Just in Time Control of constrained $(max, +)$-Linear Systems

L. Houssin, S. Lahaye, J.L. Boimond
LISA
62 avenue Notre Dame du Lac
49000 Angers-France
[houssin,lahaye,boimond]@istia.univ-angers.fr

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Abstract

This paper deals with just in time control of $(max, +)$-linear systems. The output tracking problem, considered in previous studies, is generalized by considering additional constraints in the control objective. The problem is formulated as an extremal fixed point computation. This control is applied to timetables computation for urban bus networks.

1 Introduction

The functioning of Discrete Event Dynamic Systems (DEDS) subject to synchronization and delay phenomena can be described by linear models in a particular algebraic structure called *dioid*. A linear system theory has been developed over dioids by analogy with conventional theory [1]. This theory has applications in various areas such as industrial processes, communication networks or transportation systems.
We are here interested in just in time control of DEDS which can be described by $(max, +)$-linear equations ($(max, +)$ algebra is an example of dioid). This subject has been studied for the first time in [8] and has notably been extended in [14], [10] and [12]. In these works, the control objective is limited to an output tracking problem, and some results from *Residuation theory* supply an optimal solution: an input trajectory corresponding to the latest occurring dates of input events is computed such that the output events do not occur after desired dates (the output trajectory to be tracked).
In this paper, the control objective is extended with supplementary constraints. More precisely, we can consider any constraint which can be formulated as an implicit inequality over the system state vector. For example, it is possible to specify for a given event: a desired number of occurrences in an interval of fixed dates, a minimum and/or a maximum time separation between two occurrences, or a critical time constraint for an activity in the system. The problem is formulated as an extremal fixed point computation. An iterative method is proposed to solve it.
As an illustration, these results are applied to a transportation system. Indeed, the proposed approach appears to be an adequate tool for the timetables synthesis of urban bus networks. Previous attempt at solving this problem using dioids can be found in [11] and [15]. Thanks to a different formalization, the method described in this paper supplies a more general solution.
The outline of the paper is as follows. In section 2, we introduce algebraic tools used throughout the paper. In section 3, the just-in-time control problem is stated, and an optimal solution is proposed.

In section 4, we explain how this control can be applied for timetables computation in transportation networks. A numerical example is also given.

2 Preliminaries

2.1 Dioid theory

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a semi-ring in which the sum, denoted $\oplus$, is idempotent [1, §4]. The sum (resp. product) admits a neutral element denoted ε (resp. e). A dioid is said to be complete if it is closed for infinite sums and if product distributes over infinite sums too. The sum of all its elements is generally denoted $\top$ (for top).

Example 1 *The set $\mathbb{Z}_{max} = (\mathbb{Z} \cup \{-\infty\})$ endowed with the max operator as sum and the classical sum as product is a dioid. If we add $\top = +\infty$ (with convention $\top \otimes \varepsilon = +\infty + (-\infty) = -\infty = \varepsilon$) to this set, the resulting dioid, referred to as $(max, +)$ algebra, is complete and is denoted $\overline{\mathbb{Z}}_{max}$. Set $\overline{\mathbb{Z}}_{min} = \ (\mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}, min, + \ )$ is also a complete dioid in which $\varepsilon = +\infty$ and $\top = -\infty$.*

Due to the idempotency of the sum, a dioid is endowed with a partial order relation, denoted $\succeq$, defined by the following equivalence: $a \succeq b \ \Leftrightarrow \ a = a \oplus b$. The notation $a \prec b$ defines $a \preceq b$ and $a \neq b$.
A complete dioid has a structure of complete lattice, *i.e.*, every subset $\mathcal{B}$ of a compete dioid $\mathcal{D}$ admits a *least upper bound*, namely $\bigoplus_{x \in \mathcal{B}} x$, and a *greatest lower bound* denoted $\bigwedge_{x \in \mathcal{B}} x = \bigoplus_{\{x \in \mathcal{D} | \forall y \in \mathcal{B}, x \preceq y\}} x$. Note that $\wedge$ is associative, commutative, idempotent and admits as neutral element $\top$ ($T \wedge a = a, \forall a$).

2.2 Representation of DEDS in dioids

Dioids enable to obtain linear models for DEDS which involve (only) synchronization and delay phenomena, but not choice phenomena. The behavior of such systems can be represented by some discrete functions called *dater* functions. More precisely, a discrete variable $x(\cdot)$ is associated to an event labeled x. This variable represents the occurring dates of event x. The numbering conventionally begins at 0: $x(0)$ corresponds to the date of the first occurrence of event x. These variables are extended towards negative values by:

$$x(k) = -\infty = \varepsilon \quad \text{for } k < 0,$$

such that they can be manipulated as mappings from $\mathbb{Z}$ to $\overline{\mathbb{Z}}_{max}$.
The considered DEDS, often referred to as *(max,+)-linear systems*, can be modeled by a linear state representation

$$\begin{array}{rcl} x(k) & = & Ax(k-1) \oplus Bu(k), \\ y(k) & = & Cx(k), \end{array} \tag{1}$$

where x, u and y are the state vector, the input vector and the output vector respectively and A, B, C are matrices of appropriate dimensions.
The initial state of a system is defined by a vector $v(k)$ added to the dynamic equation as follows:

$$x(k) = Ax(k-1) \oplus Bu(k) \oplus v(k).$$

More precisely, $v_i(k)$ for $0 \leq k < k_{d_i}$ represents the earliest occurring dates of initial events. Index k_{d_i} corresponds to the first occurrence of event x_i induced by inputs (this definition of initial conditions is detailed in [1, §5.4.4.1]). The notion of characteristic number introduced in [3] enables to calculate this index.

Definition 1 (characteristic number) *Let $[A]_i$ be the i-th row of matrix A, the characteristic number associated with the state variable x_i of a model described by (1), if it exists, is the least integer, noted k_{d_i}, such that $[A^{k_{d_i}}]_i B \neq \varepsilon$.*

To be manipulated as a dater, each variable v_i is extended such that: $v_i(k) = \varepsilon$ for $k < 0$ and $v_i(k) = v_i(k_{d_i} - 1)$ for $k \geq k_{d_i}$. We say that initial conditions are *canonical* if $\forall k \in \mathbb{Z},\ v(k) = \varepsilon$ and the dynamic behavior of the system then obeys to state equation (1) with $x_i(k) = \varepsilon$ for $k < k_{d_i}$. The characteristic number k_{d_i} corresponds to the event shift between inputs and state x_i. Let us define now the event shift between state x_i and output y_j. We define it, if it exists, as the least integer $k_{f_{ji}}$, such that $C_j[A^{k_{f_{ji}}}]^i \neq \varepsilon$ (notation $[A]^i$ indicates the i-th column of A).
An analogous transform to $\mathcal{Z}$-transform (used to represent discrete-time trajectories in conventional theory) can be introduced for $(max, +)$-linear systems: the γ, δ-transform. This transform enables to manipulate formal power series, with two commutative variables γ and δ, representing daters trajectories. The set of these formal series is a complete dioid denoted $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ (construction of this dioid is detailed in [8]). More formally, dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ corresponds to the dioid of formal power series in γ and δ (with boolean coefficients and integer exponents) quotiented by the following equivalence relation: $x \equiv y \Leftrightarrow \gamma^*(\delta^{-1})^* x = \gamma^*(\delta^{-1})^* y$, with $\gamma^* = \bigoplus_{i \in \mathbb{N}} \gamma^i$ (this quotient enables to take into account the monotony of dater functions). In the following, we denote x the corresponding element of $\{x(k)\}_{k \in \mathbb{Z}}$ in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ and we assume that each $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is represented by its *minimum representative* (see [1, §5]).
The support of a series x is defined as $Supp(x) = \{k \in \mathbb{Z} | x(k) \neq \varepsilon\}$.
Regarding the dynamic behavior of the system, we can interpret γ as the backward shift operator in event domain and δ as the backward shift operator in time domain.
State representation (1) becomes

$$\begin{array}{rcl} x & = & Ax \oplus Bu, \\ y & = & Cx, \end{array} \tag{2}$$

in which entries of matrices A, B and C are elements of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.
Considering the *earliest functioning rule* (an event occurs as soon as possible), we select the least solution of the first equation in (2) which is given by $x = A^* B u$ with $A^* = \bigoplus_{i \in \mathbb{N}} A^i$ [1, th 4.75]. Consequently we have $y = Hu$, in which $H = CA^*B$ is called the *transfer matrix*.
We define the *causal projection* of an element $z \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ as follows:

$$Pr_+(z) = Pr_+(\bigoplus_{i \in I} z(n_i, t_i) \gamma^{n_i} \delta^{t_i}) = \bigoplus_{i \in I} z_+(n_i, t_i) \gamma^{n_i} \delta^{t_i},$$

where $z_+(n_i, t_i) = \begin{cases} z(n_i, t_i), & \text{if } (n_i, t_i) \geq (0, 0); \\ \varepsilon, & \text{otherwise.} \end{cases}$
Mapping Pr_+ simply amounts to canceling monomials which have negative exponents.

2.3 Residuation theory

Let us consider mappings defined over complete dioids. Such a mapping $f : \mathcal{D} \to \mathcal{C}$ is said to be *isotone* if $a, b \in \mathcal{D},\ a \preceq b \Rightarrow f(a) \preceq f(b)$. Moreover f is *lower-semicontinous* (l.s.c.) if $\forall \mathcal{B} \subseteq \mathcal{D}$, $f(\bigoplus_{x \in \mathcal{B}} x) = \bigoplus_{x \in \mathcal{B}} f(x)$. Residuation theory [2] defines "pseudo-inverses" for some isotone mappings defined over ordered sets such as complete dioids [1]. More precisely, if the greatest element of set $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists for all $b \in \mathcal{C}$, then it is denoted $f^\sharp(b)$ and $f^\sharp$ is called *residual* of f.

Theorem 1 *[1, th. 4.50] Let $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping, the following statements are equivalent:*

(i) f *is residuated,*

(ii) f *is l.s.c. and* $f(\varepsilon_{\mathcal{D}}) = \varepsilon_{\mathcal{C}}$*,*

(iii) there exists a unique isotone mapping $f^\sharp$ *such that* $f \circ f^\sharp \preceq Id_{\mathcal{C}}$ *and* $f^\sharp \circ f \succeq Id_{\mathcal{D}}$*.*

Theorem 2 *[1, th. 4.56] Let* $f : \mathcal{D} \to \mathcal{C}$ *and* $g : \mathcal{C} \to \mathcal{B}$*. If* f *and* g *are residuated then* $g \circ f$ *is residuated and* $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$*.*

Example 2 *Mapping* $L_a : \mathcal{D} \to \mathcal{C}, x \mapsto a \otimes x$ *is residuated. Its residual is denoted* $L_a^\sharp = a \backslash\!\!\circ x$ *([1, §4.4.4]).*

Example 3 *The valuation* $val(x)$ *of a series* $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ *is defined by [1, definition 5.19]:*

$$\begin{array}{rcccl} val & : & \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] & \longrightarrow & \overline{\mathbb{Z}}_{min} \\ & & x & \longmapsto & val(x) = Min(Supp(x)). \end{array}$$

As an example, we have $val(\gamma^3\delta^1 \oplus \gamma^5\delta^2) = 3$*.*

Proposition 1 *Mapping val is residuated and its residual is* $val^\sharp(x) = \gamma^x\delta^*$*.*

Proof : *Mapping val is l.s.c. and* $val(\varepsilon) = \varepsilon$ *(see [1, lemma 4.93]). According to item (ii) of theorem 1, it is then residuated. We check that the proposed residual satisfies item (iii) of theorem 1:*

$$\begin{array}{rcccl} val \circ val^\sharp(x) & = & val(\gamma^x\delta^*) & = & x \\ val^\sharp \circ val(x) & = & \gamma^{val(x)}\delta^* & \succeq & x. \end{array}$$

□

Example 4 *Let* $Pr_a : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] \to \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ *defined by:*

$$\begin{array}{rcccl} Pr_a & : & x & \longmapsto & Pr_a(x) = Pr_a(\bigoplus_{(n,t)\in\mathbb{Z}^2} x(n,t)\gamma^n\delta^t) \\ & & & & = \bigoplus_{(n,t)\in\mathbb{Z}^2} x_a(n,t)\gamma^n\delta^t, \end{array}$$

in which $x_a(n,t) = \begin{cases} x(n,t) & \text{if } t \geq a, \\ \varepsilon & \text{otherwise.} \end{cases}$

Given a series $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$*, mapping* $Pr_a(x)$ *consists in preserving the monomials of* x *whose exponents in* δ *are greater than or equal to* a*.*

As an example, we have $Pr_3(\gamma^1\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^5) = \gamma^3\delta^3 \oplus \gamma^4\delta^5$*.*

Proposition 2 *Mapping* Pr_a *is residuated and its residual is* $Pr_a^\sharp(x) = x \oplus (\gamma^{-1})^*\delta^{a-1}$*.*

Proof : *Obviously* Pr_a *is l.s.c. and* $Pr_a(\varepsilon) = \varepsilon$*, then* Pr_a *is residuated. We check that the proposed residual mapping satisfies item (iii) of theorem 1:*

$$\begin{array}{rcccl} Pr_a \circ Pr_a^\sharp(x) & = & Pr_a(x \oplus (\gamma^{-1})^*\delta^{a-1}) & & \\ & = & Pr_a(x) & \preceq & x \\ Pr_a^\sharp \circ Pr_a(x) & = & Pr_a^\sharp(Pr_a(x)) & & \\ & = & Pr_a(x) \oplus (\gamma^{-1})^*\delta^{a-1} & \succeq & x \end{array}$$

□

2.4 Greatest fixed point of mappings defined over dioids

We denote $\mathcal{F}_f = \{x|f(x) = x\}$ (resp. $\mathcal{P}_f = \{x|f(x) \succeq x\}$) the set of fixed points (resp. the set of post-fixed points) of an isotone mapping f defined over a complete dioid $\mathcal{D}$. We recall that $\mathcal{P}_f$ has a complete lattice structure [1, th 4.72]. Tarski's theorem [17] states that an isotone mapping defined over a complete lattice admits at least one fixed point. Moreover, it can be shown that the greatest fixed point coincides with the greatest element of $\mathcal{P}_f$ [9, th 4.11]. The last theorem applies over complete dioids due to their ordered structure:

$$Sup\ \mathcal{P}_f = Sup\ \mathcal{F}_f \quad \text{and} \quad Sup\ \mathcal{F}_f \in \mathcal{F}_f. \tag{3}$$

In the following proposition, we specify to dioids a well known method to compute the greatest fixed point of an isotone mapping.

Proposition 3 *If the following iterative computation*

$$\begin{array}{rcl} y_0 & = & \top \\ y_{k+1} & = & f(y_k) \end{array} \tag{4}$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f.

3 Just-in-time control

In this section, a just in time control is proposed for $(max, +)$-linear systems. We take inspiration from the formalism considered in [4] for the optimal control of conventional linear systems. The principle of this control can be summarized in three items:

- ▷ The process satisfies some initial conditions and some given final conditions.
- ▷ State variables are subject to some constraints.
- ▷ The control is "optimal" in the sense that it optimizes a chosen criterion.

3.1 Initial and final conditions

As discussed in §2.2, initial state of a $(max, +)$-linear system is given by the earliest occurring dates of initial events. We consider here canonical initial conditions. For each state variable x_i, we defined its characteristic number (definition 1), *i.e.* the index k_{d_i} of the first occurrence of x_i generated by the inputs of the system.
In our framework, the given final state corresponds to the last occurrences of outputs that should be controlled. From that goal, we can deduce the last occurrences of the state variables which have to be controlled. We denote $k_{f_{y_j}}$ (resp. k_{f_i}) the last occurrence of event y_j (resp. x_i) that we aim at controlling. The computation of this index is function of the shift event $k_{f_{ji}}$ between state variable x_i and output y_j of the system (*cf.* §2.2). The last occurrence of x_i generating the last occurrence $k_{f_{y_j}}$ of output y_j is given by $k'_{f_{ji}} = k_{f_{y_j}} - k_{f_{ji}}$. We then deduce that the last occurrence of x_i to control is the one which generates the last desired occurrence for all outputs, so $k_{f_i} = max(k'_{f_{1i}}, k'_{f_{2i}}, \ldots, k'_{f_{pi}})$ for a $p-$outputs system.

3.2 Constraints

In our control problem, the constraints are used to define the control objective.
We consider constraints which can be formulated as an implicit inequality over state vector x. These constraints are specified over an interval of occurrences for each state variable. More precisely, for event labeled x_i, constraints are applied only for indices of occurrences included in interval $[k_{d_i}, k_{f_i}]$. Indeed, x_i should not be constrained for indices less than k_{d_i} since these occurrences are not induced by inputs of the system (*cf.* §2.2). Furthermore, k_{f_i} corresponds to the index of the last occurrence of x_i that we aim at controlling. In order to express these constraints in dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, in which we manipulate the whole trajectory of a dater $\{x_i(k)\}_{k\in\mathbb{Z}}$ as a formal power series x_i, we use two vectors ω and ν:

$$\begin{array}{ccc} x & \preceq & (g_1(x)\wedge\omega)\oplus\nu, \\ x & \preceq & (g_2(x)\wedge\omega)\oplus\nu, \\ \vdots & \vdots & \vdots \\ x & \preceq & (g_q(x)\wedge\omega)\oplus\nu, \end{array} \tag{5}$$

in which ω (resp. ν) is a n-vector (n is the dimension of the state vector) with entries $\omega_i = \gamma^{k_{d_i}}\delta^*$ (resp. $\nu_i = \gamma^{k_{f_i}+1}\delta^*$), and each g_l, $l = 1,2,\ldots,q$, is a mapping from $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ to $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ modeling a constraint. Vectors ω and ν enable to relax constraints for the occurrences of events x_i, $i = 1,2,\ldots,n$, whose indices are not included in $[k_{d_i}, k_{f_i}]$.
In its basic form, the control objective is limited to an output tracking problem as in previous studies [8], [14], [10]. That is, for a given target z (entry z_i is a trajectory specifying the latest desired dates for occurrences of output event y_i), we aim at satisfying the constraint denoted g_1 and defined by

$$g_1(x) = C \backslash\!\!\circ\, z. \tag{6}$$

This constraint allows to computing a control such that the output events do not occur later than the target z. We assume that for each output y_i, $i \in \{1,\ldots,p\}$, the corresponding target z_i is such that $z_i(k) \prec \top$ for $k \leq k_{f_{y_i}}$ (desired occurrence dates can not be infinite).

3.3 Criterion

A relevant goal for the control of DEDS is to delay as much as possible the input events (*i.e.* to compute the greatest control vector u) while ensuring performances imposed by a specification (the specification corresponds here to initial and final conditions as well as constraints). It corresponds to the just in time control problem which commonly aims at supplying the "right quantity" (the demand) at the "desired time" (date of the demand). Therefore, the considered criterion J is $J = u$. The optimal control is the one which maximizes J.

3.4 Synthesis

In this section, we show that the synthesis of the optimal control can be formulated as the computation of the greatest fixed point of a mapping.
To be realizable, the sought control $u \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^m$ (control vector with m entries) must be causal[1], *i.e.*, $u = Pr_+(u)$ (see §2.2). One can easily check that, $\forall u$, $Pr_+(u) \preceq u$, this then leads to the following constraint:

[1]causal control u has no monomial with negative exponent. It means that no anticipation in time-domain or event-domain is accepted.

$$u \preceq Pr_+(u). \tag{7}$$

Furthermore, we want that state x resulting from control u satisfies (5). Considering the earliest functioning of the system, that is $x = A^*Bu$, and since multiplication is residuated (*cf.* example 2), control u must satisfy

$$\forall l \in [1,q], \quad u \preceq g'_l(u) \tag{8}$$

in which $g'_l(u) = A^*B \circ\backslash ((g_l(A^*Bu) \wedge \omega) \oplus \nu)$. From (7) and (8), the optimal control u is then the greatest solution of inequalities :

$$\begin{cases} u & \preceq & g'_1(u), \\ \vdots & \vdots & \vdots \\ u & \preceq & g'_q(u), \\ u & \preceq & Pr_+(u) \end{cases}$$

which is equivalent to find the greatest u satisfying

$$u \preceq g'_1(u) \wedge \ldots \wedge g'_q(u) \wedge Pr_+(u) = f(u). \tag{9}$$

Proposition 4 *The following iterative computation*

$$\begin{array}{rcl} u_0 & = & \top \\ u_{k+1} & = & f(u_k), \end{array}$$

converges in a finite number k_e of iterations and u_{k_e} is the optimal control (greatest solution of (9)).

Proof : Thanks to proposition 3, we know that if the proposed iterative computation converges in a finite number k_e of iterations then u_{k_e} is the greatest solution of (9). So let us prove that this computation converges in a finite number of iterations.
Considering only first constraint g_1 given by (6), we get a first bound:

$$f(u_k) \preceq A^*B \circ\backslash ((C \circ\backslash z \wedge \omega) \oplus \nu) \qquad \text{for all } k \geq 0. \tag{10}$$

Moreover, we observe that $u_0 \succeq A^*B \circ\backslash \nu$, and, if $u_k \succeq A^*B \circ\backslash \nu$ then
$u_{k+1} = f(u_k) = g'_1(u_k) \wedge \ldots \wedge g'_q(u_k) \wedge Pr_+(u_k) \succeq A^*B \circ\backslash \nu$ (in regard to the definition of $g'_l(u_k)$, it is obvious that $g'_l(u_k) \succeq A^*B \circ\backslash \nu$ for $l \in \{1,\ldots,q\}$ and $Pr_+(u_k) \succeq Pr_+(A^*B \circ\backslash \nu) = A^*B \circ\backslash \nu$). We deduce that

$$f(u_k) \succeq A^*B \circ\backslash \nu \qquad \text{for all } k \geq 0. \tag{11}$$

The entries of bounding vectors $A^*B \circ\backslash ((C \circ\backslash z \wedge \omega) \oplus \nu)$ and $A^*B \circ\backslash \nu$ in (10) and (11) are particular series.
Entries of $A^*B \circ\backslash ((C \circ\backslash z \wedge \omega) \wedge \nu)$ are polynomials $\gamma^{n_d}\delta^{t_d} \oplus \ldots \oplus \gamma^{n_f}\delta^{t_f} \oplus \gamma^{n_f+1}\delta^*$ with finite exponents $t_d, \ldots, t_f$ (target dates given by z are assumed to be finite, see §3.2).
Entries of $A^*B \circ\backslash \nu$ are monomials $\gamma^{n_f+1}\delta^*$.
We then have for each entry $(f(u_k))_i$

$$\gamma^{n_f+1}\delta^* \preceq (f(u_k))_i \preceq \gamma^{n_d}\delta^{t_d} \oplus \ldots \oplus \gamma^{n_f}\delta^{t_f} \oplus \gamma^{n_f+1}\delta^*. \tag{12}$$

From (12), we can deduce that series $(f(u_k))_i$ is such that:

- its valuation is equal to n_d,
- the exponents in δ for each exponent in γ between n_d and n_f is finite (since $t_d, \ldots, t_f$ are finite),
- for exponents in γ greater than n_f, exponents in δ are equal to $+\infty$.

Moreover, from (9), only causal monomials of u_k are preserved in $f(u_k)$.
From these observations, we conclude that there exists a finite number of causal series satisfying (12). In other words, each $(f(u_k))_i$ belongs to a finite set of possible series.
Finally, we notice that f is decreasing: $u_1 = f(u_0) \preceq u_0 = \top$ and if $u_n = f(u_{n-1}) \preceq u_{n-1}$ then we have $u_{n+1} = f(u_n) \preceq f(u_{n-1}) = u_n$ (f is isotone).
Previous arguments show that the proposed iterative computation is a decreasing sequence on a finite set, which proves its convergence in a finite number of iterations.

$\square$

4 Application to urban transportation networks

In this section, we are interested in transportation systems and more particularly in urban bus networks. At first a $(max, +)$-linear model of these systems is proposed.
The behavior of such systems is controlled by a timetable. The timetable defines scheduled departure times of buses for each stop. Timetables settings is a part of an optimization, generally referred to *bus planning* [7]. First phases of this activity consist in

- the construction of a global line network: stops are localized and allocated to lines,
- the setting of frequencies for each line: *minimum and maximum headways* are used to define time intervals between two successive departures.

Timetables are subsequently synthesized. Considering the proposed model, the timetable synthesis problem is decomposed as constraints on the state vector. Then, we solve this problem by applying the control method introduced in section 3.

4.1 Modeling of a bus network

A bus network can be modeled as a state representation in $\overline{\mathbb{Z}}_{max}$ by:

$$\begin{array}{rcl} x(k) & = & Ax(k-1) \oplus Bu(k), \\ y(k) & = & Cx(k), \end{array} \tag{13}$$

in which $x(k)$ is a vector such as $x_i(k)$ denotes the departure time of the $(k+1)-$th bus at stop i. Matrix A is defined such as $A_{ij} = a_{ij}$ where a_{ij} corresponds to the traveling time from stop j to stop i, $A_{ij} = \varepsilon$ otherwise. Traveling time a_{ij} may correspond either to the time spent by a bus to run from stop j preceding stop i on the same line, or to the walking time between stops j and i belonging to different lines (a connection between buses departing from j and arriving at i is then specified). Vector $y(k)$ corresponds to the vector of daters associated with stops considered as "strategic" (at which a specified level of service must be satisfied). The timetable is represented by input vector $u(k)$, and variable $u_i(k)$ denotes the scheduled departure time of the $(k+1)-$th bus at stop i. In practice, synchronizations of buses with the timetable occur only at particular stops of the network such as the beginning or the end of a line. Concerning the other stops, the timetable has only an indicative value. So, entries of matrix B are such as $B_{ii} = e$ if timetable must be respected at stop i, $B_{ij} = \varepsilon$ otherwise.

4.2 Timetable synthesis problem

We present here the timetable synthesis problem by decomposing it into several constraints on the state vector of the proposed model (13).

- In a first place, we define an expected level of service at strategic stops of the network through *expiries*: *e.g.* the departure of $k-$th bus at stop i must occur before a date t. If we denote $z(k)$ the vector defining these target dates at strategic stops, then this leads to the following constraint:

$$C \otimes x(k) \preceq z(k) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i} \tag{14}$$

  in which k_{d_i} and k_{f_i} are the bounds of the interval of indices that we want to control for event x_i (*cf* §3.1).

- The respect of a *maximum headway* between two successive departures enables to ensure a minimum departure frequency for each line. Denoting $\triangle_i^{max}$ this maximum separation time, for stop i, we get the following constraint:

$$\begin{cases} x_i(k) = x_i(k_{d_i}) & \text{for } k = k_{d_i}, \\ x_i(k) \preceq \triangle_i^{max} \otimes x_i(k-1) & \text{for } k_{d_i} < k \leq k_{f_i}, \end{cases}$$

$$\Longleftrightarrow$$

$$x_i(k) \preceq \triangle_i^{max} \otimes x_i(k-1) \oplus x_i(k_{d_i}) \quad \text{for } k_{d_i} \leq k \leq k_{f_i}, \tag{15}$$

  where k_{d_i} and k_{f_i} are the bounds of the interval of indices that we want to control for event x_i (*cf.* §3.1). Since we consider canonical initial conditions, we have $x_i(k_{d_i} - 1) = \varepsilon$ (see §2.2).

- Furthermore, *minimum headways* enable to avoid the natural tendency of transit vehicles to bunch up as soon as a bus is in late. Thus, if a bus falls slightly behind schedule for any reasons, it will have more than the average number of passengers to pick up at the next station, which causes further delays. Thus, it keeps failing further behind schedule. Conversely, the bus behind it encounters fewer passengers than usual, allowing it to catch up with the preceding bus. Denoting $\triangle_i^{min}$ the minimum separation time between two successive departures at stop i, this constraint can be written

$$x_i(k) \succeq \triangle_i^{min} \otimes x_i(k-1) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i}. \tag{16}$$

  Generally, a specific minimum headway is defined for each line.

- In the daytime, some rush hours can appear. Origins of these peaks of charge can be different: intermodal connections (for example arrival of trains for stops closed from a station) or urban activities (school at home-time, factories closing time). Their dates of occurrences are known *a priori* and in order to take those into account, one or several departure(s) must be planned. For stop i, we model this constraint by

$$\exists k \in [k_{d_i}, k_{f_i}] \text{ s.t. } x_i(k) \succeq t_j \text{ and } x_i(k+s) \prec t_j + r, \tag{17}$$

  in which s is the expected number of departure(s) at stop i during interval $[t_j, t_j + r]$ in order to absorb the peak.

- Waiting times can be limited, at some stops of the network, to achieve a quality of service or because of physical constraint (case of a stop located on a road shared with cars or other buses). Let us consider a stop x_i preceded by a stop x_j. We note d_i^{max} the maximum waiting time expected at stop x_i and κ_{ij} the number of buses initially between x_j and x_i. This constraint can be formulated as:

$$x_i(k) \preceq d_i^{max} \otimes A_{ij} \otimes x_j(k - \kappa_{ij}) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i}. \tag{18}$$

  It also can be used to bound the traveling time of buses on a path from x_l to x_i:

$$x_i(k) \preceq d_{il}^{max} \otimes x_l(k - \kappa_{il}) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i},$$

  where d_{il}^{max} the maximum traveling time on the path from x_l to x_i and κ_{il} is the number of buses initially between x_l and x_i.

  Notice that if buses running from two stops, x_j and x_l are in connection at stop x_i where a maximum waiting time d_i^{max} is specified, then we must consider two constraints:

$$x_i(k) \preceq d_i^{max} \otimes A_{ij} \otimes x_j(k - \kappa_{ij}) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i},$$

  and

$$x_i(k) \preceq d_i^{max} \otimes A_{il} \otimes x_l(k - \kappa_{il}) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i}.$$

  For a DEDS modeled by a *Timed Event Graph* (see [1, §2.5]), this constraint is equivalent to consider that sojourn times of tokens must not exceed a maximal value in particular places or paths. In [13], this kind of constraint has been considered for the synthesis of a precompensation in a model matching objective.

4.3 Resolution

In order to apply results of section 3 to the computation of the timetable, the considered constraints have to be expressed as formal power series in dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ such as (5).

- Constraint (14) is traduced in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ by: $x \preceq (C \diamond\!\!\!\setminus z \wedge \omega) \oplus \nu$.

- Constraint (15) can be formulated by inequality: $x \preceq ((\gamma \triangle^{max} x \ \oplus \ x_d) \wedge \omega) \oplus \nu$, in which $\triangle^{max} = \begin{pmatrix} \delta^{\triangle_1^{max}} & \varepsilon & \varepsilon \\ \varepsilon & \delta^{\triangle_2^{max}} & \varepsilon \\ \varepsilon & \varepsilon & \ddots \end{pmatrix}$ and x_d is defined by $x_{d_i} = \gamma^{k_{d_i}} \delta^{x_i(k_{d_i})}$.

- In the same way, we model constraint (16) by $x \preceq ((\gamma \triangle^{min}) \diamond\!\!\!\setminus x \wedge \omega) \oplus \nu$, in which matrix $\triangle^{min}$ has an analogous structure to $\triangle^{max}$.

- To formulate constraint (17) in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ we use mappings Pr_a and val introduced in §2.3. In order to point out at least one occurrence of event x_i between dates t_j and $t_j + r$, we specify that the index of the first occurrence later than $t_j + r$ (*i.e.* for $t \geq t_j + r + 1$) must be strictly greater than the index of the first occurrence later than $t_j - 1$ (*i.e.* for $t \geq t_j$).

$$\begin{aligned} & val(Pr_{t_j+r+1}(x_i)) & \prec & \quad val(Pr_{t_j}(x_i)) \\ \iff & val(Pr_{t_j+r+1}(x_i)) & \preceq & \quad 1 \otimes val(Pr_{t_j}(x_i)). \end{aligned}$$

In order to require at least s occurrences of events x_i, we use inequality:

$$val(Pr_{t_j+r+1}(x_i)) \preceq s \otimes val(Pr_{t_j}(x_i)).$$

We recall that mappings val and Pr_a are both residuated (see propositions 1 and 2). By using theorem 2, previous inequality can be rewritten:

$$\begin{array}{rcl} x_i & \preceq & Pr^\sharp_{t_j+r+1}(val^\sharp(s \otimes val(Pr_{t_j}(x_i)))) \\ \iff x_i & \preceq & \gamma^{s \otimes val(Pr_{t_j}(x_i))}\delta^* \oplus (\gamma^{-1})^* \delta^{t_j+r}. \end{array}$$

- Equation (18) leads to $x \preceq (\phi^{max} x \wedge \omega) \oplus \nu$, in which ϕ^{max} is defined as
$$\begin{cases} \phi^{max}_{ij} = \gamma^{\kappa_{ij}} \delta^{(d_i^{max} \otimes A_{ij})} & \\ \phi^{max}_{ll} = e & \text{for all } l \neq i \\ \phi^{max}_{mn} = \varepsilon & \text{otherwise.} \end{cases}$$

These different constraints are modeled with respect to (5) and are equivalent to the following inequality:

$$u \preceq g'_1(u) \wedge g'_2(u) \wedge \ldots \wedge g'_5(u) = f(u).$$

Finally, the problem comes down to finding the greatest u such that $u \preceq f(u)$. The iterative computation presented in proposition 4 converges in a finite number k_e of iterations and u_{k_e} is the optimal timetable for concerned stops. For the other stops (where timetables have only an indicative value), we deduce the scheduled departures times from a simulation of the system based on model (13) and u_{k_e}.

4.4 Discussion and remarks

Minimum and maximum headways (determining frequencies over lines) are generally set per period (typically two hours) since passengers demand can vary significantly during a day [5]. The proposed iterative computation should then be used to set timetables for each period.
With the proposed model, some "hard connection" can be defined between lines: if a connection time a_{ij} is specified between stops j and i belonging to different lines (see §4.1), then each bus departing from stop j is connected with a bus arriving at stop i. This specification may appear to be "too rigid" since, in practice, connection between lines may be partial (only some buses departing from a stop connected with buses arriving at another stop). With the existing approach, one may consider such a case if partial timetables are synthesized *a priori*. For example, the timetable of the main line of the network may be computed satisfying constraints such as latest departure dates, headways, maximal waiting times and peaks of charge, but neglecting connections with other lines. The timetables for other lines can be generated with partial connections to the main line using constraint (17). In future works, some attention should be given to improve partial connections in the proposed approach.
Specified constraints (14), (15), (17) and (18) can be used to improve above all passenger-oriented quality criteria since they contribute to minimize waiting times and travel times. On the other hand, minimum headways (constraint (16)) rather serve the interest of the company since they allow departures to be not too close to each other, and as a by-product, to limit costs. Similarly, the chosen criterion, *viz* just-in-time criterion, is a company oriented quality criterion. In fact, by maximally delaying bus departures (notably at terminuses of lines), we minimize the number of running buses while satisfying quality objectives given by constraints. The company may then limit costs with a better allocation of buses and drivers.

Among works to our knowledge, our approach is the more similar to the study of [6]. They propose procedures for constructing timetables with maximum synchronization, that is, maximization of the number of simultaneous arrivals of vehicles to connection stops. At first, they set out a mixed integer programming procedure which proves to be relevant only for small networks (5 bus routes, 5 nodes). A heuristic algorithm is also proposed to solve larger problems in a reasonable time. They notably present a real-life example consisting in a part of a bus network in Israel: 3 transfer nodes (connection stops) at which passengers can transfer from 14 routes (7 bus lines). Several differences exist with our approach, which make results difficult to compare. At first, timetables are set in [6] disregarding vehicles assignment to lines, even though buses are assumed to have been previously allocated to lines in our model. Only "partial connection" are considered in [6], while our approach rather assumes hard connections (see discussion above). Finally, as far as we know, constraints such as latest departure dates (14), peaks of charge (17) and maximum waiting times (18) are not considered in [6].

4.5 Example

We consider two lines of the bus network of Angers (France) represented on figure 1. Only stops, at which buses are synchronized with the timetable, have been drawn on the figure.
It is assumed that stops x_4 and x_{10} are respectively in connection with x_{16} and x_{22} (we consider a null connection time: $A_{41\ 6} = \gamma^0\delta^0$ and $A_{10\ 22} = \gamma^0\delta^0$).
We apply the proposed method in order to compute the timetable for the first two hours period of a day, that is from 6:30 to 8:30.

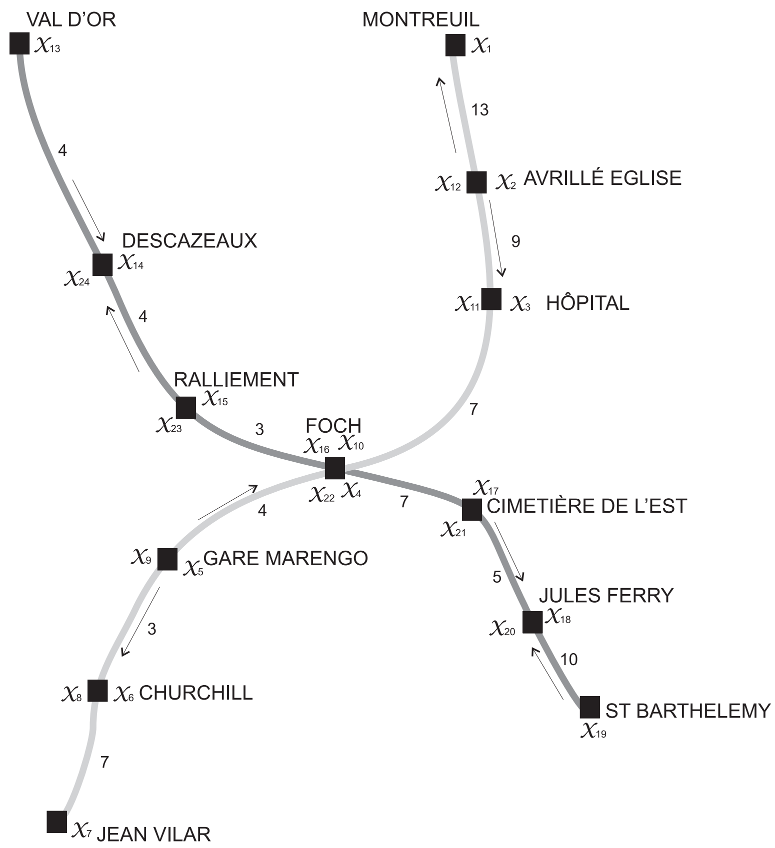

Figure 1: Two lines of the urban bus network of Angers

The dynamic behavior of the system is described by (13) with $B = Id$ (timetable is respected at each stop selected on figure 1) and we define C in such a way that strategic stops are x_1 and x_{13}.
We assume that initially (*i.e.* at 6:30), 6 buses are parked at station x_1, 6 buses at station x_7, 6 buses at station x_{13} and 6 buses at station x_{19}.
This leads to vector ω such that $\forall i,\ \omega_i = \gamma^0\delta^*$.
Considering that we want 14 (resp. 17) departures from stop x_1 (resp. x_{13}) during the considered period, the following table furnishes the targets for stops $y_1 = x_1$ and $y_2 = x_{13}$ (one departure from x_1 expected between 6:30 and 6:50, four departures from x_1 expected between 6:50 and 7:10, and so on).

	6:30	6:50	7:10	7:30	7:50	8:10	8:30
x_1 (Montreuil)	1	4	3	3	1	2	
x_{13} (Val d'or)	1	2	3	3	4	4	

Table 1: Expected departures at strategic stops.

We then obtain the following vector ν: $\nu_i = \gamma^{14}\delta^\top$ for $1 \leq i \leq 12$ (line Montreuil - Jean Vilar) and $\nu_i = \gamma^{17}\delta^\top$ for $13 \leq i \leq 24$ (line Val d'or - St Barthélemy).
The following constraints are specified for this system:

- ▷ The constraint (14) is directly traduced from table 1. We obtain:

  $$z_1 = \gamma^0\delta^{410} \oplus \gamma^1\delta^{430} \oplus \gamma^5\delta^{450} \oplus \gamma^8\delta^{470} \oplus \gamma^{11}\delta^{490} \oplus \gamma^{12}\delta^{510}$$

  and

  $$z_2 = \gamma^0\delta^{410} \oplus \gamma^1\delta^{430} \oplus \gamma^3\delta^{450} \oplus \gamma^6\delta^{470} \oplus \gamma^9\delta^{490} \oplus \gamma^{13}\delta^{510}$$

  in which δ exponents correspond to dates given in minutes elapsed from 00:00 (*e.g.* δ^{410} denotes 6:50)

- ▷ Minimum headways are given by $\triangle_{ii}^{min} = \delta^5$ for $1 \leq i \leq 12$ and $\triangle_{ii}^{min} = \delta^4$ for $13 \leq i \leq 24$

- ▷ Maximum time separations between buses are $\triangle_{ii}^{max} = \delta^{15}$ for $1 \leq i \leq 12$ and $\triangle_{ii}^{max} = \delta^{15}$ for $13 \leq i \leq 24$,

- ▷ At known period, some peaks of charge appear at stops x_5 and x_9, departures must then occur to absorb it. For x_5, we want:

  - one departure between 7:20 and 7:25,
  - one departure between 7:30 and 7:35,
  - one departure between 7:40 and 7:42.

  For x_9, we want:

  - one departure between 7:15 and 7:20,
  - one departure between 7:25 and 7:30,
  - one departure between 7:55 and 7:60.

▷ Buses are not allowed to stop more than 2 minutes at station "Foch" which corresponds to stops x_4, x_{10}, x_{16} and x_{22}. Considering the travel times, we have $\phi_{4\ 3} = \gamma^0\delta^9$, $\phi_{10\ 9} = \gamma^0\delta^6$, $\phi_{16\ 15} = \gamma^0\delta^5$ and $\phi_{22\ 21} = \gamma^0\delta^9$.

The iterative computation defined in proposition 4 has been implemented with the C++ library `libminmaxgd` [16] handling formal power series in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. For example, it converges in 16 iterations (computation time equal to 53 sec. on a Pentium 4 2.4 GHz)

Montreuil ⟶ Jean Vilar														
Montreuil	6:47	6:55	7:00	7:05	7:10	7:20	7:25	7:30	7:40	7:45	7:50	8:05	8:20	8:30
Avrillé église	7:00	7:15	7:22	7:37	7:52	8:07	8:22	8:37	8:52	9:07	9:22	9:37	9:52	10:07
Hôpital	7:09	7:24	7:31	7:46	8:01	8:16	8:31	8:46	9:01	9:16	9:31	9:46	10:01	10:16
Foch	7:16	7:31	7:38	7:53	8:08	8:23	8:38	8:53	9:08	9:23	9:38	9:53	10:08	10:23
Gare Marengo	7:20	7:35	7:42	7:57	8:12	8:27	8:42	8:57	9:12	9:27	9:42	9:57	10:12	10:27
Churchill	7:23	7:38	7:53	8:08	8:23	8:38	8:53	9:08	9:23	9:38	9:53	10:08	10:23	10:38

Jean Vilar ⟶ Montreuil														
Jean Vilar	6:42	6:47	6:57	7:02	7:07	7:20	7:30	7:45	8:00	8:15	8:30	8:45	9:00	9:15
Churchill	6:49	6:54	7:04	7:09	7:14	7:27	7:37	7:52	8:07	8:22	8:37	8:52	9:07	9:22
Gare Marengo	6:52	6:57	7:07	7:12	7:17	7:30	7:40	7:55	8:10	8:25	8:40	8:55	9:10	9:25
Foch	6:56	7:01	7:11	7:16	7:21	7:36	7:46	8:01	8:16	8:31	8:46	9:01	9:16	9:31
Hôpital	7:03	7:08	7:18	7:23	7:28	7:43	7:58	8:08	8:23	8:38	8:53	9:08	9:23	9:38
Avrillé église	7:12	7:17	7:27	7:32	7:37	7:52	8:07	8:17	8:32	8:47	9:02	9:17	9:32	9:47

Val d'or ⟶ St Barthélemy																	
Val d'or	6:50	7:05	7:10	7:20	7:24	7:30	7:42	7:46	7:50	7:58	8:02	8:06	8:10	8:18	8:22	8:26	8:30
Descazeaux	6:55	7:10	7:20	7:24	7:28	7:43	7:58	8:13	8:28	8:43	8:58	9:13	9:28	9:43	9:58	10:13	10:28
Ralliement	6:59	7:14	7:24	7:28	7:32	7:47	8:02	8:17	8:32	8:47	9:02	9:17	9:32	9:47	10:02	10:17	10:32
Foch	7:02	7:17	7:27	7:31	7:35	7:50	8:05	8:20	8:35	8:50	9:05	9:20	9:35	9:50	10:05	10:20	10:35
Cimetière	7:09	7:24	7:34	7:38	7:42	7:57	8:12	8:27	8:42	8:57	9:12	9:27	9:42	9:57	10:12	10:27	10:42
Jules Ferry	7:14	7:29	7:39	7:43	7:47	8:02	8:17	8:32	8:47	9:02	9:17	9:32	9:47	10:02	10:17	10:32	10:47

St Barthélemy ⟶ Val d'or																	
St Barthlemy	6:34	6:39	6:49	6:54	6:59	7:14	7:24	7:39	7:49	7:53	7:57	8:12	8:27	8:42	8:57	9:12	9:27
Jules Ferry	6:44	6:49	6:59	7:04	7:09	7:24	7:34	7:49	7:59	8:03	8:07	8:22	8:37	8:52	9:07	9:22	9:37
Cimetière	6:49	6:54	7:04	7:09	7:14	7:29	7:39	7:54	8:04	8:08	8:12	8:27	8:42	8:57	9:12	9:27	9:42
Foch	6:56	7:01	7:11	7:16	7:21	7:36	7:46	8:01	8:11	8:15	8:19	8:34	8:49	9:04	9:19	9:34	9:49
Ralliement	7:34	7:38	7:42	7:50	7:54	7:58	8:02	8:10	8:14	8:18	8:22	8:37	8:52	9:07	9:22	9:37	9:52
Descazeaux	7:38	7:42	7:46	7:54	7:58	8:02	8:06	8:14	8:18	8:22	8:26	8:41	8:56	9:11	9:26	9:41	9:56

5 conclusion

We have introduced a new method to compute just in time control for $(max, +)-$linear systems. Originality of this control is the possibility to take into account any constraint which can be expressed as an implicit inequality involving state vector. We also prove the convergence of the computation. We apply this method to the timetable synthesis of urban transportation systems. Future works should refine the control method as well as the model for urban bus networks, in order to take into account notably several connection modes between lines.

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