MODELLING AND CONTROL OF URBAN BUS NETWORKS IN DIOIDS ALGEBRA

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Abstract: We aim at applying dioids algebraic tools to the study of urban bus networks. In particular, we show how they can be used for timetables synthesis.

Keywords: Discrete Event System, dioid, bus networks, timetables synthesis

1. INTRODUCTION

Discrete Event Dynamic Systems (DEDS) subject to synchronization phenomena can be modeled by linear equations in a particular algebraic structure called *dioid* or *idempotent semi-ring*. For about twenty years, this property has motivated the elaboration of a "new" linear system theory. Indeed, concepts such as state representation, transfer matrix, optimal control, correctors synthesis and identification theory have been transposed from conventional linear system theory to the considered algebraic structure (Baccelli et al., 1992). Applications of this theory have essentially concerned manufacturing systems (Menguy et al., 2000; Lahaye et al., 2003a), communication networks (Le Boudec and Thiran, 2001) and transportation networks (Braker, 1993; de Vries et al., 1998). In the latter, the focus has been on modelling, performance evaluation and stability analysis of railway networks.

For our concern, we are interested in applying dioids algebraic tools to the study of urban bus networks. The piloting of these networks can be seen as a two steps process. Firstly, a so-called *operating schedule* is defined for "mean conditions". In particular, timetables are defined at each stop to specify times at which buses should theoretically run. These timetables are used to inform passengers and, at particular stops, to re-synchronize buses. The second step consists in *regulation operations*: in reaction to current conditions (breakdown of a bus, modifications of traffic flows, *etc.*), a supervisor [1] may decide of adjustments from the operating schedule (transfer passengers, stop or reroute buses, *etc.*).

In (Lahaye et al., 2003b), we have proposed a first approach aiming at modelling some of these regulation operations. In practice, such a model may be used as a tool to aid decisions since it allows evaluating relevance of adjustments.

In this paper, the focus is rather on the first stage described above. In fact, we propose a model for urban bus networks functioning according to their operating schedule. We then show how dioid techniques [2] can be used for the synthesis of timetables. As described in (Ceder et al., 2000) this problem is crucial, and aims notably at maximizing connections between buses from different lines. The outline of the paper is as follows. In §2, we recall elements of dioid theory as well as principles of DEDS description and control over dioids. In §3, we describe how bus networks operate, and a model is introduced. In §4, the method for timetables synthesis is proposed. An application to an elementary bus network is proposed in §5.

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[1] Visualizing evolutions inside the network and communicating with bus drivers.
[2] The proposed approach is mainly based on residuation theory.

2. PRELIMINARIES

2.1 Elements of dioid theory

Definition 1. (Dioid). A dioid is a set $\mathcal{D}$ with two inner operations denoted $\oplus$ and $\otimes$. The sum is associative, commutative, idempotent ($\forall a \in \mathcal{D}$, $a \oplus a = a$) and admits a neutral element denoted ε. The product is associative, distributes over the sum and admits a neutral element denoted e. The element ε is absorbing for the product.

In a dioid $\mathcal{D}$, the equivalence : $a \succeq b \Leftrightarrow a = a \oplus b$ defines a partial order relation.

Definition 2. (Complete dioid). A dioid is said to be complete if it is closed for infinite sums and if product distributes over infinite sums too. The sum of all its elements is generally denoted $\top$.

A complete dioid has a structure of complete lattice (Baccelli et al., 1992, §4) , *i.e.* two elements in a complete dioid always have a *least upper bound* namely $a \oplus b$ and a *greatest lower bound* denoted $a \wedge b = \bigoplus_{\{x | x \preceq a, \ x \preceq b\}} x$.

Example 1. The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ endowed with the max operator as sum and the classical sum as product is a complete dioid, denoted $\overline{\mathbb{Z}}_{max}$ and usually called $(max, +)$ algebra, with $\varepsilon = -\infty$ and $e = 0$.

Theorem 1. The implicit equation $x = ax \oplus b$ defined over a complete dioid admits $x = a^*b$ as least solution with $a^* = \bigoplus_{i=0}^{+\infty} a^i$ and $a^0 = e$. The star operator $*$ is usually called *Kleene star*.

2.2 Residuation theory

A mapping f defined from a complete dioid $\mathcal{D}$ into a complete dioid $\mathcal{C}$ is said to be *isotone* if $a, b \in \mathcal{D}$, $a \preceq b \Rightarrow f(a) \preceq f(b)$. Residuation theory (Blyth and Janowitz, 1972) defines "pseudo-inverses" for some isotone mappings defined over ordered sets such as dioids (Cohen, 1998). More precisely, if the greatest element of set $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists for all $b \in \mathcal{C}$, then it is denoted $f^\sharp(b)$ and $f^\sharp$ is called *residual* of f.

The isotone mapping $L_a : x \mapsto a \otimes x$ defined in a complete dioid is residuated. The greatest solution to $a \otimes x \preceq b$ exists and is equal to $L_a^\sharp(b)$, also denoted $\frac{b}{a}$ or $a \backslash b$. We furthermore have the following *formulæ* (see (Baccelli et al., 1992, §4.4)).

$$\frac{x}{ab} = \frac{a \backslash x}{b} \ \ (1) \quad \frac{x}{a^*} \preceq x \ \ (2) \quad \frac{a^* \backslash x}{a^*} = \frac{x}{a^*} \ \ (3)$$

Theorem 2. (Baccelli et al., 1992, Th. 4.56) Let $f : \mathcal{D} \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{B}$. If f and g are residuated then $g \circ f$ is residuated and $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$.

Definition 3. (Mapping restriction). Let $f : \mathcal{D} \to \mathcal{C}$ a mapping and $\mathcal{A} \subseteq \mathcal{D}$. We denote $f_{|\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ the mapping defined by equality $f_{|\mathcal{A}} = f \circ Id_{|\mathcal{A}}$ where $Id_{|\mathcal{A}} : \mathcal{A} \to \mathcal{D}$ is the canonical injection.

A *constrained residuation problem* (Cohen, 1998) consists in finding the greatest solution to $f(x) \preceq b$ not in the whole $\mathcal{D}$ but only in a subset $\mathcal{A}$ of $\mathcal{D}$. More precisely, being given a residuated mapping f we aim at solving:

$$f_{|\mathcal{A}}(x) = f \circ Id_{|\mathcal{A}}(x) \preceq b. \qquad (4)$$

Proposition 1. If the canonical injection $Id_{|\mathcal{A}}$ is residuated, then the constrained residuation problem (4) admits an optimal solution denoted $f_{|\mathcal{A}}^\sharp(b)$.

Proof :

Thanks to theorem 2, if $Id_{|\mathcal{A}}$ is residuated, then so is $f \circ Id_{|\mathcal{A}}$, and the greatest solution to (4) is given by

$$f_{|\mathcal{A}}^\sharp(b) = (f \circ Id_{|\mathcal{A}})^\sharp(b) = Id_{|\mathcal{A}}^\sharp \circ f^\sharp(b) \qquad (5)$$

$\square$

2.3 Representation of DEDS using dioids

Dioids algebra enables to model DEDS involving (only) synchronization phenomena. For instance, by dating each event, *i.e.* by associating with each event a discrete function x (resp. u and y for an input and an output) called *dater* [3] , it is possible to get a linear state representation:

$$\begin{cases} x(k) = Ax(k-1) \oplus Bu(k) \\ y(k) = Cx(k). \end{cases} \qquad (6)$$

An analogous transform to $\mathcal{Z}$-transform (used to represent discrete-time trajectories in classical theory) can be introduced for daters. The γ-transform of a dater $x(k)$ is defined as the following formal power series: $x = \bigoplus_{k \in \mathbb{Z}} x(k) \gamma^k$. Indeed, variable γ can be interpreted as the backward shift operator in event domain. The set of formal power series in one variable γ and coefficients in $\overline{\mathbb{Z}}_{max}$ is a dioid denoted $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$.

State representation (6) becomes in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$:

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[3] $x(k)$ is the time of the $k + 1$-th occurrence of event x.

$$\begin{cases} x = \gamma A x \oplus B u \\ y = C x. \end{cases} \quad (7)$$

Considering the earliest functioning of the system, we select the least solution of the first equation of (7) which is given according to theorem 1 by $x = (\gamma A)^* B u$. This leads to $y = H u = C(\gamma A)^* B u$, in which $H = C(\gamma A)^* B$ is called the *transfer matrix*.

2.4 Just In Time control

The *Just In Time (JIT) control* of a DEDS consists in computing the latest input dates which cause output dates to be earlier than or equal to a given target. Let us define a dater specifying the desired outputs and z its γ-transform. From the representation in dioid algebra, this control problem is formulated as an inequality, and residuation theory leads to optimal solution u_{opt}:

$$u_{opt} = \bigoplus_{\{u | L_H(u) = H u \preceq z\}} u = \frac{z}{H} = \frac{z}{C(\gamma A)^* B}. \quad (8)$$

3. MODELLING OF URBAN BUS NETWORKS

Many transportation systems may be studied as DEDS. With this point of view, their evolution is conditioned by events such as arrivals or departures of vehicles. In next sections, the focus is on urban bus networks. We first describe how they operate. Then, we propose a model in dioid $\overline{\mathbb{Z}}_{max}$.

3.1 Functioning of urban bus networks

As presented in (Hayat and Maouche, 1997), piloting of public transportation networks can be decomposed in two stages : the "operating schedule" and the "regulation".

Definition of an operating schedule The operating schedule is established with the aim of optimizing the offer of service according to objectives and constraints (bus fleet, line layouts, staff hours of work, *etc*). It is calculated for mean functioning conditions. In practical terms, this optimization results in:

(1) The definition of the bus routes and the bus stops for each line.
(2) The *choice of a level-of-service* for each line: distribution of resources (buses, drivers), definition of *the minimum and maximum headways* [4] (*i.e.* the expected minimum and maximum time separations between two buses).

(3) The *synthesis of timetables* defining times at which buses should theoretically run at each stop. Timetables are used to inform passengers and, at some stops, to re-synchronize buses.

Regulation This stage corresponds to adjustments or adaptations from the operating schedule in reaction to current functioning conditions. Common conditions leading to such adjustment operations are *disturbances*: breakdown of buses, modifications of traffic flows (for instance due to accidents), *etc*. A supervisor may then decide to transfer passengers, stop or reroute buses...

In the following, we are only interested in modelling the operating schedule. In §4, we furthermore show how dioid techniques can be used for the synthesis of timetables.

3.2 Modelling of a bus network

In this section, we propose a model for urban bus networks functioning according to their operating schedule. We assume that each line i includes n_i bus stops. The network is composed by M lines and $N = n_1 + \ldots + n_M$ stops. In the following, let $x_i(k)$ denote the departure time of the $(k+1)$-st bus at stop i. Without loss of generality, we assume that at the beginning of operation a bus departs from each stop [5]. Suppose that the bus coming from stop j reaches stop i. Then we have the following inequation: $x_i(k) \geq a_{ij} + x_j(k-1), k > 1$, in which a_{ij} denotes the travelling time from stop j to i. Let $x(k) = (x_1(k), x_2(k), \ldots, x_N(k))^t$, for the whole network this inequation can be written in max-algebraic matrix notation

$$x(k) \geq A \otimes x(k-1), \quad (9)$$

in which $A_{ij} = a_{ij}$ if stop j precedes stop i, otherwise $A_{ij} = \varepsilon$.

In practice, a public transportation network operates under a timetable which schedules the departure times of every bus. Buses respect timetables

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[4] A minimum headway is defined to counteract the natural tendency of transit vehicles to bunch up. Thus, if a bus falls

slightly behind schedule for any reasons, it will have more than the average number of passengers to pick up at the next station, which causes further delays. Thus, it keeps failing further behind schedule. Conversely, the bus behind it encounters fewer passengers than usual, allowing it to catch up with the preceding bus. Such bunching tends to appear if no minimum headway is defined. Symmetrically, a maximum headway specifies the desired level-of-service since it defines the minimum frequency of buses on the line.
[5] If no or several bus(es) initially depart from stops, then this results only in indexes modification. These cases can be dealt exactly as cases of places initially containing no or several token(s) for the modelling of timed event graph (Baccelli et al., 1992, §2.5.2).

only at specific stops[6] such as terminus or departure of lines or main stations. At the other stops, timetables are only used to inform passengers. We denote $u_l(k)$ the scheduled departure time for the $(k+1)$-st bus at stop l and S_u the set of specific stops, this leads to

$$x(k) \geq Bu(k), \qquad (10)$$

in which B is a $N \times N$ matrix where $B_{ii} = e$ if stop $i \in S_u$, $B_{ij} = \varepsilon$ otherwise.

Considering inequations (9) and (10) and assuming that buses depart as soon as possible, a scheduled transportation network is modelled in dioid $\overline{\mathbb{Z}}_{max}$ by a state equation as in (6).

4. TIMETABLES SYNTHESIS OF URBAN BUS NETWORKS

4.1 Problem description

In a bus network, it may be profitable to minimize waiting times at a given stop for passengers arriving from another given stop. Such stops, generally close from a geographic point of view[7], belong to an itinerary which should be promoted[8] and are called *connection stops*.

With the aim of doing so, we consider a problem formulated in (Ceder et al., 2000)[9]. This problem consists in finding timetables which

(i) maximize synchronizations at connection stops,

(ii) ensure an expected level-of-service at particular stops of the network,

(iii) satisfy planning constraints (*i.e.* minimum and maximum headways, see §3.1).

In order to compute timetables at each stop so as to synchronize bus departures at connection stops (objective mentioned at item (i)), we have to find input $\{u(k)\}_{k \in \mathbb{Z}}$ such that state $\{x(k)\}_{k \in \mathbb{Z}}$ satisfies:

$$x(k) = A'x(k-1) \oplus B'u(k) \qquad (11)$$

where evolution matrix A' takes into account connections between lines and $B' = B$. More precisely, $A'_{ij} = A_{ij}$ if stop j precedes stop i on the same line; $A'_{ij} = w_{ij}$ denotes the walking time

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[6] Where buses can park without blocking or disturbing traffic.

[7] Walking time between these stops is small (they are sometimes located at a same node of the network).

[8] Due to the important flow of users and/or for commercial reasons.

[9] Tools used in (Ceder et al., 2000) are essentially based on linear programming methods

from stop j to stop i if i is in connection with j; $A'_{ij} = \varepsilon$ otherwise.

The desired level-of-service (mentioned at item (ii)) is specified for some particular stops (we denote S_o the set of these stops). More precisely, for such a stop $i \in S_o$, a dater $\{z_i(k)\}_{k \in \mathbb{Z}}$ can be defined: $z_i(k)$ specifies the latest date at which the $k+1$-st bus should depart from stop i. From equation (11), departure times at stops belonging to S_o are given by:

$$y(k) = C'x(k), \qquad (12)$$

where C' is a $q \times N$ matrix ($q = Card(S_o)$) in which $C'_{ij} = e$ if i, j are indexes in y and x of a same stop in S_o, $C'_{ij} = \varepsilon$ otherwise.

We then aim at finding $\{u(k)\}_{k \in \mathbb{Z}}$ such that

$$y(k) = C'x(k) \preceq z(k). \qquad (13)$$

According to item (iii), timetables given by dater $\{u(k)\}_{k \in \mathbb{Z}}$ must furthermore satisfy the planning constraints, *i.e.*

$$u(k+1) \quad \geq \quad \triangle^{min}u(k) \qquad (14)$$

$$\triangle^{max}u(k) \quad \geq \quad u(k+1), \qquad (15)$$

where $\triangle^{min}$ and $\triangle^{max}$ are diagonal matrices traducing minimum and maximum headways for each line. For all stops s of line i, the element $\triangle^{min}_{ss}$ (resp. $\triangle^{max}_{ss}$) is equal to the minimum (resp. maximum) headway $\triangle^{min}_i$ (resp. $\triangle^{max}_i$) of line i.

4.2 Formalization as a constrained residuation problem

From γ-transforms of equations (11) and (12), we obtain the expected *input/output behavior* (*cf.* § 2.3): $y = Hu = L_H(u)$, with $H = C'(\gamma A')^* B'$.

Equation (13) can then be written

$$y = L_H(u) \preceq z. \qquad (16)$$

More precisely, we are interested in finding the greatest solution to this inequation. This solution allows buses to stay at stops belonging to S_u as late as possible while satisfying desired departures dates at stops in S_o (given by z). With this formulation, the problem given by equation (16) then appears to be a JIT control problem (*cf.* § 2.4).

Moreover, the searched solution u must satisfy inequations (14) and (15). Denoting $\mathcal{A}[\![\gamma]\!]$ the subset of $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$ composed of series satisfying γ-transforms of (14) and (15), we can rewrite the

considered problem as a constrained residuation problem (*cf.* § 2.2):

$$\begin{cases} L_H(u) \preceq z \\ u \in \mathcal{A}[\![\gamma]\!] \end{cases} \text{(admissible domain).} \qquad (17)$$

Canonical injection of the subset $\mathcal{A}[\![\gamma]\!]$ in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$ is denoted $Id_{|\mathbb{A}}$ and $L_{H_{|\mathbb{A}}}$ is the mapping L_H restricted to the domain $\mathcal{A}[\![\gamma]\!]$. Regarding definition 3, the constrained problem is equivalent to solve:

$$L_{H_{|\mathbb{A}}}(u) = L_H \circ Id_{|\mathbb{A}}(u) \preceq z . \qquad (18)$$

4.3 Solving the problem

From equation (18), we can apply proposition 1 if $Id_{|\mathbb{A}}$ is residuated:

$$L_{H_{|\mathbb{A}}}^\sharp(z) = Id_{|\mathbb{A}}^\sharp \circ L_H^\sharp(z). \qquad (19)$$

The considered problem will then be solved in two stages:

(*i*) compute the solution to the "relaxed" problem $u_{opt} = L_H^\sharp(z)$ (*i.e* without planning constraints),
(*ii*) find the best approximation of (*i*) in admissible domain $u_{opt_{\mathbb{A}}} = Id_{|\mathbb{A}}^\sharp(L_H^\sharp(z))$.

To apply this reasoning, it remains to show that canonical injection $Id_{|\mathbb{A}}$ is residuated. With that goal, we first characterize the subset $\mathcal{A}[\![\gamma]\!]$.

Subset $\mathcal{A}[\![\gamma]\!]$ is composed of formal power series which satify planning constraints (14) and (15). Since $\triangle^{max}$ is diagonal, it is invertible and we obtain the two following inequations:

$$\begin{cases} \triangle^{min} \, u(k-1) & \preceq \, u(k) \\ (\triangle^{max})^{-1} \, u(k+1) & \preceq \, u(k), \end{cases}$$

The γ-transforms of these inequations lead to

$$\begin{cases} \triangle^{min} \, \gamma u & \preceq \, u \\ (\triangle^{max})^{-1} \, \gamma^{-1} u & \preceq \, u, \end{cases}$$

and $(\triangle^{min} \, \gamma \oplus (\triangle^{max})^{-1} \, \gamma^{-1}) \, u \preceq u$. Since product is residuated, we equivalently have

$$u = \frac{u}{\triangle^{min} \, \gamma \oplus (\triangle^{max})^{-1} \, \gamma^{-1}} \wedge u.$$

The greatest solution is given by (see (Baccelli et al., 1992, Th. 4.7.3))

$$u = \frac{u}{(\gamma \triangle^{min} \oplus \gamma^{-1}(\triangle^{max})^{-1})^*}. \qquad (20)$$

With the aim of abbreviating notations, we below denote $p = \gamma \triangle^{min} \oplus \gamma^{-1}(\triangle^{max})^{-1}$. Equality (20) allows the following definition of subset $\mathcal{A}[\![\gamma]\!]$:

$$A[\![\gamma]\!] = \{x \in \overline{\mathbb{Z}}_{max}[\![\gamma]\!] \mid x = \frac{x}{p^*}\}.$$

In order to show the residuability of canonical injection $Id_{|\mathbb{A}}$, we study the quotient of $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$ by a particular equivalence relation (this idea was inspired by (Cohen, 1993, Th. 30)).

Proposition 2. Let Π be the mapping from $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$ into itself defined by $\Pi : x \mapsto \frac{x}{p^*}$. We define the following equivalence relation

$$\{x \, \mathcal{R} \, y\} \Longleftrightarrow \{\frac{x}{p^*} = \frac{y}{p^*}\}.$$

(1) Each equivalence class of $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]_{/\mathcal{R}}$ contains one and only one element belonging to $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ and this element is explicitly given by $\frac{x}{p^*}$ for any x in the class.
(2) Element $\frac{x}{p^*}$ is the least element in $[x]_{\mathcal{R}}$, and it is the greatest element among those of $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ which are less than x.

Proof :

(1) For a given element $x \in \overline{\mathbb{Z}}_{max}[\![\gamma]\!]$, the element $\frac{x}{p^*}$ clearly belongs to $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ and since $\frac{p^* \, \grave{} x}{p^*} = \frac{x}{p^*}$ (using (3)), it also belongs to $[x]_{/\mathcal{R}}$. Moreover, suppose that there is another element $x_1 \in [x]_{\mathcal{R}}$ which also belongs to $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$: $\Pi(x_1') = x_1 \Leftrightarrow$ $x_1 = \frac{x_1'}{p^*} = \frac{p^* \, \grave{} x_1'}{p^*} = \frac{x_1}{p^*}$, however $x_1 \mathcal{R} x$, so $\frac{x_1}{p^*} = \frac{x}{p^*}$ hence $x_1 = \frac{x}{p^*}$ which proves uniqueness of the element belonging to $[x]_{\mathcal{R}}$ and $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$.
(2) Since $p^* \succeq e$, for all $y \in [x]_{\mathcal{R}}$, we have $\frac{x}{p^*} = \frac{y}{p^*} \preceq y$ (see (2)). Consequently $\frac{x}{p^*}$ is the least element of $[x]_{\mathcal{R}}$. For all $z \in \Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ such that $z \preceq x$, since Π is isotone, we obtain $\frac{z}{p^*} \preceq \frac{x}{p^*}$, hence $\frac{x}{p^*}$ is the greatest element of $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ which is less than x.
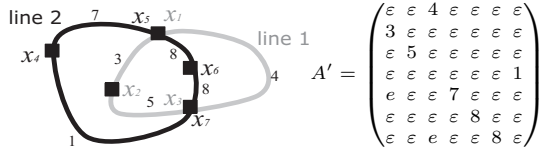
$\square$

Note that for any element $x \in \Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ there exists y such that $x = \frac{y}{p^*}$, and we have $\frac{x}{p^*} = \frac{p^* \, \grave{} y}{p^*} = \frac{y}{p^*} = x$ (using (3)), which shows that $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!]) \subseteq \mathcal{A}[\![\gamma]\!]$. The converse is obvious and subsets $\Pi(\overline{\mathbb{Z}}_{max}[\![\gamma]\!])$ and $\mathcal{A}[\![\gamma]\!]$ are then identical.

Last assertion of proposition 2 consequently proves that canonical injection $Id_{|\mathbb{A}}$ is residuated and with $x' \in \overline{\mathbb{Z}}_{max}[\![\gamma]\!]$, its residual is

$$Id_{|\mathbb{A}}^\sharp(x') = \bigoplus_{\{x \in \mathcal{A}[\![\gamma]\!] \mid x \preceq x'\}} x = \frac{x'}{p^*}.$$

5. EXAMPLE

We now present an example of bus network composed of two lines including two connection stops (see the graph below). We assume that x_5 and x_7 are respectively in connection with x_1 and x_3 (null walking times for these connections). We then obtain the following matrix A'.

$$A' = \begin{pmatrix} \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ e & \varepsilon & \varepsilon & 7 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 8 & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & 8 & \varepsilon \end{pmatrix}$$

The desired output is specified for only one stop x_7 ($S_o = \{x_7\}$), this leads to $C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}$.

Moreover, the following tabular represents the desired level-of-service of x_7.

k	1	2	3	4	5
$z(k)$	10	20	24	28	40

Planning constraints are identical for both lines: $\triangle^{min} = 7$ and $\triangle^{max} = 10$.

We solve the problem in two stages ($cf.$ item (i) and (ii) of §4.3). The first step consists in computing the solution of the relaxed problem u_{opt}. We show the computed dater u_{opt_6} of the generated timetables. Note that planning constraints are not satisfied.

k	1	2	3	4	5
$y(k)$	10	20	24	28	40
$u_{opt_6}(k)$	12	16	20	32	$\top$

Finally we find the best approximation of u_{opt} in $\mathcal{A}[\![\gamma]\!]$.

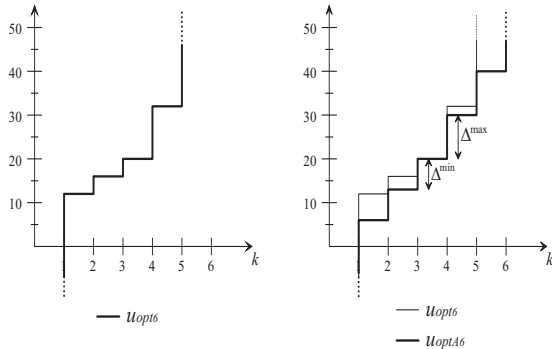| k	1	2	3	4	5
$y(k)$	7	14	21	28	38
$u_{opt_{\mathbb{A}6}}(k)$	6	13	20	30	40

Fig. 1. Computation of dater $u_{opt_{\mathbb{A}6}}(k)$ in two stages

Remark 1. For the computation, we have used software tools dedicated to calculus in dioids (SW2004, 2004).

6. CONCLUSION

We show that dioids algebra can be suitable for bus networks modelling. First, we have described the functioning of such systems. A $(max, +)$ modelling is established to simulate such particular systems which are conditioned by timetables. We suggest a solution using residuation theory to the problem of timetables generation. We are currently trying to extend this work by considering "more general" synchronizations at connection stops (*e.g.* not occurring at each departure).

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