^n$, x_i denotes the i-th entry, $|x|_1, |x|_2, |x|_\infty$ are respectively the $1, 2, \infty$ norms of x, and $\operatorname{diag}(x) \in \mathbb{R}^{n\times n}$ is the diagonal matrix whose i-th diagonal element is x_i. Given two vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\operatorname{col}(x, y) := [x^\top, y^\top]^\top \in \mathbb{R}^{n+m}$. For a matrix $M \in \mathbb{R}^{n\times m}$, $M^\top$ denotes its transpose. For square invertible matrices $M \in \mathbb{R}^{n\times n}$, M^{-1} denotes the inverse of M and $M^{-\top}$ its inverse transpose, $M > 0$ ($M \geq 0$) denotes positive definiteness (semi-definiteness) of M, $\operatorname{spec}(M) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ denotes the spectrum, i.e., the set of the eigenvalues of matrix M, finally $\operatorname{He}(M) := (M + M^\top)$ is the Hermitian component of matrix M. If matrix $M \in \mathbb{R}^{n\times n}$ is symmetric the eigenvalues are real and can be always arranged in algebraically non decreasing order as follows $\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_n(M)$. Given $M \in \mathbb{R}^{n\times m}$, M_{ij} denotes the ij component of M with $i = 1, \dots, n$ and $j = 1, \dots, m$, while $M^\top_{ij}$ denotes the ij component of $M^\top$ with $i = 1, \dots, m$ and $j = 1, \dots, n$. The operator $\operatorname{diag}(M_1, M_2) \in \mathbb{R}^{n\times n}$ denotes the block-wise concatenation of matrices $M_1 \in \mathbb{R}^{n_1\times n_1}$ and $M_2 \in \mathbb{R}^{n_2\times n_2}$ where $n := n_1 + n_2$. Matrix $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix of order n but often we will drop the subscript n if the dimension is clear from the context. Given a function $f(x, y)$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ we use the following notation: $\nabla_x f(x, y) := \operatorname{col}\left(\frac{\partial f}{\partial x_1}(x, y), \dots, \frac{\partial f}{\partial x_n}(x, y)\right) \in \mathbb{R}^n$, $\nabla_y f(x, y) := \operatorname{col}\left(\frac{\partial f}{\partial y_1}(x, y), \dots, \frac{\partial f}{\partial y_m}(x, y)\right) \in \mathbb{R}^m$, $\nabla^2_{yx} f(x, y) := \left(\frac{\partial \nabla_x f}{\partial y_1}(x, y), \dots, \frac{\partial \nabla_x f}{\partial y_m}(x, y)\right) \in \mathbb{R}^{n\times m}$, and $\nabla^2_x f(x, y) \in \mathbb{R}^{n\times n}$ represents the Hessian matrix with respect x. Finally the symbols $\mathscr{L}_\infty$, $\mathscr{C}^k$, $(k = 0, 1, \dots)$ denote respectively the set of essentially bounded and k-times differentiable functions.

2 Problem statement

We consider a modified version of the *linear robust output regulation* set up. We start by considering an uncertain plant model of the form,

$$\dot{x}_\mathrm{p} = A_\mathrm{p}(\mu)x_\mathrm{p} + B_\mathrm{p}(\mu)u + P_\mathrm{p}(\mu)w, \tag{1a}$$
$$e = C_\mathrm{p}(\mu)x_\mathrm{p} + Q_\mathrm{p}(\mu)w, \tag{1b}$$

with state $x_\mathrm{p} \in \mathscr{X}_\mathrm{p} \cong \mathbb{R}^{n_\mathrm{p}}$, control input $u \in \mathscr{U} \cong \mathbb{R}^m$ and tracking error $e \in \mathscr{E} \cong \mathbb{R}^p$. The plant matrices $A_\mathrm{p}(\mu) \in \mathbb{R}^{n_\mathrm{p}\times n_\mathrm{p}}$, $B_\mathrm{p}(\mu) \in \mathbb{R}^{n_\mathrm{p}\times m}$, $P_\mathrm{p}(\mu) \in \mathbb{R}^{n_\mathrm{p}\times s}$, $C_\mathrm{p}(\mu) \in \mathbb{R}^{p\times n_\mathrm{p}}$, $Q_\mathrm{p}(\mu) \in \mathbb{R}^{p\times s}$ depend continuously on a vector $\mu \in \mathcal{M}$, whose values are assumed to range over a known compact set $\mathcal{M} \subset \mathbb{R}^{n_\mu}$ containing the origin. Without loss of generality we assume that $\mu = 0$ corresponds to the nominal model. We assume that the plant (1) is driven by an exogenous signal $w \in \mathscr{W} \cong \mathbb{R}^s$ generated by a known *exosystem* of the form,

$$\dot{w} = Sw, \tag{2}$$

where the matrix S is assumed to be semi-simple and such that $\operatorname{spec}(S) \subset \mathbb{C}^0$. This implies that for any initial condition $w(0)$ the arising solution $w(\cdot)$ to (2) is uniformly bounded. Depending on the context, the signal w may represent references and/or disturbances. Stability and regulation for (1) are ensured by a given error feedback controller of the form

$$\dot{x}_\mathrm{c} = A_\mathrm{c}x_\mathrm{c} + B_\mathrm{c}e \tag{3a}$$
$$u_\mathrm{reg} = C_\mathrm{c}x_\mathrm{c}, \tag{3b}$$

with state $x_\mathrm{c} \in \mathscr{X}_\mathrm{c} \cong \mathbb{R}^{n_\mathrm{c}}$ and output $u_\mathrm{reg} \in \mathscr{U}$. The classical formulation of the robust linear output regulation problem is reported in the following Problem 1.

Problem 1 (Linear output regulation problem) *Given the plant model (1) with exosystem (2) find, if possible, a controller of the form (3) such that:*

(1) The closed-loop matrix

$$\begin{bmatrix} A_\mathrm{p}(\mu) & B_\mathrm{p}(\mu)C_\mathrm{c} \\ B_\mathrm{c}C_\mathrm{p}(\mu) & A_\mathrm{c} \end{bmatrix} \tag{4}$$

obtained through the interconnection $u = u_\mathrm{reg}$ is Hurwitz for all $\mu \in \mathcal{M}$.

(2) Solutions of (1)–(3)–(2) originating from any initial condition $(x_\mathrm{p}(0), x_\mathrm{c}(0), w(0)) \in \mathscr{X}_\mathrm{p} \times \mathscr{X}_\mathrm{c} \times \mathscr{W}$ satisfy $\lim_{t\to+\infty} e(t) = 0$, for all $\mu \in \mathcal{M}$.

The solution of Problem 1 relies on the well known "Internal Model Principle", see [11], and the following assumptions are necessary for its solvability.

Assumption 1 (Output regulation assumptions)

(1) the pairs $[A_\mathrm{p}(\mu), B_\mathrm{p}(\mu)]$ and $[C_\mathrm{p}(\mu), A_\mathrm{p}(\mu)]$ are respectively stabilizable and detectable for all $\mu \in \mathcal{M}$,

(2) the non resonance condition

$$\mathrm{rank} \begin{bmatrix} A_\mathrm{p}(\mu) - \lambda I & B_\mathrm{p}(\mu) \\ C_\mathrm{p}(\mu) & 0 \end{bmatrix} = n_\mathrm{p} + p, \quad \forall \lambda \in \mathrm{spec}(S)$$

holds for all $\mu \in \mathcal{M}$.

Assumption 1 is required for the existence of a stabilizing error-feedback controller of the form (3) that satisfies the internal model property and solves Problem (1). Moreover since we are interested in over-actuated systems, in addition to Assumption 1, following [21,17] we make the following characterizing assumptions on the class of systems under investigation:

Assumption 2 (Over-actuation) *System (1) is over-actuated, that is, $m > p$ and $\mathrm{rank}\ B_\mathrm{p}(\mu) \geq p$, for all $\mu \in \mathcal{M}$. For simplicity, we also assume that $\mathrm{rank}\ C_\mathrm{p}(\mu) = p$, for all $\mu \in \mathcal{M}$.*

Assumption 3 (Nominal right-invertibility) *The triplet $[C_\mathrm{p}(0), A_\mathrm{p}(0), B_\mathrm{p}(0)]$ is right-invertible.*

As proved in [17], Assumptions 2, 3 imply the existence of a family of inputs and state trajectories that satisfies a zero tracking error condition, i.e., $e(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$. These inputs and state trajectories can be parametrized using an *annihilator*, i.e., a dynamical system of the following form:

$$\dot{x}_\mathrm{a} = A_\mathrm{a}x_\mathrm{a} + B_\mathrm{a}v \tag{5a}$$
$$u_\mathrm{rd} = C_\mathrm{a}x_\mathrm{a} + D_\mathrm{a}v \tag{5b}$$
$$x_\mathrm{rd} = E_\mathrm{a}x_\mathrm{a} \tag{5c}$$

where $x_\mathrm{a} \in \mathscr{X}_\mathrm{a} \cong \mathbb{R}^{n_\mathrm{a}}$ is the state, $v \in \mathscr{V} \cong \mathbb{R}^q$ the input and $(x_\mathrm{rd}, u_\mathrm{rd}) \in \mathscr{X}_\mathrm{p} \times \mathscr{U}$ are the outputs. Roughly speaking, an annihilator is a dynamical system whose outputs parametrize all the possible inputs u_rd and state trajectories x_rd for (1) that are invisible from the output e. A more rigorous definition of annihilator is the following:

Definition 1 ([17,7] Nominal annihilator) *An internally stable system of the form (5) is said to be a nominal* annihilator [1] *for (1) if, for all $x_\mathrm{a}(0) \in \mathscr{X}_\mathrm{a}$ and all inputs $v(\cdot) \in \mathscr{L}_\infty$, the corresponding outputs $u_\mathrm{rd}(\cdot), x_\mathrm{rd}(\cdot)$ satisfy*

$$\begin{aligned} \dot{x}_\mathrm{rd}(t) &= A_\mathrm{p}(0)x_\mathrm{rd}(t) + B_\mathrm{p}(0)u_\mathrm{rd}(t) \\ 0 &= C_\mathrm{p}(0)x_\mathrm{rd}(t), \end{aligned}$$

for all $t \in \mathbb{R}_{\geq 0}$.

Under Assumptions 2, 3 a nominal annihilator for (1) can be designed using geometric tools. The construction, which stems from the results in [17], is reported below for sake of completeness. Let us denote by $\mathscr{R}^\star \subset \mathscr{X}_\mathrm{p}$ the supremal controllability subspace contained in $\ker C_\mathrm{p}(0)$, and let $\mathbb{F}(\mathscr{R}^\star)$ be the set of *friends* of $\mathscr{R}^\star$ (see [20, Ch.5]). Define $\mathscr{V} := B_\mathrm{p}(0)^{-1}\mathscr{R}^\star$. It is known that $\ker B_\mathrm{p}(0) \subset \mathscr{V}$, and $\dim \mathscr{V}/\ker B_\mathrm{p}(0) = \mathrm{rank}\ B_\mathrm{p}(0) - p$, see [12].

$$\begin{tikzcd}
\mathscr{V}/\ker B_\mathrm{p}(0) \arrow[dr, "B_\mathrm{a}"] & & & \\
\mathscr{V} \arrow[u, "\pi"] \arrow[r, "\overline{B}_\mathrm{p}(0)"] \arrow[d, "D_\mathrm{a}"'] & \mathscr{R}^\star \arrow[r, "A_\mathrm{a}"] \arrow[d, "E_\mathrm{a}"'] & \mathscr{R}^\star \arrow[d, "E_\mathrm{a}"'] \arrow[dr, "0"] & \\
\mathscr{U} \arrow[r, "B_\mathrm{p}(0)"] & \mathscr{X}_\mathrm{p} \arrow[r, "A_K(0)"] & \mathscr{X}_\mathrm{p} \arrow[r, "C_\mathrm{p}(0)"] & \mathscr{E}
\end{tikzcd}$$

Figure 1. Commutative diagram of the nominal annihilator.

Select, arbitrarily, a symmetric set $\Omega_\rho \subset \mathbb{C}^-$ and let $K \in \mathbb{F}(\mathscr{R}^\star)$ be such that $\mathrm{spec}\, A_K|\mathscr{R}^\star = \Omega_\rho$, where $A_K := A_\mathrm{p}(0) + B_\mathrm{p}(0)K$. Let $\overline{B}_\mathrm{p}(0)$ denote the co-domain restriction to $\mathscr{R}^\star$ of the domain restriction of $B_\mathrm{p}(0)$ to the subspace $\mathscr{V}$, i.e., $\overline{B}_\mathrm{p}(0) = \mathscr{R}^\star|B_\mathrm{p}(0)|\mathscr{V}$, D_a be the insertion map of $\mathscr{V}$ in $\mathscr{U}$, and $B_\mathrm{a} : \mathscr{V}/\ker B_\mathrm{p}(0) \to \mathscr{R}^\star$ be the unique map satisfying $B_\mathrm{a}\pi = \overline{B}_\mathrm{p}(0)$, where $\pi : \mathscr{V} \to \mathscr{V}/\ker B_\mathrm{p}(0)$ denotes the canonical projection modulo $\ker B_\mathrm{p}(0)$. Furthermore, let $E_\mathrm{a} : \mathscr{R}^\star \to \mathscr{X}_\mathrm{p}$ denote the insertion map of $\mathscr{R}^\star$ in $\mathscr{X}_\mathrm{p}$, let $C_\mathrm{a} : \mathscr{R}^\star \to \mathscr{U}$ be defined

[1] The interconnection of (5) and the internal model unit of (3) is termed *extended reference model* in [17].

as $C_a = KE_a$ and let $A_a := A_K|\mathscr{R}^\star$. With this assignment, the 5-ple $(A_a, B_a, C_a, D_a, E_a)$ defines a *nominal annihilator* for (1). According to the commutative diagram in Figure 1 we derive the following identities:

$$
\begin{aligned}
(A_p(0) + B_p(0)K)E_a &= E_a A_a && \text{(7a)}\\
C_a &= KE_a && \text{(7b)}\\
B_p(0)D_a &= E_a B_a && \text{(7c)}\\
C_p(0)E_a &= 0. && \text{(7d)}
\end{aligned}
$$

Remark 1 Notice that the nominal annihilator is defined on the basis of the nominal plant and the invisibility property reported in Definition 1 is, in general, no longer preserved when $\mu \neq 0$. ⌟

By construction the signal u_{rd} does not affect the tracking error e, therefore it can be added to the contribution provided by (3) in order to optimize the total input. Following [21,17], the total input u is realized as a sum of two contributions

$$u := u_{reg} + u_{rd}, \tag{8}$$

where $u_{reg} \in \mathscr{U}$ represents the *regulation component*, able to zeroing asymptotically the error signal e, and $u_{rd} \in \mathscr{U}$ represents a *redundant component* that can be selected by manipulating the input v on the basis of some optimality criterion. The interconnection of (1), (3) and (5) through (8) leads to the following linear system:

$$
\begin{aligned}
\dot{\overline{x}} &= A(\mu)\overline{x} + B(\mu)v + P(\mu)w && \text{(9a)}\\
e &= C(\mu)\overline{x} + Q(\mu)w, && \text{(9b)}
\end{aligned}
$$

where $\overline{x} := \mathrm{col}(x_p, x_c, x_a) \in \overline{\mathscr{X}}$ and the augmented state space is defined as $\overline{\mathscr{X}} := \mathscr{X}_p \times \mathscr{X}_c \times \mathscr{X}_a$. The matrices $A(\mu) \in \mathbb{R}^{n\times n}$, $B(\mu) \in \mathbb{R}^{n\times q}$, $P(\mu) \in \mathbb{R}^{n\times s}$, $C(\mu) \in \mathbb{R}^{p\times n}$, $Q(\mu) \in \mathbb{R}^{p\times s}$, where $n := n_p + n_c + n_a$, are reported below:

$$
\left[\begin{array}{c|c|c} A(\mu) & B(\mu) & P(\mu) \\ \hline C(\mu) & 0 & Q(\mu) \end{array}\right] :=
\left[\begin{array}{ccc|c|c}
A_p(\mu) & B_p(\mu)C_c & B_p(\mu)C_a & B_p(\mu)D_a & P_p(\mu) \\
B_cC_p(\mu) & A_c & 0 & 0 & B_cQ_p(\mu) \\
0 & 0 & A_a & B_a & 0 \\ \hline
C_p(\mu) & 0 & 0 & 0 & Q_p(\mu)
\end{array}\right].
$$

Remark 2 It is important to notice that under Assumption 1 and the interconnection (8), the matrix $A(\mu)$ defined above is Hurwitz for all $\mu \in \mathcal{M}$. This is clear noticing that the state matrix of the annihilator is Hurwitz by construction, and that the upper left block is Hurwitz by hypothesis, see (4). ⌟

Remark 3 In nominal conditions ($\mu = 0$) the controller trajectory response $x_c(\cdot)$ does not depend on the selection of $v(\cdot)$. This is straightforward remembering that the controller (3) is in a error-feedback form and the tracking error $e(\cdot)$ is not affected by $v(\cdot)$. As a consequence the pair $[A(0), B(0)]$ is not controllable but it is stabilizable. ⌟

Once the annihilator (5) is included into the plant-controller loop, the signal v represents a degree of freedom that can be used for optimization purposes without affecting the tracking error of the nominal model. Let us suppose we want to optimize a function

$$\overline{J} : \mathscr{V} \times \overline{\mathscr{X}} \to \mathbb{R}_{\geq 0}, \tag{11}$$

which depends on the augmented state $\overline{x}$ and the annihilator input v. Our goal is to find a suitable selection of v that leads to an instantaneous minimization of the value of $\overline{J}$. This selection is performed using a dynamic allocation unit that is explicitly presented in Section 3. However a selection of v that instantaneously minimize $\overline{J}$ may lead to destabilizing allocation strategies. Indeed, the response forced by $v(\cdot)$ on the state trajectory $x(\cdot)$ is not taken into account. To overcome this issue, and to ensure internal stability of the plant, it is useful to require a weaker form of optimality: an *asymptotic optimality.* By asymptotic optimality we mean that only the asymptotic behavior of (9) matters. More explicitly we aim to optimizing $x_{ss} = -A(\mu)^{-1}B(\mu)v - A(\mu)^{-1}P(\mu)w$, which represents the steady-state value reached by (9) when forced by constant $v(\cdot)$ and $w(\cdot)$. Asymptotic optimality allows us to derive closed form solutions for a large class of strongly convex problems, whose explicit design is presented in Section 4.

However, since the value of μ and w are not available for measurements, the best that we can do is to consider the following change of coordinates that highlights the effects of a non zero μ and non constant v on the nominal dynamics

$$\overline{x} := -A(0)^{-1}B(0)v + x. \tag{12}$$

Here $x \in \mathscr{X}$ represents a time varying signal due to the transient response of (9) and $-A(0)^{-1}B(0)v$ represents the corresponding nominal steady-state value forced by a constant v. Assuming w constant and only looking at steady-state conditions may appear quite restrictive; however in many cases it is possible to speed up the allocator dynamics in order to induce a proper time scale separation property. In this way the slow dynamics, that corresponds to the plant plus the annihilator, can be considered almost in steady-state conditions and the change of coordinates in (12) is not too restrictive. Plugging (12) into (9) yields the following representation,

$$\dot{x} = A(\mu)x + B_1(\mu)v + B_2\dot{v} + P(\mu)w, \tag{13}$$

where matrices $B_1(\mu) \in \mathbb{R}^{n\times q}$, $B_2 \in \mathbb{R}^{n\times q}$ are defined below:

$$B_1(\mu) := B(\mu) - A(\mu)A(0)^{-1}B(0)$$
$$B_2 := A(0)^{-1}B(0).$$

It is worth noticing that B_2 does not depend on the uncertain parameter μ and B_1 satisfies $B_1(\mu)|_{\mu=0} = 0$. Substituting (12) into (11) we define a *steady-state* cost function $J : \mathscr{V} \times \mathscr{X} \to \mathbb{R}_{\geq 0}$ as

$$J(v, x) := \overline{J}(v, -A(0)^{-1}B(0)v + x). \tag{15}$$

Taking advantage of (15), the selection of the most suitable input v is performed as an on-line minimization of J with respect to v, regarding x as perturbation independent of v. More precisely, for any $x^\star \in \mathscr{X}$ we aim at solving the following optimization problem:

$$\underset{v \in \mathscr{V}}{\text{minimize}} \; J(v, x^\star). \tag{16}$$

Remark 4 Optimization problem (16) can be also tackled in a different way using Model Predictive Control (MPC). MPC is rather popular in the allocation literature [14,2,23,22], since optimization of the actuators redundancy can be often formulated as an MPC problem. In our setup, a possible MPC formulation is the following:

$$\begin{aligned} &\underset{\dot{v}(\cdot) \in \mathscr{L}_\infty}{\text{minimize}} && \int_0^T J(v, x)\, \mathrm{d}t \\ &\text{subject to} && (13), (2), \end{aligned} \tag{17}$$

where we considered v as a part of an augmented state through a dynamic extension and we optimize over $\dot{v} \in \mathscr{L}_\infty$. Clearly (17) is parametrized by the initial conditions $\overline{x}(0) = \overline{x}_0 \in \overline{\mathscr{X}}$ and $w(0) = w_0 \in \mathscr{W}$, plus some positive time horizon $T \in \mathbb{R}_{>0}$. However, there are remarkable differences between (17) and (16). First, (16) is a punctual minimization, while in (17) the running cost J is minimized over a time horizon T. Second, (16) does not require a precise model of (13), but treats x as a disturbance affecting the objective function J, while in (16) the presence of the exosystem is explicitly taken into account (reference/disturbance signal w is assumed to be available). We also remark that (17) is not a standard LQR problem, not even in the case of quadratic running cost $J(v, x)$ and $T = \infty$, since, in general, the presence of the marginally stable exosystem implies $J(v, x) \nrightarrow 0$ for $T \to \infty$. ⌟

In the dynamic allocation spirit we assume that the minimizer of Equation (16) is dynamically tracked by a possible nonlinear *redundancy allocator* of the form:

$$\dot{v} = \gamma g(v, x), \tag{18}$$

where $v \in \mathscr{V}$ is the redundancy allocator state, $x \in \mathscr{X}$ is the input and $\gamma \in \mathbb{R}_{>0}$ is a time scaling parameter. We assume that the vector field $g(\cdot, \cdot)$ is smooth, so that, solutions to (18) exist uniquely for any initial conditions.

We are now ready to formally state the *optimal dynamic allocation problem*:

Problem 2 (*Optimal dynamic allocation)* *Given the plant model (1), the regulator (3), the nominal annihilator (5) and the cost function (15), find a redundancy allocator of the form (18) that satisfies the following requirements:*

i) Nominal output regulation: *The matrix in (4) is Hurwitz for all $\mu \in \mathcal{M}$ and asymptotic regulation is achieved for $\mu = 0$.*

ii) Nominal invisibility: *For $\mu = 0$ the tracking error e defined in (9) does not depend on v.*

iii) Structurally well defined steady-state: *For any initial condition $(x(0), v(0), w(0)) \in \mathscr{X} \times \mathscr{V} \times \mathscr{W}$ the solution to*

$$\dot{x} = A(\mu)x + B_1(\mu)v + B_2\dot{v} + P(\mu)w \tag{19a}$$
$$\dot{v} = \gamma g(v, x) \tag{19b}$$
$$\dot{w} = Sw, \tag{19c}$$

is well defined for all μ in a sufficiently small neighborhood of $\mu = 0$ and converges exponentially to a unique steady-state [16].

iv) Structural asymptotic optimality: *There exists $\gamma^\star \in \mathbb{R}_{>0}$ such that for all $\gamma \in \mathbb{R}_{(0,\gamma^\star]}$ in (19) and for each μ in a sufficiently small neighborhood of $\mu = 0$, the point $(-A(\mu)^{-1}B_1(\mu)v^\star(\mu), v^\star(\mu)) \in \mathscr{X} \times \mathscr{V}$, where*

$$v^\star(\mu) := \arg\min_{v \in \mathscr{V}} J(v, -A(\mu)^{-1}B_1(\mu)v),$$

exists, is unique, and constitutes a globally exponentially stable equilibrium for (19) when the exosystem is disconnected, i.e., $w = 0$.

The relevance of the requirements defined in Problem 2 is commented below.

i) Nominal output regulation ensures that in the nominal case ($\mu = 0$) the tracking error e asymptotically vanishes. In the perturbed case ($\mu \neq 0$), because the allocator unit (18) is a nonlinear, the steady-state trajectories may not provide output zeroing, as the linear internal model unit of the regulator may not be able to offset higher-order harmonics of the exogenous signals generated by (18). However, a practical regulation holds, since the global contractivity property in the proof of Proposition 3 ensures that the interconnection between (19a), (19b) is ISS with respect to the the exogenous signal w, see [16, Theorem 2.29]. We conclude that, because the output map is linear, the tracking error is bounded. In the special case of quadratic allocation, see Section 4, the allocator unit is linear and the steady-state entails $e(t) \to 0$ due to the strong properties of robust linear regulators [11].

ii) Nominal invisibility ensures that when the plant is known ($\mu = 0$), the addition of the redundant component u_{rd} at the controller output u_{reg} according to (8) has no effect on the tracking error e.

iii) Structurally well defined steady-state entails the property that, for arbitrary initial conditions, the trajectories of the closed-loop system driven by the exosystem converge exponentially to a unique steady-state. We show that this property can be obtained using a global *contraction* property of (13), which ensures a number of desiderable properties.

iv) Finally structural asymptotic optimality ensures that for any initial condition there exist a unique globally exponentially stable equilibrium for (13), and this equilibrium solves the optimization problem defined in (16). Moreover the structural requirement ensures that this equilibrium point is robustly exponentially stable for sufficiently small values of the parameter μ.

Remark 5 Although Problem 2 has been set up in the output regulation framework, as done in [17], the result presented is of more general validity, and can be applied to other tracking problems where a stable right-inverse of the plant is available, as for instance done in [7]. ⌟

Remark 6 The global contractivity associated to iii) ensures that the steady-state is periodic if the eigenvalues of the exosystem matrix have commensurate angular frequencies. ⌟

3 Allocator design and analysis

Following the paradigm given in [18] for the redundancy allocator in (18) we adopt a gradient descent flow dynamics defined as follows

$$\dot{v} = \gamma g(v, x) := -\gamma \nabla_v J(v, x), \tag{20}$$

where $\gamma \in \mathbb{R}_{>0}$ is a positive gain tuning the speed of the gradient descent. In order for (20) to make sense we need differentiability of the cost function J and in addition to this we also assume some other desiderable properties of J useful to solve Problem 2.

Assumption 4 (Regularity of the cost function) *The cost function $J(\cdot, \cdot)$ is twice continuously differentiable and satisfies the following properties:*

- *uniformly bounded curvature: there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that:*

$$c_1 I \leq \nabla_v^2 J(v, x) \leq c_2 I, \tag{21}$$

for all $(v, x) \in \mathscr{V} \times \mathscr{X}$.

- *bounded mixed interaction: there exists $c_3 \in \mathbb{R}_{>0}$ such that:*

$$\left|\nabla_{xv}^2 J(v, x)\right| \leq c_3,$$

for all $(v, x) \in \mathscr{V} \times \mathscr{X}$.

- *limited curvature: there exists $c_4 \in \mathbb{R}_{>0}$ such that:*

$$\left|\nabla_x^2 J(v, x)\right| \leq c_4 \tag{22}$$

for all $(v, x) \in \mathscr{V} \times \mathscr{X}$.

Remark 7 Alternative choices for the strategy in (20) are possible. For example one may think to employ higher order minimization methods, that exploit knowledge of the Hessian matrix to speed up the convergence process. Some well known examples are the the steepest descent method, where $g(\cdot, \cdot)$ is defined as follows

$$g(v, x) := \frac{\nabla_v J(v, x)^\top \nabla_v J(v, x)}{\nabla_v J(v, x)^\top \nabla_v^2 J(v, x) \nabla_v J(v, x)} \nabla_v J(v, x),$$

and the Newton method, where $g(\cdot, \cdot)$ has an expression that we do not report explicitly here. These optimization techniques are faster but require additional assumptions, e.g., Lipschitz continuity of the Hessian matrix, see [3, Page 488], and complicate the analysis of the resulting closed loop. Moreover we will show in Theorem 1 that the allocation speed, which is proportional to γ, must be selected sufficiently small to guarantee stability of the allocator unit. We believe that selecting more aggressive versions of $g(\cdot, \cdot)$ will then require reducing γ in the practical tuning of our solution. ⌟

The bound on the Hessian $\nabla_v^2 J(v, x)$ ensures the existence and the uniqueness of a minimizer for (16) and guarantees that the resulting gradient-based redundancy allocator (20) is globally convergent. Furthermore the weak interaction assumption ensures that the interaction among the annihilator (5) and the gradient-based redundancy allocator (20) is not too strong so that it is possible to find γ small enough such that this interconnection is stable. Finally the bounded Hessian in the x direction ensures that also in the unnominal case the minimizer for (16) exists and is unique. Having clarified the role of Assumption 4 we are ready to state the main result of the paper.

Theorem 1 (Main result) *Under Assumptions 2, 3, 4 there exist $\gamma^\star \in \mathbb{R}_{>0}$ such that for all $\gamma \in \mathbb{R}_{(0,\gamma^\star)}$ the redundancy allocator in (20) satisfies the requirements of Problem 2 with respect to the performance criterion defined in (15).*

For the convenience of the reader the proof of Theorem 1 is broken into separate parts according to each individual requirement defined in Problem 2.

Proposition 1 *The annihilator defined in (5) enjoys the nominal invisibility property defined in Problem 2.*

Proof. By construction the annihilator (5) satisfies the equalities in (7). Since the invisibility holds only in the nominal case we can consider $\mu = 0$ and without loss of generality we also assume $u_{\mathrm{reg}} = 0$ and $w = 0$. Under these assumptions the plant dynamics can be written as

$$\begin{aligned}\dot{x}_{\mathrm{p}} &= A_{\mathrm{p}}(0)x_{\mathrm{p}} + B_{\mathrm{p}}(0)C_{\mathrm{a}}x_{\mathrm{a}} + B_{\mathrm{p}}(0)D_{\mathrm{a}}v\\ e &= C_{\mathrm{p}}(0)x_{\mathrm{p}}.\end{aligned}$$

Consider the change of coordinates $\tilde{x}_{\mathrm{p}} = x_{\mathrm{p}} - E_{\mathrm{a}}x_{\mathrm{a}}$. Using the identities in (7) and the annihilator dynamics (5), after some manipulations one obtains,

$$\begin{aligned}\dot{\tilde{x}}_{\mathrm{p}} &= A_{\mathrm{p}}(0)\tilde{x}_{\mathrm{p}}\\ e &= C_{\mathrm{p}}(0)\tilde{x}_{\mathrm{p}},\end{aligned}$$

which shows that the output e is not affected by v. □

The following proposition establishes useful properties of the cost function J that are direct consequences of Assumption 4.

Proposition 2 *Under Assumption 4, the function $J(v,x)$ is strongly convex and radially unbounded with respect to v. Moreover, for all $x^\star \in \mathscr{X}$ the minimizer $v^\star = \arg\min_{v\in\mathscr{V}} J(v,x^\star)$ exists and is unique.*

Proof. By Assumption 4, $J(\cdot,\cdot)$ is twice differentiable and with positive definite Hessian with respect to v, therefore strong convexity of $J(\cdot,x)$ holds for each x by definition. Given any x, consider an arbitrary direction $v^\star \in \mathscr{V}$ and the function $\tau \mapsto J(\tau v^\star, x)$ where $\tau \in \mathbb{R}_{\geq 0}$. The second derivative with respect to τ yields $\mathrm{d}^2 J(\tau v^\star,x)/\,\mathrm{d}\tau^2 = (v^\star)^\top \nabla_v^2 J(v,x)|_{v=\tau v^\star} v^\star$ and since

$$c_1|v^\star|^2 \leq (v^\star)^\top \nabla_v^2 J(v,x)|_{v=\tau v^\star} v^\star \leq c_2|v^\star|^2,$$

integrating with respect to τ twice we obtain the following quadratic bounds

$$\begin{aligned}c_1|v^\star|^2\frac{\tau^2}{2} + c_{11}\tau + c_{12} &\leq J(\tau v^\star, x)\\ &\leq c_2|v^\star|^2\frac{\tau^2}{2} + c_{21}\tau + c_{22},\end{aligned}$$

for some $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$. Since the direction $v^\star$ is arbitrary and c_1, c_2 are positive by Assumption 4, $J(v,x)$ is radially unbounded with respect to v. Moreover, for any $x^\star \in \mathscr{X}$, the function $J(\cdot,x^\star)$ is continuous, positive definite and radially unbounded, thus its level sets are compact and by the extreme value theorem for each level set there exist at least one minimizer. Finally since every strongly convex function is also strictly convex, the minimizer $v^\star = \arg\min_{v\in\mathscr{V}} J(v,x^\star)$ is unique and satisfies $\nabla_v J(v,x^\star)|_{v=v^\star} = 0$. □

Now we tackle the problem of the existence of a structurally unique steady-state for (19). Toward this goal it is useful to introduce the notion of global contractivity.

Definition 2 (Global contractivity) *A smooth nonlinear system $\dot{\xi} = f(\xi,d)$, with state ξ and input d, is globally contracting with respect to the (constant) metric $M = M^\top > 0$ with contraction rate $\kappa \in \mathbb{R}_{>0}$ if the condition*

$$\mathrm{He}(\nabla_\xi f(\xi,d)M) \leq -\kappa I, \tag{25}$$

holds for all (ξ,d).

The notion of contractivity was introduced in [15] and many extensions have been proposed, see for example the recent work [10] and references therein. For our purposes it is sufficient to recall that globally contracting systems converges exponentially to a unique steady-state, [15], and therefore a sufficient condition for the existence of a unique steady-state can be cast as a global contractivity requirement. This idea is formalized in the following proposition.

Proposition 3 (Structural steady-state) *For μ and γ sufficiently small the interconnection between (20) and (13) is globally contracting and satisfies the structurally well defined steady-state property in Problem 2.*

Proof. According to Definition 2 we consider a metric $M(\mu) = M(\mu)^\top > 0$ (to be specified) and a generic $\kappa \in \mathbb{R}_{>0}$. The interconnection between (20) and (13) is globally contractive if

$$\mathrm{He}\left(\begin{bmatrix}\nabla_x\dot{x} & \nabla_v\dot{x}\\ \nabla_x\dot{v} & \nabla_v\dot{v}\end{bmatrix} M(\mu)\right) \leq -\kappa I. \tag{26}$$

To establish (26) we consider the congruence transformation $T := \left[\begin{smallmatrix} I & -B_2\\ 0 & I\end{smallmatrix}\right]$ and focus on dynamics (13), (20) to get:

$$T\begin{bmatrix}\nabla_x\dot{x} & \nabla_v\dot{x}\\ \nabla_x\dot{v} & \nabla_v\dot{v}\end{bmatrix} = \begin{bmatrix} A(\mu) & B_1(\mu)\\ -\gamma\nabla_{xv}^2 J(v,x) & -\gamma\nabla_v^2 J(v,x)\end{bmatrix}. \tag{27}$$

Pick $M(\mu) := \mathrm{diag}\,(P(\mu), I) > 0$, where $P(\mu)$ is the unique solution of the following Lyapunov equation

$$A(\mu)P(\mu) + P(\mu)A(\mu)^\top = -I, \tag{28}$$

then using (28) and (27), we get

$$\begin{aligned}\Xi(x,v,\gamma,\mu) :&= \mathrm{He}\left(T\begin{bmatrix}\nabla_x\dot{x} & \nabla_v\dot{x}\\ \nabla_x\dot{v} & \nabla_v\dot{v}\end{bmatrix} M(\mu)T^\top\right)\\ &= \mathrm{He}\left(\begin{bmatrix}-I/2 - M_1(\mu)/2 & 0\\ -\gamma M_2(x,v,\gamma,\mu) & -\gamma\nabla_v^2 J(v,x)\end{bmatrix}\right),\end{aligned} \tag{29}$$

where for conciseness of notation we introduced $M_1(\mu) \in \mathbb{R}^{n\times n}$ and $M_2(x,v,\mu,\gamma) \in \mathbb{R}^{q\times n}$ defined as follows:

$$
\begin{aligned}
M_1(\mu) &:= B_1(\mu)B_2^\top + B_2B_1(\mu)^\top, && \text{(30a)}\\
M_2(x,v,\mu,\gamma) &:= \nabla^2_{xv}J(v,x)P(\mu) \\
&\quad - \nabla^2_v J(v,x)B_2^\top + \gamma^{-1}B_1(\mu)^\top. && \text{(30b)}
\end{aligned}
$$

By assumption $M_1(\mu)$ depends in a continuous way on μ, and since μ ranges over a compact set $|B_1(\mu)|$ is bounded. Moreover for any fixed selection of γ thanks to Assumption 4 also $|M_2(x,v,\mu,\gamma)|$ is uniformly bounded for all $(x,v) \in \mathscr{X}\times\mathscr{V}$. Simple manipulations involving (30) and (22) show that the following inequality holds

$$
\begin{aligned}
|M_2(x,v,\mu,\gamma)| &\le |\nabla^2_{xv}J(x,v)P(\mu)| + |\nabla^2_v J(x,v)B_2^\top| \\
&\quad + \gamma^{-1}|B_1^\top(\mu)| \\
&\le c_3|P(\mu)| + c_2|B_2^\top| + \gamma^{-1}|B_1(\mu)^\top|,
\end{aligned}
$$

where c_2, c_3 are defined in Assumption 4. Since $B_1(0) = 0$, $M_1(0) = 0$ and $B_1(\cdot)$, $M_1(\cdot)$ are continuous, then for each γ there exists a small enough $\mu^\star \in \mathbb{R}_{>0}$ such that for all $|\mu| \le \mu^\star$ we have,

$$
\begin{aligned}
I + M_1(\mu) &\ge m_1 I && \text{(31a)}\\
|M_2(x,v,\mu,\gamma)| &\le c_3|P(\mu)| + c_2|B_2^\top| + 1 \le m_2, && \text{(31b)}
\end{aligned}
$$

where $m_1 := \min_{|\mu|\le\mu^\star}\lambda_1(I+M_1(\mu)) > 0$, and $m_2 := c_3\max_{|\mu|\le\mu^\star}|P(\mu)| + c_2|B_2^\top| + 1 > 0$. Using (31) we show below that there exists a small enough γ and $\kappa' \in \mathbb{R}_{>0}$ such that

$$
\Xi(x,v,\gamma,\mu) \le -\kappa' I, \quad \forall\mu : |\mu| \le \mu^\star. \tag{32}
$$

Due to the structure of $\Xi(x,v,\gamma,\mu)$ in (29), via the congruence transformation T we readily obtain (26) for a small enough $\kappa \in \mathbb{R}_{>0}$. Then a well defined steady-state follows from (25) and Definition 2. We complete the proof by showing (32). By a Schur complement applied to (29), (32) holds if

$$
\begin{aligned}
\nabla^2_v J - \frac{\kappa' I}{2\gamma} &\ge \frac{c_1}{2} I \\
-I - M_1 + \kappa' I + \frac{\gamma}{2}M_2^\top\left(\nabla^2_v J - \frac{\kappa' I}{2\gamma}\right)^{-1} M_2 &\le 0
\end{aligned}
$$

where the arguments of M_1, M_2 and $\nabla_v J$ have been omitted for brevity. The first equation above follows readily from the lower bound in (21) after choosing $\kappa' \le c_1\gamma$. For the second one we use the bounds in (31) and again the upper bound in (21) with $\kappa' \le c_1\gamma$ to obtain the following sufficient condition

$$
-m_1 + \kappa' + \gamma\frac{m_2^2}{c_1} \le 0, \quad \forall|\mu| \le \mu^\star, \tag{33}
$$

which is satisfied for a small enough $\gamma \in \mathbb{R}_{>0}$ and small enough $\kappa' \in \mathbb{R}_{>0}$. □

Remark 8 The proof of Proposition 3 entails the essential derivations employed to show contractivity and then robust stability "in the small" of our scheme. From the interplay between γ and $\mu^\star$, we emphasize that there is a trade-off between the speed of the allocation (γ large) and the level of the plant uncertainty that we can tolerate before losing stability of the loop. This consideration is consistent with the intuitive idea that allocation must be performed in the right "directions" (which are unknown in the unnominal case) otherwise the invisibility property defined in Problem 2 is no longer preserved and stability may be lost. In our preliminary work [5], we favored robustness versus performance by enforcing a restricted robust allocation policy minimizing a cost J evaluated with $u = u_{\mathrm{reg}} + \beta u_{\mathrm{rd}}$, with $\beta \in \mathbb{R}_{(0,1]}$ being a parameter establishing the trade-off between robustness (stability is guaranteed for any $\mu \in \mathcal{M}$ if β is small enough) and restricted optimality (if $\beta = 1$ then the original optimization problem is recovered by the allocator). In that setting the proof of Proposition 3 is not much different because parameter β appears in (32), which is guaranteed by a suitable interplay between γ and β without any restriction on $\mu \in \mathcal{M}$. ⌟

Finally we solve the structural asymptotic optimality requirement defined in Problem 2. The precise statement is the following one.

Proposition 4 (Structural asymptotic optimality) *For μ and γ sufficiently small and a disconnected exosystem ($w = 0$), the interconnection between (20) and (13) converges exponentially to the the unique equilibrium point defined in Problem 2.*

Proof. First, let us notice that in nominal conditions $B_1(0) = 0$ and by Proposition 2, the point $v^\star(0) \in \mathscr{V}$ defined as

$$
v^\star(0) = \arg\min_{v\in\mathscr{V}} J(v,0),
$$

exists and is unique. We now claim that also in the unnominal case the perturbed minimizer $v^\star(\mu)$ exists, is unique and satisfies

$$
v^\star(\mu) = \arg\min_{v\in\mathscr{V}} J(v, -A(\mu)^{-1}B_1(\mu)v).
$$

For this purpose let us define the following quantities

$$
\begin{aligned}
\tilde{J}(v) &:= J(v, -A(\mu)^{-1}B_1(\mu)v) \\
R(\mu) &:= -A(\mu)^{-1}B_1(\mu),
\end{aligned}
$$

which represent a perturbed cost function J and a shorthand symbol for $-A(\mu)^{-1}B_1(\mu)$. Then consider the following quantity

$$\begin{aligned}\nabla_v^2 \tilde{J}(v) &:= \nabla_v^2 J(v, R(\mu)v) + R(\mu)^\top \nabla_{vx}^2 J(v, R(\mu)v) \\ &\quad + \nabla_{xv}^2 J(v, R(\mu)v) R(\mu) \\ &\quad + R(\mu)^\top \nabla_x^2 J(v, R(\mu)v) R(\mu).\end{aligned}$$

Since $|\nabla_{xv}^2 J(v, R(\mu)v)| = |\nabla_{vx}^2 J(v, R(\mu)v)| \leq c_3$ and $|\nabla_x^2 J(v, R(\mu)v)| \leq c_4$ are bounded by Assumption 4, observing that $R(\mu) \to 0$ for $\mu \to 0$, it is concluded that $\nabla_v^2 \tilde{J}(v)$ is positive definite and $\tilde{J}(\cdot)$ is strongly convex for all μ sufficiently close to the origin. Then following the same argument of Proposition 2 we obtain the existence and the uniqueness of the perturbed minimizer $v^\star(\mu)$. Moreover $\nabla_v J(v^\star(\mu), -A(\mu)^{-1}B_1(\mu)v^\star(\mu)) = 0$ and the point $(-A(\mu)^{-1}B_1(\mu)v^\star(\mu), v^\star(\mu)) \in \mathscr{X} \times \mathscr{V}$ is the unique equilibrium for (19) with the allocator choice in (20) when the exosystem is disconnected ($w = 0$). In Proposition 4 we already established that the interconnection between (20) and (13) is globally contractive. By establishing the existence of an equilibrium, the global contraction property is sufficient to say that all trajectories exponentially converge to this equilibrium, see [15,16], which completes the proof. $\square$

4 Design examples

In this section we develop three design example of practical interest, explicitly showing the resulting allocator structure. The three cost functions that we consider involve only the overall input u, therefore the setup of Problem 2 is specialized into the *dynamic input optimization* framework, see [21]. Since the input u is directly available, knowledge of the plant state x_{p} is not required and the corresponding redundancy allocator (18) can be implemented in an error-feedback form. The three cost functions are relaxed versions of the 1, 2 and ∞ norms of u that are especially suitable to optimize sparsity, power and maximum amplitude of u. The resulting redundancy allocator could be useful in different applications. For example 1-norm is a good way to measure energy consumptions in satellites actuated by thrusters, see [19,4], while consumptions in hybrid cars [6] is related to the 2-norm of the input. Finally the ∞-norm is very useful to cope with saturation limits, see for example [21,8], or to optimize the internal forces in multi-robot systems [1]. We show that under a mild assumption these relaxed norms satisfy Assumption 3 with known bounds and consequently the corresponding dynamic input allocation problems are well posed and admit an explicit solution.

We start by solving the problem in a fairly general setting and we will show that the relaxed 1, 2 and ∞ cases follow as special cases. For clarity of the presentation it is useful to introduce an intermediate performance criterion $J_u : \mathscr{U} \to \mathbb{R}_{\geq 0}$. According to the change of coordinates in (12) the input (8) can be expressed in the (x, v) coordinates as follows

$$u = \phi(x, v) := Fx + Gv \tag{34}$$

where $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{m \times q}$ are properly defined matrices whose explicit expression is the following $F := [0, C_{\mathrm{c}}, C_{\mathrm{a}}]$, $G := D_{\mathrm{a}} - C_{\mathrm{a}} A_{\mathrm{a}}^{-1} B_{\mathrm{a}}$. Thanks to (34) and according to (15) the steady-state performance criterion is written as

$$J(v, x) := J_u(\phi(v, x)).$$

The gradient descent strategy defined in (20) and the corresponding *input redundancy allocator* yields,

$$\begin{aligned}\dot{v} &= -\gamma \nabla_v J(v, x) \\ &= -\gamma \nabla_v \phi(v, x) \nabla_u J_u(\phi(v, x)) \\ &= -\gamma G^\top W(u) u|_{u = \phi(v,x)}, \\ &= -\gamma G^\top W(\phi(v, x)) \phi(v, x),\end{aligned} \tag{35}$$

where $W(u) \in \mathbb{R}^{m \times m}$ is an input dependent matrix that "weighs" the allocation directions, see also its use in [21,1]. Notice that there is no real loss of generality in assuming that $\nabla_u J_u(u)^\top := W(u)u$, but some attention may be required in the definition of $W(u)$. The terminology weight matrix is borrowed from the 2-norm minimization problem, see Subsection 4.2; indeed, taking $J_u := u^\top \overline{W} u/2$, with $\overline{W} = \overline{W}^\top > 0$, the corresponding allocator has the expression $\dot{v} = -\gamma G^\top \overline{W} \phi(v, x)$. Toward the goal of checking Assumption 3, we provide an explicit expression for the curvature and for the mixed interaction as follows:

$$\nabla_v^2 J(v, x) = G^\top \nabla_u^2 J_u(u)|_{u=\phi(v,x)} G \tag{36a}$$
$$\nabla_{xv}^2 J(v, x) = G^\top \nabla_u^2 J_u(u)|_{u=\phi(v,x)} F \tag{36b}$$
$$\nabla_x^2 J(v, x) = F^\top \nabla_u^2 J_u(u)|_{u=\phi(v,x)} F. \tag{36c}$$

To avoid trivial scenarios, the following assumption is made:

Assumption 5 *Matrix G has full column rank.*

Assumption 5 involves no loss of generality, as otherwise one can reduce the dimension of v by projection modulo $\ker G$ (note that $q < m$, as shown in [17]). Thanks to Assumption 5 we can state the following:

Proposition 5 *If Assumption 5 holds true and there exist c_5, $c_6 \in \mathbb{R}_{>0}$ such that*

$$c_5 I \leq \nabla_u^2 J_u(u) \leq c_6 I, \quad \forall u \in \mathscr{U}, \tag{37}$$

then Assumption 4 is satisfied.

The proof is a trivial consequence of matrix properties involving (36) and is omitted. In the following subsections we show that (37) holds so that Proposition 5 applies for three peculiar choices of the cost function J_u and we derive then an explicit representation for the redundancy allocator structure.

4.1 Relaxed Manhattan norm (1-norm)

In this subsection we consider a smoothed version of the 1-norm of the input u. The cost function $J_u : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is defined as follows

$$J_u(u) := \sum_{k=1}^{m} \sqrt{u_k^2 + \epsilon_2^2} + \frac{\epsilon_3}{2} u_k^2, \tag{38}$$

where $\epsilon_2 \in \mathbb{R}_{>0}$ is a smoothing constant and the $\epsilon_3 \in \mathbb{R}_{>0}$ term is used to enforce the strong convexity property required in Assumption 3. Then according to the gradient-based allocation strategy defined in (38) the explicit expression for the components of $\nabla_u J_u(u)$ is

$$\nabla_{u_i} \left(\sum_{k=1}^{m} \sqrt{u_k^2 + \epsilon_2^2} + \frac{\epsilon_3}{2} u_k^2 \right) = \frac{u_i}{\sqrt{u_i^2 + \epsilon_2^2}} + \epsilon_3 u_i, \tag{39}$$

for $i = 1, \ldots, m$. We may also notice that the above expression can be put in the form (35) by defining the weight matrix as follows

$$W(u) := \begin{bmatrix} \frac{1}{\sqrt{u_1^2+\epsilon_2^2}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{u_m^2+\epsilon_2^2}} \end{bmatrix} + \epsilon_3 I.$$

It is also interesting to notice that the eigenvalues of $W(u)$ satisfy the inequality

$$\epsilon_3 \leq \lambda_i(W(u)) \leq \epsilon_3 + \epsilon_2^{-1},$$

for $i = 1, \ldots, m$ and for all $u \in \mathscr{U}$. According to Proposition 5 we want prove that $\nabla_u^2 J_u(u)$ is uniformly upper and lower bounded as in (37). We start by noticing that due to the special structure of the gradient (39), $\nabla_{u_i} J_u(u)$ depends only on u_i as a consequence $\nabla_u^2 J_u(u)$ is diagonal and its (i, i) entry is given by

$$\begin{aligned} \nabla_{u_i,u_i}^2 J_u(u) &= -\frac{u_i^2}{(u_i^2 + \epsilon_2^2)^{\frac{3}{2}}} + \frac{1}{\sqrt{u_i^2 + \epsilon_2^2}} + \epsilon_3 \\ &= \frac{\epsilon_2^2}{(u_i^2 + \epsilon_2^2)^{\frac{3}{2}}} + \epsilon_3. \end{aligned} \tag{40}$$

for $i = 1, \ldots, m$. By inspection we can verify that $\nabla_u^2 J_u(u)$ is positive definite, the eigenvalues are uniformly bounded and satisfy the inequality

$$\epsilon_3 \leq \lambda_i \left(\nabla_u^2 J_u(u) \right) \leq \epsilon_3 + \epsilon_2^{-1}, \tag{41}$$

for $i = 1, \ldots, m$ and for all $u \in \mathscr{U}$. According to Proposition 5 the cost function defined in (38) satisfies Assumption 4 with bounds $c_5 = \epsilon_3$ and $c_6 = \epsilon_3 + \epsilon_2^{-1}$ and then Theorem 1 applies.

4.2 Quadratic norm (2-norm)

In this subsection we consider the quadratic norm (or weighted Euclidean norm) of the input u. The quadratic norm is a very special case and is considerably easier to manage compared to other norms. We define the cost function $J_u : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ as follows

$$J_u(u) := \frac{1}{2} \sum_{k=1}^{m} \sum_{\ell=1}^{m} u_k u_\ell \overline{W}_{k\ell} = \frac{1}{2} u^\top \overline{W} u, \tag{42}$$

where $0 < \overline{W} = \overline{W}^\top \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix. The cost function (42) may represent the instantaneous power of u and could be useful optimize consumptions. According to (20) the gradient-based dynamic input optimizer takes the form

$$\dot{v} = -\gamma G^\top \overline{W} u|_{u=\phi(v,x)},$$

where $\nabla_u^2 J(u) = \overline{W}$ is positive definite thanks to the following inequality

$$\lambda_1(\overline{W}) \leq \lambda_i \left(\nabla_u^2 J_u(u) \right) \leq \lambda_m(\overline{W}),$$

for $i = 1, \ldots, m$. Again according to Proposition 5 the cost function defined in (42) satisfies Assumption 4 with bounds $c_5 = \lambda_1(\overline{W})$ and $c_6 = \lambda_m(\overline{W})$ and Theorem 1 applies.

4.3 Max norm (∞-norm)

Finally we consider a smoothed version of the ∞ norm of the input u. This allocation strategy is especially useful to avoid the occurrence of input saturation. The cost function $J_u(u)$ employs a modified version of the *softmax* function, plus a differentiable approximations of the absolute value. For our purposes we define $J_u : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ as follows,

$$\begin{aligned} J_u(u) = \epsilon_1 \log &\left(\sum_{i=1}^{m} \exp \left(\epsilon_1^{-1} \sqrt{u_i^2 + \epsilon_2^2} \right) \right) \\ &+ \frac{\epsilon_3}{2} u^\top u, \end{aligned} \tag{43}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}_{>0}$ are small smoothing constants and $\epsilon_3 \in \mathbb{R}_{>0}$ is a small regularizing term used to enforce strong convexity. The expression in (43) approximates the infinity norm $|u|_\infty$ for $\epsilon_1, \epsilon_2, \epsilon_3$ sufficiently small. Again we choose the dynamic input allocator according to (35) and the explicit expression of the components of the gradient are

$$\nabla_{u_i} J_u(u) = \frac{\exp\left(\epsilon_1^{-1}\sqrt{u_i^2+\epsilon_2^2}\right) u_i / \sqrt{u_i^2+\epsilon_2^2}}{\sum_{k=1}^m \exp\left(\epsilon_1^{-1}\sqrt{u_k^2+\epsilon_2^2}\right)} + \epsilon_3 u_i$$

for $i = 1, \ldots, m$. We may also notice that the above expression can be written in the form (35) defining the weight matrix $W(u)$ as follows

$$W(u) := \Lambda(u) + \epsilon_3 I. \tag{44}$$

where $\Lambda(u) \in \mathbb{R}^{m\times m}$ is a diagonal matrix depending on a normalization function $\rho : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ as reported below:

$$\begin{aligned}
\rho(u) &:= \frac{1}{\sum_{k=1}^m \exp\left(\epsilon_1^{-1}\sqrt{u_k^2+\epsilon_2^2}\right)} \\
\sigma(\nu) &:= \frac{\exp\left(\epsilon_1^{-1}\sqrt{\nu^2+\epsilon_2^2}\right)}{\sqrt{\nu^2+\epsilon_2^2}} \\
\Lambda(u) &:= \rho(u) \begin{bmatrix} \sigma(u_1) & & 0 \\ & \ddots & \\ 0 & & \sigma(u_m) \end{bmatrix}.
\end{aligned}$$

Similarly to the 1-norm case, see Subsection 4.1, the eigenvalues of $W(u)$ are uniformly bounded and satisfy the inequality

$$\epsilon_3 \leq \lambda_i\left(W(u)\right) \leq \epsilon_3 + \epsilon_2^{-1}$$

for $i = 1, \ldots, m$ and for all $u \in \mathscr{U}$. Now in order to check Assumption 3 we need an explicit expression for $\nabla_u^2 J_u(u)$ and for simplicity we proceed component wise,

$$\begin{aligned}
\nabla^2_{u_j,u_i} J_u(u) &= \nabla_{u_j}\rho(u)\sigma(u_i)u_i + \rho(u)\nabla_{u_j}\sigma(u_i) \\
&\quad + \epsilon_3 \nabla_{u_j} u_i \\
&= -\epsilon_1^{-1}\rho(u)^2\sigma(u_j)u_j u_i^\top \sigma(u_i)^\top \\
&\quad + \left(\frac{1+\epsilon_1^{-1}\epsilon_2^{-2}u_i^2\sqrt{u_i^2+\epsilon_2^2}}{1+\epsilon_2^{-2}u_i^2}\right)\rho(u)\sigma(u_i)\delta_{ij} \\
&\quad + \epsilon_3\delta_{ij},
\end{aligned}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, m$ and where δ_{ij} denotes the usual Kronecker delta function. This expression can be also written in a more compact form as follows:

$$\nabla_u^2 J_u(u) = V(u) - b(u)b(u)^\top + \epsilon_3 I, \tag{46}$$

where $b(u) := \epsilon_1^{-1/2}\Lambda(u)u$ and $V(u) \in \mathbb{R}^{m\times m}$ is a diagonal matrix, whose diagonal elements are defined as:

$$V_{ii}(u) := \left(\frac{1+\epsilon_1^{-1}\epsilon_2^{-2}u_i^2\sqrt{u_i^2+\epsilon_2^2}}{1+\epsilon_2^{-2}u_i^2}\right)\Lambda_{ii}(u),$$

for $i = 1, \ldots, m$. Equation (46) shows that $\nabla_u^2 J_u(u)$ is a diagonal uniformly positive definite matrix, $V(u) + \epsilon_3 I$, plus a rank one perturbation term $bb^\top$. Thanks to this special structure we can employ a large number of results regarding rank one perturbations. Now the goal is to show that Proposition 5 holds for the cost function (43). The proof comprises three steps. In Step 1, we show that $V(u)$ is positive definite for all $u \in \mathscr{U}$. In Step 2, using Weyl's inequality we prove that the rank one perturbed matrix $V(u)-b(u)b(u)^\top$ has $m-1$ bounded and positive eigenvalues. In Step 3, thanks to the matrix determinant lemma, see [9], we show that $\det\left(V(u) - b(u)b(u)^\top\right)$ has the same sign as $\det V(u)$ so that we can conclude that all the eigenvalues are positive and bounded. Finally we notice that the $\epsilon_3 I$ term enforces the desired uniformity property.

Step 1. We start noticing that $V(u)$ is diagonal and by inspection the elements $V_{ii}(u)$ satisfies the following bound

$$0 \leq V_{ii}(u) \leq \max\{\epsilon_1^{-1}, \epsilon_2^{-1}\} \tag{47}$$

for $i = 1, \ldots, m$ and for all $u \in \mathscr{U}$, thus matrix $V(u)$ is positive definite.

Step 2. For the convenience of the reader, we recall an adapted version of Weyl's theorem, see [13, Theorem 4.3.1, page 239]. For the sake of simplicity we drop the u dependence for matrices $V = V(u)$ and $b = b(u)$.

Theorem 2 (Weyl) *Let V, $bb^\top \in \mathbb{R}^{m\times m}$ be symmetric and let the respective eigenvalues of V, $bb^\top$ and $V - bb^\top$ be $\left[\lambda_i(V)\right]_{i=1}^m$, $\left[\lambda_i(-bb^\top)\right]_{i=1}^m$ and $\left[\lambda_i(V-bb^\top)\right]_{i=1}^m$, each list being algebraically ordered. Then for each $i = 1, \ldots, m$ the following inequality*

$$\lambda_i(V - bb^\top) \leq \lambda_{i+j}(V) + \lambda_{m-j}(-bb^\top), \tag{48}$$

holds for all $j = 0, \ldots, m-i$ and

$$\lambda_{i-j+1}(V) + \lambda_j(-bb^\top) \leq \lambda_i(V - bb^\top), \tag{49}$$

for all $j = 1, \ldots, i$.

Inequality (48) can be used to upper bound the eigenvalues of $V - bb^\top$, indeed taking $j = 0$, using (47) and the fact that $\lambda_m\left(-bb^\top\right) = 0$, we readily obtain

$$\begin{gathered}
\lambda_1(V - bb^\top) \leq \lambda_1(V) \leq \max\{\epsilon_1^{-1}, \epsilon_2^{-1}\} \\
\vdots \\
\lambda_m(V - bb^\top) \leq \lambda_m(V) \leq \max\{\epsilon_1^{-1}, \epsilon_2^{-1}\}.
\end{gathered}$$

For a lower bound we may similarly use (49) by taking $j = i$, which yields

$$\begin{aligned}\lambda_1(V) + \lambda_1(-bb^\top) &\leq \lambda_1(V - bb^\top)\\ \lambda_1(V) &\leq \lambda_2(V - bb^\top)\\ &\vdots\\ \lambda_1(V) &\leq \lambda_m(V - bb^\top).\end{aligned}$$

Recalling that V is positive definite, this shows that $\lambda_i(V - bb^\top) \geq 0$ for $i = 2, \ldots, m$.

Step 3. To show positivity of the last eigenvalue $\lambda_1(V - bb^\top)$, we can use the *matrix determinant lemma* [9] reported next.

Lemma 1 (Matrix determinant lemma) *Let $V \in \mathbb{R}^{m\times m}$ be an invertible matrix, and $bb^\top \in \mathbb{R}^{m\times m}$ a rank-one matrix obtained as a dyadic product between vectors. Then*

$$\det(V - bb^\top) = \left(1 - b^\top V^{-1} b\right)\det(V).$$

Thus it is sufficient to show that the following holds

$$b^\top(u)V^{-1}(u)b(u) < 1 \tag{52}$$

for all $u \in \mathscr{U}$. Since $b(u)^\top V(u)^{-1} b(u)$ involves the product of diagonal matrices the products is also diagonal and the elements are given by

$$\epsilon_1^{-1}\Lambda_{ii}(u)V_{ii}(u)^{-1}\Lambda_{ii}(u) = \frac{(1+\epsilon_2^{-2}u_i^2)\Lambda_{ii}(u)}{\epsilon_1^{-1} + \epsilon_2^{-2}u_i^2\sqrt{u_i^2+\epsilon_2^2}}.$$

After some manipulations the value of the quadratic form (52) yields

$$b(u)^\top V(u)^{-1} b(u) = \frac{\sum_{i=1}^m \alpha(u_i)\exp\left(\epsilon_1^{-1}\sqrt{u_i^2+\epsilon_2^2}\right)}{\sum_{k=1}^m \exp\left(\epsilon_1^{-1}\sqrt{u_k^2+\epsilon_2^2}\right)}$$

where the coefficients $\alpha(u_i) \in \mathbb{R}$, are defined below

$$\alpha(u_i) := \frac{u_i^2(1+\epsilon_2^{-2}u_i^2)}{u_i^2(1+\epsilon_2^{-2}u_i^2) + \epsilon_1\sqrt{u_i^2+\epsilon_2^2}} < 1 \tag{53}$$

for $i = 1, \ldots, m$. Thanks to (53), (52) is satisfied and $\lambda_1(V - bb^\top) > 0$. Combining the results of Weyl's inequality and the matrix determinant lemma we obtain the following inequality

$$\epsilon_3 I \leq \nabla_u^2 J_u(u) \leq I(\epsilon_3 + \max\{\epsilon_1^{-1}, \epsilon_2^{-1}\}) \tag{54}$$

for all $u \in \mathscr{U}$. Then according to Proposition 5 the cost function defined in (43) satisfies Assumption 4 with bounds $c_5 = \epsilon_3$ and $c_6 = (\epsilon_3 + \max\{\epsilon_1^{-1}, \epsilon_2^{-1}\})$, and implies that Theorem 1 holds also for the relaxed max norm defined in (43).

5 Simulations

We present an academic example to illustrate the effectiveness of the proposed approach. We consider an exosystem of the form $S := \mathrm{diag}\left(\left[\begin{smallmatrix} 0 & 1/2 \\ -1/2 & 0\end{smallmatrix}\right], 0\right)$ with initial condition $w(0) := (1, 0, 1) \in \mathbb{R}^3$. The arising solution is of the form $w(t) = (\sin(t/2), \cos(t/2), 1)$, providing a sinusoidal reference to track and a constant disturbance to reject. Matrices $A_\mathrm{p}(0)$, $B_\mathrm{p}(0)$, $C_\mathrm{p}(0)$ have been randomly generated and are reported in (55) at the top of the next page. Matrix $P_\mathrm{p}(0)$ is designed to extract from w the constant part while $Q_\mathrm{p}(0)$ selects the sinusoid as a reference to track for the output. The nominal transfer matrix $H_0(s) := C(0)(sI - A(0))^{-1}B(0)$ has been perturbed $H(s) := H_0(s) + \Delta(s)$ by a norm-bounded additive disturbance $\Delta(s)$, satisfying $|\Delta(j\omega)| \leq 0.1$ for all $\omega \in \mathbb{R}_{\geq 0}$. Regarding the allocator, because the convergence rate of a gradient descent strategy depends on c_6/c_5, see [3, Page 466], we suggest to select the smoothing parameters ϵ_1, ϵ_2, and ϵ_3 in such a way that c_6/c_5 is not excessively large. In order to facilitate the selection we may use the explicit bounds reported in (41) and (54). For this specific example we chose $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.05$. The selection of γ can be performed in the nominal scenario using (33) and afterwards decreasing its value in order to compensate for uncertainties. In this example we selected $\gamma_1 = 4$, $\gamma_2 = 10$, and $\gamma_\infty = 8$, respectively for the 1-2 and ∞ norm case. For the 2-norm, $\overline{W} = \epsilon_3 I$. The initial condition for (1) has been set equal to $x_\mathrm{p}(0) = (-110, 12, 90) \in \mathbb{R}^3$, which is large enough so that we can illustrate global convergence towards the steady state, while $x_\mathrm{c}(0)$ and $x_\mathrm{a}(0)$ have been set to zero.

Figure 2 shows the input u for the three allocation strategies. For the 1 and ∞-norm allocation we observe a nonlinear response due to the nonlinear gradient of the cost function J. It is also interesting to point out that the 1-norm can be considered as a rough approximation of the 0-norm [2]. Finally we observe that, thanks to the properties of contracting systems [15], the steady-state response is periodic. Figure 3 shows a comparison between the norm of u_reg and u. We notice that the redundant control action u_rd helps in reducing the peak values, but we remark that the performance strongly depends on the uncertainty level. If the plant is too uncertain, a slow allocation must be performed and we can only hope to optimize a constant steady state value. Figure 4 shows

[2] Given a vector $x \in \mathbb{R}^n$, the zero norm $|x|_0$ is the number of non-zero elements of x. It can be considered as a special case of the Hamming distance. Notice that technically speaking the 0-norm is not a norm because it is not homogeneous.

$$\left[\begin{array}{c|c|c} A(0) & B(0) & P(0) \\ \hline C(0) & 0 & Q(0) \end{array}\right] = \left[\begin{array}{ccc|cccc|ccc} 0.68 & 0.27 & 0.45 & 0.32 & -0.35 & -0.30 & 0.45 & 0 & 0 & -0.57 \\ -0.28 & 0.07 & -0.44 & -0.44 & 0.92 & 0.79 & 0.19 & 0 & 0 & -0.08 \\ 0.70 & -0.46 & -0.18 & 0.48 & 0.03 & -0.10 & -0.05 & 0 & 0 & 0.18 \\ \hline 0.90 & 0.51 & -0.26 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \tag{55}$$

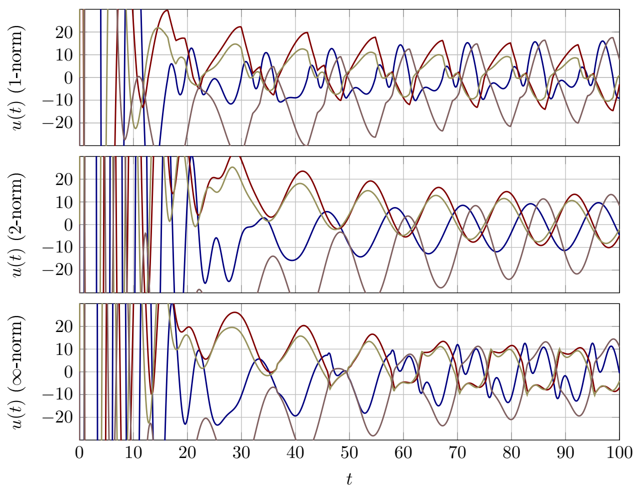

Figure 2. Comparison of the plant input u for the three allocation strategies presented in Section 4. Respectively from above: 1, 2 and ∞-norm allocation.

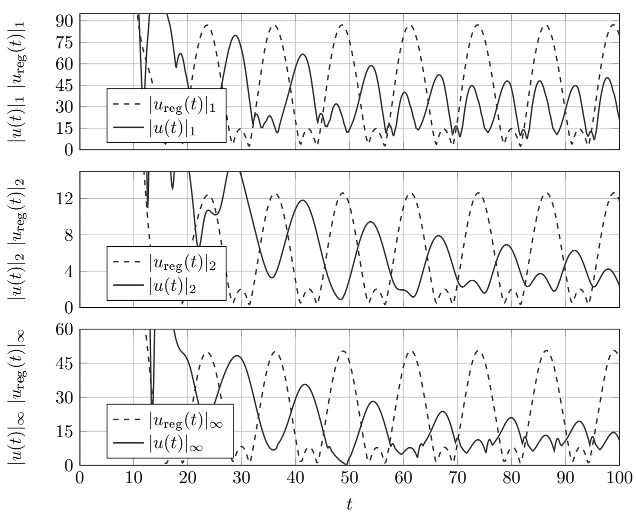

Figure 3. Comparison among norms of the controller output u_{reg} and the net plant input u. From above 1, 2, and ∞ norm.

the convergence of the tracking error e. Only for the 2-norm allocation we recover asymptotic regulation, while in the other cases a spurious periodic perturbation due to higher order harmonics, results into a practical regulation. This is, however, a small price to pay for a considerable improvement of the response in terms of sparsity (1-norm) and overall amplitude (∞-norm) of the control effort, which are inherently nonlinear performance metrics. Finally, it is interesting to compare the gradi-

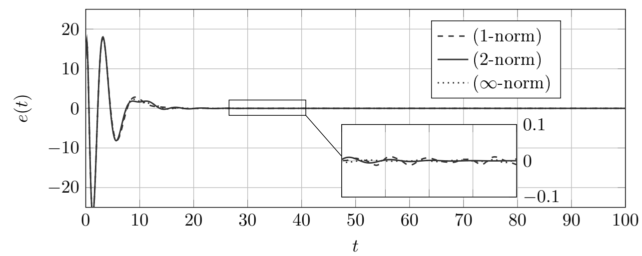

Figure 4. Euclidean norm of the tracking error in the case of an uncertain plant ($\mu \neq 0$).

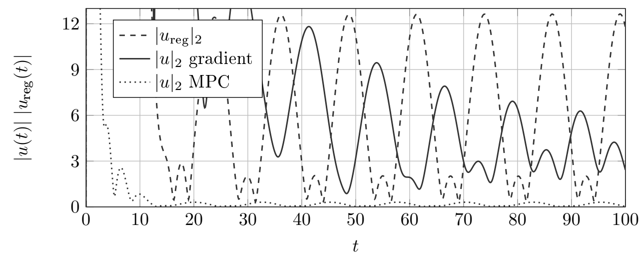

Figure 5. Gradient allocation vs MPC allocation for the 2-norm optimization problem.

ent based allocation scheme with the optimal solution to (17) obtained numerically with an MPC scheme, formulated according to the discussion in Remark 4. Toward this goal, we discretized (17) using a Tustin algorithm selecting a sampling time $T_{\mathrm{s}} = 0.005$ s and a time horizon $T = 5$ s (large enough to guarantee stability of the closed-loop). The discretized quadratic allocation problem in Section 4.2 results into an unconstrained least square problem which can be solved explicitly. Minimization of the 1 and ∞ norms can be formulated as Linear Programming (LP) problems, see [3, Exercise 4.11], and consequently used into a MPC framework. However over-actuated systems are often large and the resulting LP problem may be extremely demanding from a numerical point of view.

From Figure 5 we observe that the MPC allocation performs much better, but in contrast to the gradient allocation strategy, the MPC algorithm has full access to the plant, controller and exosystem state. On the contrary, the gradient allocation requires only u_{reg} to work. Finally we remark that the gradient allocation strategy developed in this paper has a plug and play structure that can easily connected/disconnected from an pre-designed control loop and is associated to computationally attractive implementations not requiring online optimizations.

6 Conclusions

We defined the dynamic redundancy allocation problem for linear uncertain over-actuated systems in the framework of robust linear output regulation. We presented a procedure to solve this problem under some regularity conditions on the cost function and we derived explicit designs for three case of practical interest. We shown that the proposed dynamic allocation strategy ensures structural stability and contractivity of the closed-loop system and implies the existence of a unique attractive steady-state motion. Simulations performed on an academic example indicate that the method is effective in providing on-line minimization of the desired cost function. It is noted, however, that nonlinear allocation strategies may produce a steady-state error with perturbed plants, pointing to the need for a redesign of the internal model unit of the regulator to offset the spurious harmonics generated by the dynamic optimizer. The redesign of the internal model unit and the development of better allocation strategies using the knowledge of the controller state to improve performance will be the subject of future works.

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