TIMETABLE SYNTHESIS USING $(MAX, +)$ ALGEBRA

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Abstract: This paper deals with control of transportation systems. We propose an approach based on dioid theory to compute timetables of a transportation network. We generalize the problem by considering additional constraints for the control objective.

Keywords: Discrete Event Systems, dioids, transportation systems, control

1. INTRODUCTION

The functioning of Discrete Event Dynamic Systems (DEDS) subject to synchronization and delay phenomena can be described by linear models in a particular algebraic structure called dioid. A linear system theory has been developed over dioids by analogy with conventional theory (Baccelli *et al.*, 1992).

Many transportation systems can be described as a DEDS. As in (Goverde *et al.*, 1998), (de Vries *et al.*, 1998) and (Nait-Sidi-Moh, 2003), we consider transportation networks as $(max, +)$-systems where state variables represent departure times of vehicles at some stations ($(max, +)$ algebra is an example of dioid).

We are here interested in the control of these particular systems. In normal conditions, the control of urban transportation systems is realized by the timetables. They are defined at each stop to specify times at which buses should theoretically run. They are used to inform passengers and, at particular stops, to synchronize vehicles. In (Houssin *et al.*, 2004), a solution has already been proposed in which timetables was subject to two kinds of constraints.

In this paper, we generalize the problem of timetable computation by considering additional constraints for the control objective. More precisely, we can consider any constraint which can be expressed by an implicit inequality over the system state. For example, it is possible to specify for a given stop, with this formalism: a desired number of departures in an interval of given dates, a minimum and/or a maximum time separation between two departures, or a critical time constraint on a path of the system. The problem is formulated as an extremal fixed point computation. An iterative method is proposed to solve it.

2. PRELIMINARIES

2.1 Dioid theory

A dioid $(\mathcal{D}, \oplus, \otimes)$ is an idempotent semi-ring with neutral elements denoted ε and e (Baccelli *et al.*, 1992, §4). A dioid is said to be complete if it is closed for infinite sums and if product distributes over infinite sums too. The sum of all its elements is generally denoted $\top$ (for top).

Example 1. The set $\overline{\mathbb{Z}}_{max}$ endowed with the max operator as sum and the classical sum as product is a complete dioid in which $\varepsilon = -\infty$, $e = 0$ and $\top = +\infty$. Set $\overline{\mathbb{Z}}_{min} = (\mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}, min, +)$ is also a complete dioid in which $\varepsilon = +\infty$, $e = 0$ and $\top = -\infty$.

Due to the idempotency of the sum, a dioid is endowed with a partial order relation, denoted $\succeq$, defined by the following equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$. The notation $a \prec b$ defines $a \preceq b$ and $a \neq b$. A complete dioid has a structure of complete lattice, *i.e.*, two elements in a complete dioid always have a *least upper bound*, namely $a \oplus b$, and a *greatest lower bound* denoted $a \wedge b = \bigoplus_{\{x|x \preceq a,\ x \preceq b\}} x$.

2.2 Representation of DEDS in dioids

Dioid algebra enables to model DEDS which involve synchronization phenomena. The behavior of such systems can be represented by some discrete functions called *dater* functions. More precisely, a discrete variable $x(\cdot)$ is associated to an event labeled x. This variable represents the occurring dates of event x. The numbering conventionally begins at 0: $x(0)$ corresponds to the date of the first occurrence of x. These variables are extended towards negative values by: $x(k) = -\infty = \varepsilon$ for $k < 0$ such that they can be manipulated as mappings from $\mathbb{Z}$ to $\overline{\mathbb{Z}}_{max}$.

The considered DEDS can be modeled by a linear state representation

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k), \\ y(k) &= Cx(k), \end{aligned} \qquad (1)$$

where x, u and y are the state vector, the input vector and the output vector respectively.

The initial sate of a system is defined by a vector $v(k)$ added to the dynamic equation as follows:

$$x(k) = Ax(k-1) \oplus Bu(k) \oplus v(k).$$

More precisely, $v_i(k)$ for $0 \leq k < k_{d_i}$ represents the earliest occurring dates of initial events. To be manipulated as a dater, each variable v_i is extended such that: $v_i(k) = \varepsilon$ for $k < 0$ and $v_i(k) = v_i(k_{d_i} - 1)$ for $k \geq k_{d_i}$. We say that initial conditions are *canonical* if $\forall k \in \mathbb{Z},\ v(k) = \varepsilon$ and the dynamic behavior of the system then obeys to state equation (1). Index k_{d_i} denotes the first occurrence of event x_i induced by inputs (this definition of initial conditions is more detailed in (Baccelli *et al.*, 1992, §5.4.4.1)). The notion of characteristic number introduced in (Boimond and Ferrier, 1996) enables to calculate this index.

Definition 1. (characteristic number). Let $[A]_i$ be the i-th row of matrix A, the characteristic number associated with the state variable x_i of a model described by (1), if it exists, is the least integer, noted k_{d_i}, such that $[A^{k_{d_i}}]_i B \neq \varepsilon$.

The characteristic number k_{d_i} corresponds to the event shift between inputs and a state x_i. Let us define now the event shift between a state x_i and an output y_j. We define it, if it exists, as the least integer $k_{f_{ji}}$, such that $C_j[A^{k_{f_{ji}}}]^i \neq \varepsilon$ (notation $[A]^i$ indicates the i-th column of A).

The γ, δ-transform enables to manipulate formal power series, with two commutative variables γ and δ, representing daters trajectories. The set of these formal series is a complete dioid denoted $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In the following, we denote x the corresponding element of $\{x(k)\}_{k\in\mathbb{Z}}$ in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The support of a series x is defined as $Supp(x) = \{k \in \mathbb{Z} | x(k) \neq \varepsilon\}$.

In $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, state representation (1) becomes

$$\begin{aligned} x &= Ax \oplus Bu, \\ y &= Cx, \end{aligned} \qquad (2)$$

in which entries of matrices are elements of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

In accordance with the earliest functioning rule (an event occurs as soon as possible), we select the least solution of the first equation in (2) which is given by $x = A^*Bu$ with $A^* = \bigoplus_{i\in\mathbb{N}} A^i$. Consequently we have $y = Hu$, in which $H = CA^*B$ is called the transfer matrix.

2.3 Residuation theory

Let us consider mappings defined over complete dioids. Such a mapping $f : \mathcal{D} \to \mathcal{C}$ is said to be *isotone* if $a, b \in \mathcal{D},\ a \preceq b \Rightarrow f(a) \preceq f(b)$. Moreover f is *lower-semicontinous* (l.s.c.) if $\forall a, b \in \mathcal{D}, f(a \oplus b) = f(a) \oplus f(b)$. Residuation theory (Blyth and Janowitz, 1972) defines "pseudo-inverses" for some isotone mappings defined over ordered sets such as complete dioids (Baccelli *et al.*, 1992). More precisely, if the greatest element of set $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists for all $b \in \mathcal{C}$, then it is denoted $f^\sharp(b)$ and $f^\sharp$ is called *residual* of f.

Theorem 1. (Baccelli *et al.*, 1992, th. 4.50) Let $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping, the following statements are equivalent:

(i) f is residuated,
(ii) f is l.s.c. and $f(\varepsilon_{\mathcal{D}}) = \varepsilon_{\mathcal{C}}$,
(iii) there exists a unique mapping $f^\sharp$ such that $f \circ f^\sharp \preceq Id_{\mathcal{C}}$ and $f^\sharp \circ f \succeq Id_{\mathcal{D}}$.

Theorem 2. (Baccelli *et al.*, 1992, th. 4.56) Let $f : \mathcal{D} \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{B}$. If f and g are residuated then $g \circ f$ is residuated and $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$.

Example 2. The valuation $val(x)$ of a series $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is defined by (Baccelli *et al.*, 1992, definition 5.19):

$$\begin{aligned} val : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] &\longrightarrow \overline{\mathbb{Z}}_{min} \\ x &\longmapsto val(x) = Min(Supp(x)). \end{aligned}$$

As an example, we have $val(\gamma^3\delta^1 \oplus \gamma^5\delta^2) = 3$.

Proposition 1. Mapping val is residuated and its residual is $val^\sharp(x) = \gamma^x\delta^*$.

Proof : Mapping val is l.s.c. and $val(\varepsilon) = \varepsilon$ (see (Baccelli *et al.*, 1992, lemma 4.93)). According to item (ii) of theorem 1, it is then residuated. We check that the proposed residual satisfies item (iii) of theorem 1:

$$\begin{aligned} val \circ val^\sharp(x) &= val(\gamma^x\delta^*) = x \\ val^\sharp \circ val(x) &= \gamma^{val(x)}\delta^* \succeq x. \end{aligned}$$

Example 3. Let $Pr_a : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] \rightarrow \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ defined by:

$$\begin{aligned} Pr_a : x \longmapsto Pr_a(x) &= Pr_a(\bigoplus_{(n,t)\in\mathbb{Z}^2} x(n,t)\gamma^n\delta^t) \\ &= \bigoplus_{(n,t)\in\mathbb{Z}^2} x_a(n,t)\gamma^n\delta^t, \end{aligned}$$

in which $x_a(n,t) = \begin{cases} x(n,t) & \text{if } t \geq a, \\ \varepsilon & \text{otherwise.} \end{cases}$

Given a series $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the mapping $Pr_a(x)$ consists in preserving the monomials of x whose exponents in δ are greater than or equal to a. As an example, we have $Pr_3(\gamma^1\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^5) = \gamma^3\delta^3 \oplus \gamma^4\delta^5$.

Proposition 2. The mapping Pr_a is residuated and its residual is $Pr_a^\sharp(y) = y \oplus (\gamma^{-1})^*\delta^{a-1}$.

Proof : According to item *(iii)* of theorem 1, we check that $Pr_a \circ Pr_a^\sharp(y) \preceq y$ and $Pr_a^\sharp \circ Pr_a(y) \succeq y$:

$$\begin{aligned} Pr_a \circ Pr_a^\sharp(y) &= Pr_a(y \oplus (\gamma^{-1})^*\delta^{a-1}) \\ &= Pr_a(y) \preceq y \end{aligned}$$

$$\begin{aligned} Pr_a^\sharp \circ Pr_a(y) &= Pr_a^\sharp(Pr_a(y)) \\ &= Pr_a(y) \oplus (\gamma^{-1})^*\delta^{a-1} \succeq y \end{aligned}$$

2.4 Greatest fixed point of mappings defined over dioids

We denote $\mathcal{F}_f = \{x | f(x) = x\}$ (resp. $\mathcal{P}_f = \{x | f(x) \succeq x\}$) the set of fixed points (resp. the set of post-fixed points) of an isotone mapping f defined over a complete dioid $\mathcal{D}$. We recall that $\mathcal{P}_f$ has a complete lattice structure (Baccelli *et al.*, 1992, th 4.72). Tarski's theorem (Tarski, 1955) states that an isotone mapping defined over a complete lattice admits at least one fixed point. Moreover, it can be shown that the greatest fixed point coincides with the greatest element of $\mathcal{P}_f$:

$$Sup\ \mathcal{P}_f = Sup\ \mathcal{F}_f \quad \text{and} \quad Sup\ \mathcal{F}_f \in \mathcal{F}_f. \tag{3}$$

In the following proposition, we specify to dioids a well known method to compute greatest fixed point of isotone mappings defined over lattices.

Proposition 3. If the following iterative computation

$$\begin{aligned} y_0 &= \top \\ y_{k+1} &= f(y_k) \wedge y_k \end{aligned} \tag{4}$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f.

3. OPTIMAL CONTROL

The principle of the proposed control can be summarized in three items:

- ▷ The process verifies some initial conditions and some given final conditions.
- ▷ State variables are subject to some constraints.
- ▷ The control is "optimal" in the sense that it optimizes a chosen criterion.

3.1 Terminal conditions

We consider here canonical initial conditions. For each state variable x_i, we defined its characteristic number (definition 1), *i.e.* the index k_{d_i} of the first occurrence of x_i generated by the inputs of the system.

In our framework, the given final state corresponds to the last occurrences of outputs that we want to control. From that goal, we can deduce the last occurrences of the state variables which have to be controlled. We denote $k_{f_{y_j}}$ (resp. k_{f_i}) the last occurrence of event y_j (resp. x_i) that we aim at controlling. The computation of this index is function of the shift event $k_{f_{ji}}$ between a state variable x_i and an output y_j of the system (*cf.* §2.2). The last occurrence of x_i generating the last occurrence $k_{f_{y_j}}$ of output y_j is given by $k'_{f_{ji}} = k_{f_{y_j}} - k_{f_{ji}}$. We then deduce that the last occurrence of x_i to control is the one which generates the last desired occurrence for all outputs, so $k_{f_i} = max(k'_{f_{1i}}, k'_{f_{2i}}, \ldots, k'_{f_{pi}})$ for a $p-$outputs system.

3.2 Constraints

We consider constraints which can be formulated as an implicit inequality in x. These constraints are applied to an interval of occurrences for each concerned state variable. More precisely, for an event labeled x_i, constraints are applied only for indices of occurrences included in interval

$[k_{d_i}, k_{f_i}]$. Indeed, x_i should not be constrained for indices less than k_{d_i} since these are not induced by the inputs of the system (*cf.* §2.2). Furthermore, k_{f_i} corresponds to the index of the last occurrence of x_i that we aim at controlling. In order to express these constraints in dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, in which we manipulate the whole trajectory of a dater $\{x_i(k)\}_{k\in\mathbb{Z}}$ as a formal power series x_i, we use two vectors ω and ν:

$$x \preceq (g_l(x) \wedge \omega) \oplus \nu, \quad \text{for } l \in \{1, \ldots, q\} \qquad (5)$$

in which ω (resp. ν) is a n-vector (n is the dimension of the state vector) with entries $\omega_i = \gamma^{k_{d_i}}\delta^*$ (resp. $\nu_i = \gamma^{k_{f_i}+1}\delta^*$), and each g_l, $l = 1, 2, \ldots, q$, is a mapping from $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^n$ to $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^n$ modeling a constraint. Vectors ω and ν enable to relax constraints for the occurrences of events x_i, $i = 1, 2, \ldots, n$, whose indices are not included in $[k_{d_i}, k_{f_i}]$.

3.3 Criterion

A relevant goal for the control of DEDS is to delay as much as possible the input events occurrences (*i.e.* to compute the greatest control vector u) while ensuring performances imposed by a specification (the specification corresponds here to terminal conditions and constraints). It corresponds to the just in time control problem which commonly aims at supplying the "right quantity" (the demand) at the "desired time" (date of the demand). Therefore, the considered criterion J is $J = u$. The optimal control is the one which maximizes J.

3.4 Synthesis

Considering the earliest functioning rule of the system, we obtain from (2) $x = A^*Bu$ and using the following notation $g'_l = A^*B \lcirclearrowleft ((g_l(A^*Bu) \wedge \omega) \oplus \nu)$, the optimal control u is the greatest solution of inequalities :

$$u \preceq g'_l(u), \quad \text{for } l \in \{1, \ldots, q\} \qquad (6)$$

which is equivalent to find the greatest u satisfying

$$u \preceq g'_1(u) \wedge \ldots \wedge g'_q(u) = f(u). \qquad (7)$$

Proposition 4. If the following iterative computation converges in a finite number k_e of iterations

$$\begin{aligned} u_0 &= \top \\ u_{k+1} &= f(u_k) \wedge u_k, \end{aligned}$$

then u_{k_e} is the control which both respects terminal conditions and constraints traduced by f (equation (7)) and optimizes the criterion J.

Proof : The set of controls u which satisfy the constraints is the set $\mathcal{P}_f$ of post fixed points of f. The iterative computation defined in proposition 3, if it terminates, gives its greatest element $Sup\ \mathcal{P}_f$.

4. APPLICATION TO URBAN TRANSPORTATION NETWORKS

In this section, we first present a $(max, +)$-linear model for urban bus networks. The timetable synthesis problem ((Nait-Sidi-Moh, 2003)) is then decomposed as constraints on state vector. We solve it by applying the method introduced in section 3.

4.1 Modeling of a bus network

A transportation system can be modeled as a state representation in $\overline{\mathbb{Z}}_{max}$ by:

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k), \\ y(k) &= Cx(k), \end{aligned} \qquad (8)$$

in which $x(k)$ is a vector such as $x_i(k)$ denotes the departure time of the $(k+1)-$th bus at stop i. Matrix A is defined such as $A_{ij} = a_{ij}$ where a_{ij} corresponds to the traveling time from stop j to stop i, $A_{ij} = \varepsilon$ otherwise. Travelling time a_{ij} may correspond either to the time spent by a bus to run from stop j preceding stop i on the same line, or to the walking time between stops j and i belonging to different lines (a connexion between buses departing from j and arriving at i is then specified). Vector $y(k)$ corresponds to the vector of daters associated with stops considered as "strategic" (at which the level of service must be respected more particularly). The timetable is represented by input vector $u(k)$, and variable $u_i(k)$ denotes the scheduled departure time of the $(k+1)-$th bus at stop i. In practice, synchronizations of buses with timetable occur only at particular stops of the network such as the beginning or the end of a line. Concerning the other stops, the timetable has only an indicative value. So, entries of matrix B are such as $B_{ii} = e$ if timetable must be respected at stop i, $B_{ij} = \varepsilon$ otherwise.

4.2 Timetable synthesis problem

We present here the timetable synthesis problem by decomposing it into several constraints on the state vector of the proposed model.

- In a first place, we define an expected level of service at strategic stops of the network. This quality is specified by a target vector denoted $z(k)$ which contains the latest departure dates for buses at strategic stops. This constraint leads to :

$$C \otimes x(k) \preceq z(k) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i}. \quad (9)$$

- For each line, we define a maximum headway (*i.e.* the expected maximum time separation between two buses departures at a stop). Maximum headways enable to define a minimum departure frequency for each line. For stop i, this constraint can be written:

$$\begin{cases} x_i(k) = x_i(k_{d_i}) & \text{for } k = k_{d_i}, \\ x_i(k) \preceq \triangle_i^{max} \otimes x_i(k-1) & \text{for } k_{d_i} < k \leq k_{f_i}, \end{cases}$$

$$\Leftrightarrow x_i(k) \preceq \triangle_i^{max} \otimes x_i(k-1) \oplus x_i(k_{d_i}) \quad (10)$$

for $k_{d_i} \leq k \leq k_{f_i}$, where k_{d_i} and k_{f_i} are the bounds of the interval of indices that we want to control for event x_i (*cf.* §3.1).

- Furthermore, minimum headways enable to avoid the natural tendency of transit vehicles to bunch up as soon as a bus is in late. For stop i, this constraint can be written

$$x_i(k) \succeq \triangle_i^{min} \otimes x_i(k-1) \qquad \text{for } k_{d_i} \leq k \leq k_{f_i}. \quad (11)$$

  Generally, a specific minimum headway is defined for each line.

- In the daytime, some rush hours appear. Origins of these peaks of charge can be different: intermodal connections or urban activities (school at home-time, factories closing time) but they are generally planned. In this case, it is wanted that one or several departure(s) occur(s) in an interval of given dates in order to quickly absorb peaks of charge. For stop i, we model this constraint by: $\exists k \in [k_{d_i}, k_{f_i}]$, s.t.

$$x_i(k) \succeq t_j \text{ and } x_i(k+s) \prec t_j + r, \quad (12)$$

  in which s is the expected number of departure(s) at stop i during interval $[t_j, t_j + r]$ in order to absorb the peak.

- At some stops of the network, we want to limit waiting times to achieve a quality of service or because of physical constraint (case of a stop located on a road shared with cars). We note ϕ_{ji}^{max} the sum of the traveling time from i to j with the maximum waiting time expected at stop i. This constraint can be formulated as:

$$x_j(k) \preceq \phi_{ji}^{max} \otimes x_i(k-1) \quad \text{for } k_{d_j} \leq k \leq k_{f_j}. \quad (13)$$

4.3 Resolution

In order to apply results of section 3 to these systems, constraints have to be expressed as formal power series in dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ such as (5).

- Constraint (9) is traduced in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ by : $x \preceq (C \diamond\!\!\!\backslash z \wedge \omega) \oplus \nu$.
- Constraint (10) can be formulated by inequality: $x \preceq ((\gamma \triangle^{max} x \oplus x_d) \wedge \omega) \oplus \nu$, in which $\triangle^{max} = \begin{pmatrix} \delta^{\triangle_1^{max}} & \varepsilon & \varepsilon \\ \varepsilon & \delta^{\triangle_2^{max}} & \varepsilon \\ \varepsilon & \varepsilon & \ddots \end{pmatrix}$ and x_d is defined by $x_{d_i} = \gamma^{k_{d_i}} \delta^{x_i(k_{d_i})}$.
- In the same way, we model constraint (11) by $x \preceq (\gamma \triangle^{min} \diamond\!\!\!\backslash x \wedge \omega) \oplus \nu$, in which $\triangle^{min}$ has an analogous structure to $\triangle^{max}$.
- To formulate constraint (12) in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ we use mappings Pr_a and val previously defined in §2.3. In order to point out at least one occurrence of event x_i between dates t_j and $t_j + r$, we specify that the index of the first occurrence later than $t_j + r$ (*i.e.* for $t \geq t_j + r + 1$) must be strictly greater than the index of the first occurrence later than $t_j - 1$ (*i.e.* for $t \geq t_j$).

$$\begin{aligned} & val(Pr_{t_j+r+1}(x_i)) \prec \quad val(Pr_{t_j}(x_i)) \\ \Longleftrightarrow \; & val(Pr_{t_j+r+1}(x_i)) \preceq 1 \otimes val(Pr_{t_j}(x_i)). \end{aligned}$$

  In order to require the occurring of at least s events, we use inequality:

$$val(Pr_{t_j+r+1}(x_i)) \preceq s \otimes val(Pr_{t_j}(x_i)).$$

  We recall that mappings val and Pr_a are both residuated. By using theorem 2, previous inequality can be rewritten:

$$x_i \preceq Pr^{\sharp}_{t_j+r+1}(val^{\sharp}(s \otimes val(Pr_{t_j}(x_i)))).$$

- The γ, δ-transform of (13) leads to $x \preceq ((\phi^{max}\, x) \wedge \omega) \oplus \nu$, in which ϕ^{max} is defined as

$$\begin{cases} \phi_{ll}^{max} = e & \text{for } l \neq j \\ \phi_{ji}^{max} = \delta^{\phi_{ji}^{max}} & \\ \phi_{\alpha\beta}^{max} = \varepsilon & \text{otherwise.} \end{cases}$$

Constraints have then been modeled with respect to (5). With notations of §3.4, these inequalities are equivalent to the following inequality:

$$u \preceq g_1'(u) \wedge g_2'(u) \wedge \ldots \wedge g_5'(u) = f(u).$$

Finally, the problem comes down to finding the greatest u such that $u \preceq f(u)$. If the iterative computation presented in proposition 4 converges in a finite number k_e of iterations then u_{k_e} is the optimal timetables for stops at which synchronizations with timetables is respected. For the other stops (where timetables have only an indicative value), we deduce the scheduled departures times from a simulation of the system based on model (8) and u_{k_e}.

4.4 Example

We consider the urban transportation network represented on figure 1 and composed of two lines. It is assumed that stops x_2 and x_6 are respectively in connection with x_8 and x_{10}.

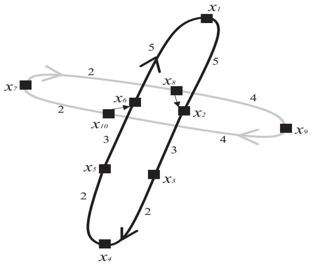

Fig. 1. A simple urban bus network

The dynamic behavior of the system is described by (8) with

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^6\delta^5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^0 & \varepsilon & \varepsilon \\ \varepsilon & \delta^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^4\delta^2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^4 & \varepsilon \end{pmatrix},$$

$B = Id$ (timetable must be respected at each stop) and we define C in such a way that strategic stops are x_4 and x_8:

$$C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \end{pmatrix}.$$

Initial conditions of the system lead to vector ω such that $\forall i,\ \omega_i = \gamma^0\delta^*$. We want 10 departures at stop $y_1 = x_4$ and 10 departures at stop $y_2 = x_8$, then we obtain a vector ν *s.t.* $\nu_i = \gamma^{10}\delta^*$.

The following constraints are specified:

- ▷ The sixth bus at stop x_4, resp. the 10-th, must depart before 80, resp. 130 and the eighth bus at stop x_8, resp. the 10-th, must depart before 60, resp. 100, then we have $z_1 = \gamma^0\delta^{80} \oplus \gamma^6\delta^{130}$ and $z_2 = \gamma^0\delta^{60} \oplus \gamma^8\delta^{100}$,
- ▷ minimum headways correspond to $\triangle_{ii}^{min} = \delta^6$ for $1 \leq i \leq 6$ and $\triangle_{ii}^{min} = \delta^5$ for $7 \leq i \leq 10$,
- ▷ maximum time separation between buses is $\triangle_{ii}^{max} = \gamma^0\delta^9$ for $1 \leq i \leq 6$ and $\triangle_{ii}^{max} = \gamma^0\delta^7$ for $7 \leq i \leq 10$,
- ▷ a departure at stop x_2 must occur between dates 105 and 107 and a departure at stop x_6 must occur between dates 60 and 62,
- ▷ buses are not allowed to stop more than 2 time units at stop x_9. Considering the travel time between x_8 and x_9, we have $\phi_{98} = \delta^6$.

The iterative computation defined in prop. 4 converges in 10 iterations providing the following timetable.

k	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
1	40	45	48	50	59	62	23	25	31	41
2	46	51	54	56	68	71	28	30	36	46
3	52	57	60	62	77	80	33	35	41	51
4	58	63	66	68	86	89	48	40	46	56
5	64	69	72	74	95	98	43	45	51	63
6	70	75	78	80	104	107	48	50	56	70
7	79	84	87	89	113	116	53	55	61	77
8	88	93	96	98	122	125	58	60	66	84
9	96	101	105	107	131	134	65	67	73	91
10	102	107	114	116	140	143	72	74	80	98

5. CONCLUSION

We have introduced a new method to compute just in time control for DEDS. Originality of this control is the possibility to take into account any constraint which can be expressed as an implicit inequality involving state vector. We apply this method to transportation systems, more particularly to the problem of timetable synthesis. The convergence of the iterative computation has ever since been proved and a real-life example has also been proposed in (Houssin *et al.*, 2006).

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