Cyclic jobshop problem and (max,plus) algebra

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AbstractIn this paper, we focus on the cyclic job-shop problem. This problem consists in determining the order of a set of generic tasks on machines in order to minimize the cycle time of the sequence. We propose an exact method to solve this problem. For each solution, a linear max-plus model (possibly non causal) is obtained. To evaluate the performance of a considered schedule, we build the causal max-plus representation and compute the eigenvalue of the evolution matrix. A branch and bound procedure is presented.

Keywords: Cyclic scheduling, (max,plus) algebra

1. INTRODUCTION

The main idea of cyclic (or periodical) scheduling is to perform a set of generic tasks infinitely often (or with a infinite time horizon). The schedule has no end and the production is specified by cycle periodically repeated one behind the other. This kind of problems arises in many contexts such as robotics (Kats and Levner [1997]), manufacturing systems (Pinedo [2005], Hillion and Proth [1989]) or multiprocessor computing (Hanen and Munier [1995b]). Up to now, periodic scheduling problems have been studied from several points of view. Most of them consider graph based approaches, integer linear programming, Petri nets. Although these approaches are different, these tools usually leads to the same results. An overview of the cyclic scheduling problem is available in Hanen and Munier [1995a] and Brucker and Kampmeyer [2008].

In this paper, we tackle the problem of cyclic jobshop scheduling which is proven to be NP-hard by Hanen [1994]. More precisely, we propose a max-plus linear model (possibly non causal) for each schedule. This paper presents a method to obtain a simple (and causal) recurrence of the form $x(k) = Ax(k-1)$ for the date of each task. The cycle time of the schedule is given by computing the eigenvalue of the evolution matrix. More precisely, decision variables correspond to an order of tasks on each machine (or dedicated processor). For each assignement of these decison variables, we associate a max-plus linear model. A branch and bound procedure is then presented to obtain the schedule characterized by the minimum cycle time. The remainder of this paper is as follows. In section 2, we introduce algebraic tools used throughout the paper and $(max,+)$-linear systems. In section 3, the cyclic scheduling problem problem is stated. We first describe the basic scheduling problem and then we propose an optimal solution for the cyclic scheduling problem with resource constraints. A simple numerical example is also developed during all of the paper.

2. MAX-PLUS MODEL FOR DISCRETE EVENTS SYSTEMS

Max-plus algebra enables to model discrete events systems that involve synchronization and delay phenomena. Applications of this theory have essentially concerned manufacturing systems Menguy et al. [2000], Lahaye et al. [2003], communication networks Boudec and P. [2001] and transportation networks Houssin et al. [2007], de Vries et al. [1998]. Few works on scheduling can be found however we can mention Gaubert and Mairesse [1999] and more recently Bouquard et al. [2006].

2.1 Max-plus algebra and discrete event systems

We first recall some algebraic tools. An exhaustive presentation of this theory can be found in Baccelli et al. [1992]. The $(max,+)$ semiring is the set $\mathbb{R}\cup\{-\infty\}$ endowed with the max operator, written $a \oplus b = max(a, b)$, and the usual sum written $a \otimes b = a + b$. The sum (resp. product) admits a neutral element denoted $\varepsilon = -\infty$ (resp. $e = 0$), it leads to $a \oplus \varepsilon = a$ and $a \otimes e = a$. For matrices, additions and products give $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$.

The behavior of such systems can be represented by some discrete functions called *dater* functions. More precisely, a discrete variable $x(\cdot)$ is associated to an event labeled x. This variable represents the occurring dates of event x. The numbering conventionally begins at 0: $x(0)$ corresponds to the date of the first occurrence of x. In this way, autonomous DEDS can be modeled by a linear state representation

$$x(k) = Ax(k-1), \qquad (1)$$

where x is a vector and A is the evolution matrix. Moreover, to each square matrix A, a graph $G = (N, E)$ can be associated. Indeed, there is no arc (i, j) if $A_{ji} = \varepsilon$, and (i, j) is labelled with A_{ji} otherwise. In the same way, a dedicated $(max, +)$ matrix is associated to each graph.

We now recall the classical spectral theorem of an irreducible square matrix.

Theorem 1. Any irreducible matrix A of size $n \times n$ with entries in $\mathbb{R} \cup \{-\infty\}$ has a unique eigenvalue which is the maximal cycle mean of the graph associated to A.

Several algorithms for computing the eigenvalue are possible. We can mention the Karp's algorithm in $\mathcal{O}(|n|^3)$ (Karp [1978], Dasdan and Gupta [1998]) and the Howard's algorithm (Cochet-Terrasson et al. [1998]) of unproved complexity but showing excellent performance.

2.2 Max-plus model for non causal systems

We now focus on non causal $(max, +)$-systems and we establish conditions on the existence of a causal representation of these systems.

The general form of an autonomous $(max, +)$-system can be written as

$$x(k) = \bigoplus A_q x(k - q), \quad q \in \mathbb{Z}.$$

By means of an extension of the state vector, last equation leads to this compact form

$$x(k) = A_1 x(k - 1) \oplus A_0 x(k) \oplus A_{-1} x(k + 1). \qquad (2)$$

We are confronted to a fixed point equation (see [Baccelli et al., 1992, Th. 4.70]) and the least solution (that corresponds to the earliest functionning) is given by

$$x(k) = A'_1 x(k - 1) \oplus A'_{-1} x(k + 1), \qquad (3)$$

with $A'_1 = A_0^* A_1$ and $A'_{-1} = A_0^* A_{-1}$ considering $X^* = \bigoplus_{i \in \mathbb{N}} X^i$.

Proposition 1. If the matrix A_{-1} contains only one element i, j such that $A_{-1_{ij}} \neq \varepsilon$ and $i \neq j$, the causal representation of the system is given by

$$x(k) = (A'_{-1} A'_1)^* A'_1 x(k - 1). \qquad (4)$$

Proof : First, we state that if A'_1 has also only one element different from ε and if it is not a digonal element, it involves that $(A'_{-1})^2 = \varepsilon$. If it is a diagonal element, the system can not admit a causal representation since it leads to $x_i(k) \succeq A_{ii} x_i(k + 1)$. From (3), we deduce

$$\begin{aligned} x(k) &= A'_1 x(k-1) \oplus A'_{-1} x(k+1) \\ &= A'_1 x(k-1) \oplus A'_{-1}(A'_1 x(k) \oplus A'_{-1} x(k+2)) \\ &= A'_1 x(k-1) \oplus A'_{-1} A'_1 x(k) \oplus (A'_{-1})^2 x(k+2)) \end{aligned}$$

Last equality leads to

$$x(k) = A'_1 x(k - 1) \oplus A'_{-1} A'_1 x(k),$$

since $(A'_{-1})^2 = \varepsilon$. In accordance with the earliest functionning rule (an event occurs as soon as possible), we select the least solution of last equality which is given by

$$x(k) = (A'_{-1} A'_1)^* A'_1 x(k - 1).$$

□

Remark 1. For the meantime, we didn't prove neatly the general case, *i.e.* no restrictions on the number of elements of A_{-1} different from ε.

3. CYCLIC SCHEDULING PROBLEMS

This section is devoted to cyclic scheduling problems. A schedule is called periodic (or cyclic) with cycle time ω if $x_i(k) = \omega + x_i(k - 1) = k \times \omega + x_i(0)$ where $x_i(k)$ denotes the k-th occurrence of task i. We describe these problems from a $(max, +)$ point of view. In $(max, +)$-algebra the last equation becomes $x_i(k) = \omega \otimes x_i(k - 1) = \omega^k \otimes x_i(0)$.

We first recall the basic periodic scheduling problem and then we tackle the problem of the jobshop cyclic scheduling problem.

3.1 The basic cyclic scheduling problem

We recall the basic cyclic scheduling problem. In this framework, there is no resource constraints since each task is performed without preemption on a dedicated machine. In this problem we have:

- operations (or tasks) $i = 1, \ldots, n$ and the associated processing times $p_1, \ldots, p_n$,
- generalized precedence constraints (or conjunctive constraints) between the k-th occurrence of task i and the $k + H_{ji}$-th occurrence of task j given by

$$x_i(k + H_{ij}) \geq p_j x_j(k).$$

The event shift function H is called the height. We denote T the set of operations. A uniform graph $G_u(T, U)$ can be associated to the generalized precedence constraints in which a node is a task and an arc corresponds to a constraints. Each arc $(i, j) \in U$ is supplied by two values p_i and H_{ji}.

A cyclic scheduling problem is said to be consistent, if and only if every circuit of G_u has a positive height (Hanen and Munier [1995a], Brucker and Kampmeyer [2008]). Let us define a matrix H' such that $H'_{ij} = -H_{ij}$ if $(i, j) \in U$ and $H'_{ij} = \varepsilon$, otherwise.

Proposition 2. The basic scheduling problem is consistent if and only if the largest eigenvalue of H' is negative.

Proof : It also means that the problem is consistent if and only if the minimum cycle mean of G_u is positive. We can build a conjugate graph $G'_u(T, U)$ in which every $(i, j) \in U$ is labeled with $-H_{ji}$. Consequently, the problem is consistent if and only if the maximum cycle mean of G'_u is negative. Then, the matrix associated to G'_u is H'. Hence, the consistency of the problem can easily checked by means of Karp's algorithm or Howard algorithm.

□

Of course, the $k+1$-occurrence of a task i can not start before the end of the k-occurrence of a task i. Consequently, we get the following non-reentrance constraints

$$x_i(k + 1) \geq p_i x_i(k) \qquad \forall i \in T$$

Such systems, if it is consistent, can easily be expressed as a causal $(max, +)$-linear system and the average cycle time is given by the largest $(max, +)$-eigenvalue.

Example 1. Let us consider the following basic scheduling problem

task	1	2	3	4	5
processing time	2	3	1	2	2

and the following uniform constraints :

$$\begin{aligned} x_1(k) &\geq 2x_4(k-1) \\ x_2(k) &\geq 2x_1(k) \\ x_3(k) &\geq 3x_2(k) \\ x_3(k) &\geq 2x_5(k-1) \\ x_4(k) &\geq 3x_2(k) \\ x_4(k) &\geq 2x_5(k-1) \\ x_5(k) &\geq 2x_1(k+1). \end{aligned}$$

We first check the consistency of the problem. There is a solution since the eigenvalue of the matrix H' (in which non-reentrance constraints are included) associated to the constraints is equal to $-\frac{1}{3}$. The $(max, +)$ formulation leads to a non causal system of the form (2). Nonetheless, a causal representation of the earliest schedule can be obtained as shown in §2.2. The new associated model is

$$x(k) = \begin{pmatrix} 2 & \varepsilon & \varepsilon & 2 & \varepsilon \\ 4 & 3 & \varepsilon & 4 & \varepsilon \\ 7 & 6 & 1 & 7 & 2 \\ 7 & 6 & \varepsilon & 7 & 2 \\ 11 & 10 & \varepsilon & 11 & 6 \end{pmatrix} x(k-1).$$

The eigenvalue of the evolution matrix leads to 7. It corresponds to the optimal cycle time of this basic scheduling problem. The solution of the schedule is depicted in figure 1.

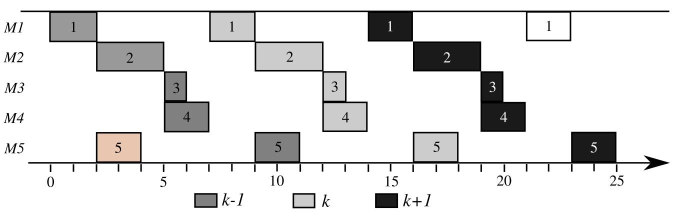

Figure 1. Optimal solution of the basic cyclic scheduling problem.

Remark 2. Notice that the number of work-in-process is not limited to one. For instance, the solution of example 1 presents two work-in-process for $t \in [7, 11]$.

3.2 The jobshop cyclic scheduling problem

In a jobshop cyclic scheduling problem, tasks are mapped onto m machines. Naturally, we have $m < n$. It means that the number of machine is lower than the number of tasks ($n = |T|$). A presentation of this problem is proposed by Hanen and Munier [1995a].

We denote $M(i) \in \{1, \ldots, m\}$ the machine dedicated to the task i. Occurrences of operations to be processed on the same machine can not overlap. Indeed, new constraints have to be took in account to consider this allocation. Let us consider two tasks i, j performed on the same machine such that the next occurrence of j to be performed after the l-th task i is the k-th occurrence. Therefore, we can consider $x_j(k) \geq p_i + x_i(l)$. Since the schedule is periodic, last equation is equivalent to $x_j(0) + k\omega \geq p_i + x_i(0) + l\omega$. It is also equivalent to $x_j(0) \geq p_i + x_i(0) + (l-k)\omega$ and we can state

$$x_j(s) \geq p_i + x_i(s + l - k) \qquad \forall s \in \mathbb{N}. \tag{5}$$

Moreover, the next occurrence of i to be performed after the k-th task j is the $l+1$-th occurrence. So, we have $x_i(l+1) \geq p_j + x_j(k)$. In the same manner of (5), we obtain

$$x_i(s) \geq p_j + x_j(s + k - l - 1) \qquad \forall s \in \mathbb{N}. \tag{6}$$

Constraints (5) and (6) can be rewritten as follows : for all different tasks $i, j \in T$ such that $M(i) = M(j)$, we have

$$\exists K_{ij} \in \mathbb{Z} \qquad x_i(s) \geq p_j + x_j(s - K_{ij}) \; \forall s \in \mathbb{N} \tag{7}$$

and $K_{ij} + K_{ji} = 1$. The event shifts K_{ij} and K_{ji} can be interpreted as the ordering on machine $M(i)$. For instance, the selection $K_{ij} = 0$ and $K_{ji} = 1$ means that the first occurrence if task i starts after the first occurrence of task j and, since the schedule is periodic, we can state that the k-occurrence if task i starts after the k-occurrence of task j. A solution of the jobshop cyclic scheduling problem is a complete selection of all K_{ij}. These selection constraints, also called disjunctive constraints, defines a selection graph $G_s(T, S)$. This graph is non connected and there is one non connected component per machine.

Thus, the jobshop cyclic scheduling problem leads to two graphs $G_s(T, S)$ corresponding to selection constraints and $G_u(T, U)$ that characterizes the conjunctive constraints as in the basic cyclic problem. The total graph, denoted $G_t(T, U \cup S)$ of the problem can be obtain through merging $G_s(T, S)$ and $G_u(T, U)$. If $(i, j) \in U$ and $(i, j) \in S$, arc (i, j) is supplied by p_i and $min(H_{ij}, K_{ij})$ (by selecting the minimum value of H_{ij} and K_{ij}, both constraints are met). As for the basic cyclic scheduling problem, we introduce the matrix H'_t defined such that $H'_{t_{ij}} = min(H_{ij}, K_{ij})$ if $(i, j) \in U \cup S$ and $H'_{t_{ij}} = \varepsilon$ otherwise. Nonetheless not all values for each K_{ij} are possible and we now propose a consistency property.

Proposition 3. A selection is consistent if and only if the largest eigenvalue of H'_t is negative.

Proof : As for proposition 2, we can build the graph G'_t and the problem is consistent if and only if the maximum mean cycle of this graph is negative.

□

Example 2. Let us consider the cyclic scheduling problem of example 1. We now consider that tasks 1,4 and 5 are performed by machine 1 and tasks 2 and 3 are performed by machine 2. The selection graph (resp. uniform graph) of this problem is depicted in dashed lines (resp. solid line) of figure 2. For more clarity, we don't report in figure 2 arcs from non reentrance constraints). The merge of these two graphs leads to the total graph.

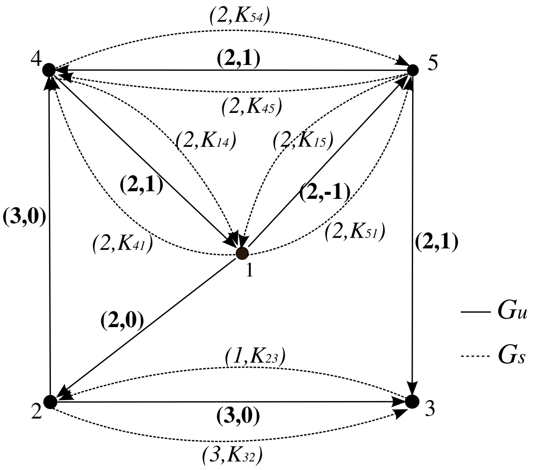

Figure 2. Total graph of the cyclic jobshop problem.

After a complete selection, *i.e.* a choice for each K_{ij}, the functionning of the system is described by an equation of the form (2). Therefore, a causal representation of the form (4) can be obtained. As in the basic cyclic scheduling problem, the optimal period of the schedule is given by the largest eigenvalue of the system. Therefore the problem can be summarized as follows : find a complete selection that minimizes the largest eigenvalue of the evolution matrix. Let us denote A_u the matrix of conjunctive constraints such that $x(k) \geq A_u x(k-1)$ and A_s the evolution matrix of selection constraints such that $x(k) \geq A_s x(k-1)$. We now introduce the following property that bounds the optimal period of this problem.

Proposition 4. The largest eigenvalue of A_u and the largest eigenvalue of A_s are lower bounds of the optimal cycle time of the periodic jobshop problem.

Proof : Considering the conjunctive constraints, this lower bound is the optimal solution of the basic cyclic scheduling problem, *i.e.* without resource constraints. The optimal schedule of a complete selection is given by the largest eigenvalue of matrix $A = A_u \oplus A_s$. Hence, we have $x(k) \geq A_u x(k-1)$ and consequently $x(k) \geq \lambda(A_u)x(k-1)$ where $\lambda(A_u)$ denotes the largest eigenvalue of A_u. The same reasoning holds for A_s. However, the computation of $\lambda(A_s)$ is not so easy because A_s is not defined while a complete selection has not been realized. We propose a simple way to compute it. Let us consider the set of tasks $\{i_1, i_2, \ldots, i_l\}$ performed on a machine. It corresponds to a strongly connected part of A_s. The functionning of a sequence on this machine is obviously limited by the overall time of processing of all the tasks $\{i_1, i_2, \ldots, i_l\}$, so the cycle time of this machine cannot exceed the sum $p_{i_1} + p_{i_2} + \ldots + p_{i_l}$ that is the eigenvalue of this component. Thus, the largest eigenvalue of A_s is given by maximum of the overall processing times of each machine. It also could be proven that every circuit of A_s is of height 1 and the sum of all processing times is an eigenvalue of the component.

□

Example 3. Considering the example 2, we get $\lambda(A_u) = 7$ (already computed in example 1). For machine 1 (resp. 2), the overall processing time is 6 (resp.4). So we can state that 7 is a lower bound of the optimal cycle time for this cyclic jobshop problem.

As seen previously, every $K_{ij} \in \mathbb{Z}$, so we now propose some bounds on this event shift.

If the basic scheduling problem (*i.e.* without constraints) of the jobshop scheduling problem is consistent, then H'^* can be computed (the computation of H'^* converges since the largest eigenvalue is negative, see Baccelli et al. [1992]). Therefore, $(-H'^*)_{ij}$, if it is not equal to ε, is the minimum shift event between task i and task j induced by the uniform constraints. More precisely, we have $x_i(k) \geq P_{ij} + x_j(k - -H'^*_{ij})$ and so

$$x_i(k) \geq P_{ij} + x_j(k + H'^*_{ij}), \tag{8}$$

in which P_{ij} is a sum of several processing times (at least p_j).

Let us consider two tasks i and j such that $M(i) = M(j)$ and suppose that $(-H'^*)_{ij} \neq \varepsilon$, consequently we get (8) and since

$$x_j(k) \geq p_i + x_i(k - K_{ji}),$$

we can state

$$\begin{aligned} x_i(k - H'^*_{ij}) &\geq P_{ij} + x_j(k), \\ \Rightarrow x_i(k - H'^*_{ij}) &\geq P_{ij} + p_i + x_i(k - K_{ji}), \\ \Rightarrow x_i(k - H'^*_{ij}) &> x_i(k - K_{ji}). \end{aligned}$$

As $x(k)$ is monotone, it requires $K_{ji} > H'^*_{ij}$ and for integrity reason we obtain the following bound

$$K_{ji} \geq H'^*_{ij} + 1.$$

Consequently, we also have a bound for K_{ij} since $K_{ij} + K_{ji} = 1$.

Example 4. Let us consider tasks 1 and 5 of example 2. We have $H'^*_{15} = -2$. It leads to $K_{51} \geq -1$ and $K_{15} \leq 2$. Moreover $H'^*_{51} = 1$, and we get $K_{15} \geq 2$ and $K_{51} \leq -1$. We can conclude that $K_{51} = -1$ and $K_{15} = 2$. Generally, we only obtain upper bounds and lower bounds for the K_{ij}.

3.3 A branch and bound procedure for the jobshop cyclic scheduling problem

In previous section, we have seen that the definition domain of the variables K_{ij} can be reduced by using the uniform constraints. Besides, a lower bound of the optimal cycle time has been computed. A branch and bound procedure (not detailed here) can be realized to solve this kind of problem. The branching is realized on variables K_{ij}. Since, there is no need to branch K_{ji} if K_{ij} is already branched, the depth of the exploration tree is equal to the half of the number of selection variables. For each node, the consistency is checked. If the solution is consistent, we build the causal $(max, +)$-representation of the system and compute the largest eigenvalue of the evolution matrix.

Example 5. For our example, we obtain the following results : $K_{14} = 1$, $K_{41} = 0$, $K_{15} = 2$, $K_{51} = -1$, $K_{32} = 0$, $K_{23} = 1$, $K_{45} = 1$ and $K_{54} = 0$. The lower bounds is reached and we obtain an optimal cycle time of 7. The corresponding schedule can be found in figure 3.

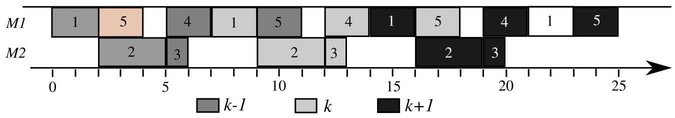

Figure 3. Optimal solution of the jobshop cyclic scheduling problem.

4. CONCLUDING REMARKS

We presented a general $(max, +)$-linear model for cyclic scheduling problems. This class of systems leads frequently to a non causal representation in $(max, +)$-algebra. In light of those observations, we propose a way to obtain a causal representation with recurrence of order 1. However, this method involves strong constraints on the system structure. We first apply it to the basic scheduling problem and then we try to formalize the cyclic jobshop problem with $(max, +)$-algebra. Bounds are proposed, both for the optimal cycle time and for the selection variables K_{ij}. Future works should refine the method and compare it to mixed integer linear programs of Hanen [1994] and Brucker and Kampmeyer [2008] for larger instances of the problem. We also plan to solve the general case of proposition 1.

One may note that this kind of approach suggests that some performance evaluation could be performed (robustness to delay for instance) and it is not necessary relevant to schedule the minimum cycle time if the solution is not robust. It might be interesting, in some cases, to minimize the deviation of the cycle time when delays occurrs. It is worthwhile to study this connection in more details.

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