Inner approximations of the maximal positively invariant set for polynomial dynamical systems

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Abstract

The Lasserre or moment-sum-of-square hierarchy of linear matrix inequality relaxations is used to compute inner approximations of the maximal positively invariant set for continuous-time dynamical systems with polynomial vector fields. Convergence in volume of the hierarchy is proved under a technical growth condition on the average exit time of trajectories. Our contribution is to deal with inner approximations in infinite time, while former work with volume convergence guarantees proposed either outer approximations of the maximal positively invariant set or inner approximations of the region of attraction in finite time.

1 Introduction

This paper is an effort along a research line initiated in [5] for developing convex optimization techniques to approximate sets relevant to non-linear control systems subject to non-linear constraints, with rigorous proofs of convergence in volume.

The approximations are obtained by solving numerically a hierarchy of semidefinite programming or linear matrix inequality (LMI) relaxations, as proposed originally by Lasserre in the context of polynomial optimization [8]. Convergence proofs are achieved by exploiting duality between non-negative continuous functions and Borel measures, approximated respectively with sums of squares (SOS) of polynomials and moments, justifying the terminology moment-sum-of-square or Lasserre hierarchy. In the context of control systems, the primal moment formulation builds upon the notion of occupation measures [9] and the dual SOS formulation can be classified under Hamilton-Jacobi techniques [1].

Previous works along this line include inner approximations of the region of attraction [6], outer approximations of the maximal positively invariant (MPI) set [7], as well as outer approximations of the reachability set [3]. These techniques were applied e.g. in robotics [10] and biological systems [13]. In [5,6] the regions of attraction are defined for a finite time horizon, which is a technical convenient framework since the occupation measures have then finite mass. To cope with an infinite time horizon and MPI sets, a discount factor was added in [7] so that the mass of the occupation measure decreases fast enough when time increases. In [3], the mass was controlled by enforcing a growth condition on the volume of complement sets. This condition, difficult to check a priori, can be validated a posteriori using duality theory.

It must be emphasized here that, in general, the infinite time hoziron setup is more convenient for the classical Lyapunov framework and asymptotic stability, see e.g. [2] and references therein, whereas the finite time horizon setup is more convenient for approaches based on occupation measures. In the current paper, we make efforts to adapt the occupation measure framework to an infinite time horizon setup, at the price of technical difficulties similar to the ones already encountered in [3]. Contrary to the outer approximations derived in [7], we have not been able to use discounted occupation measures for constructing inner approximations. Instead, the technical device on which we relied is a growth condition of the average exit time of trajectories.

The main contributions of this work are:

1. A hierarchy for constructing inner approximations of the MPI set for a polynomial dynamic system with semialgebraic constraints;

2. A rigorous proof of convergence of the hierarchy, under an assumption on the average exit time of trajectories.

Section 2 presents the problem statement. Section 3 describes the MPI set inner approximation method. Section 4 includes the proof of convergence with appropriate assumptions. Numerical results are analyzed in Section 5. Conclusion and future work are discussed in Section 6.

2 Problem statement

Consider the autonomous system

$$\dot{x}(t) = f(x), x \in X \subset \mathbb{R}^n, t \in [0, +\infty[\qquad (1)$$

with a given polynomial vector field f of total degree δ_0. The state trajectory $x(.)$ is constrained to the interior $\text{int}(X)$ of a nonempty compact basic semi-algebraic set

$$X := \{x \in \mathbb{R}^n, g_i(x) \geq 0, i = 1, ..., n_X\}$$

where the g_i are polynomials of degree δ_i. Let $\partial X := X \setminus \text{int}(X)$ denote the boundary of X.

The vector field f is polynomial and therefore Lipschitz on the compact set X. As a result, for any $x_0 \in \text{int}(X)$, there exists a unique maximal solution $x(.|x_0)$ to ordinary differential equation (1) with initial condition $x(0|x_0) = x_0$. The time interval on which this solution is defined contains the time interval on which $x(.|x_0) \in \text{int}(X)$.

For any $t \in \mathbb{R}_+ \cup \{+\infty\}$, we define the following set:

$$X_t := \{x_0 \in \text{int}(X) : \forall s \in [0, t], x(s|x_0) \in \text{int}(X)\}.$$

With this notation, X_∞ is the set of all initial states generating trajectories staying in $\text{int}(X)$ *ad infinitum*: X_∞ is the MPI set included in $\text{int}(X)$. Indeed, for any $x_0 \in X_\infty$ and $t \geq 0$, by definition, $x(t|x_0) \in X_\infty$.

The complementary set $X_t^c := \text{int}(X) \setminus X_t$ is the set of initial conditions generating trajectories reaching the target set ∂X at any time before t: this is the region of attraction of ∂X with free final time lower than t. The complementary set X_∞^c is the region of attraction of ∂X with free and unbounded final time.

In this paper we want to approximate the MPI set X_∞ from inside as closely as possible.

3 Inner approximations of the MPI set

This section presents an infinite dimensional linear programming problem (LP) and a hierarchy of convex linear matrix inequality (LMI) relaxations yielding inner approximations of the MPI set.

3.1 Infinite dimensional LP

Consider the following infinite dimensional LP

$$
\begin{aligned}
d = \quad & \inf \quad \mu_0(X) \\
& \text{s.t.} \quad \nabla v \cdot f(x) \leq 0, \forall x \in X \\
& \qquad w(x) \geq v(x) + 1, \forall x \in X \\
& \qquad w(x) \geq 0, \forall x \in X \\
& \qquad v(x) \geq 0, \forall x \in \partial X
\end{aligned}
\tag{2}
$$

where the infimum is with respect to $v \in C^1(X)$ and $w \in C^0(X)$. It is worth noting that the constraint $\nabla v \cdot f(x) \leq 0$ is similar to the one of Lyapunov theory. However, it is here used in a completely different way, since v is not required to be positive outside ∂X.

Theorem 1 *Let (v, w) be a feasible pair for problem* (2). *Then, the set $\hat{X}_\infty := \{x \in int(X) : v(x) < 0\}$ is a positively invariant subset of X_∞.*

Proof: Since X_∞ is the MPI set included in X and $\hat{X}_\infty \subset X$ by definition, it is sufficient to prove that $\hat{X}_\infty$ is positively invariant.

Let $x_0 \in \hat{X}_\infty$. Then, for any $t > 0$, it holds $v(x(t|x_0)) = v(x_0) + \int_0^t \frac{d}{dt} \left(v(x(s|x_0)) \right) ds = v(x_0) + \int_0^t \nabla v \cdot f(x(s|x_0)) ds \leq v(x_0) < 0$ using the Lyapunov-like constraint in problem (2).

We still have to show that $x(t|x_0)$ remains in $int(X)$ at all times $t \geq 0$. If not, then there exists $t > 0$ such that $x(t|x_0) \in \partial X$ according to the intermediate value theorem, the trajectory being of course continuous in time. However, by feasibility of (v, w), one then has $v(x(t|x_0)) \geq 0$, which is in contradiction with the fact that $v(x(t|x_0)) < 0$ for all $t > 0$ which we just proved.

Thus, we obtain that for all $t > 0$, $x(t|x_0) \in int(X)$ and $v(x(t|x_0)) < 0$, i.e. $x(t|x_0) \in \hat{X}_\infty$. $\square$.

This shows that the set $\hat{X}_\infty$ is an inner approximation of X_∞.

Remark 1 *The decision variable w as well as the cost $\int_X w(x) \, d\lambda(x)$ are introduced, as in [5], to maximize the volume of the computed $\hat{X}_\infty$. It can be compared to the so-called "outer iterations" of the expanding interior algorithm presented in [11].*

3.2 SDP tightening

In what follows, $\mathbb{R}_k[x]$ denotes the vector space of real multivariate polynomials of total degree less than or equal to k, and $\Sigma_k[x]$ denotes the cone of sums of squares (SOS) of polynomials of degree less than or equal to k.

For $i = 0, \ldots, n_X$ let $k_i := \lceil \delta_i / 2 \rceil$. Let $k_{min} := \max\limits_{i=0,\ldots,n_X} k_i$ and $k \geq k_{min}$. The infinite dimensional LP (2) has an SOS tightening that can be written

$$
\begin{aligned}
d_k = \quad &\inf \quad \mathbf{w}'\mathbf{l} \\
&\text{s.t.} \quad -\nabla v \cdot f = q_0 + \sum_i q_i\, g_i \\
&\qquad\quad w - v - 1 = p_0 + \sum_i p_i\, g_i \\
&\qquad\quad w = s_0 + \sum_i s_i\, g_i \\
&\qquad\quad v = t_0 + \sum_i t_i^+\, g_i - \sum_i t_i^-\, g_i
\end{aligned}
\tag{3}
$$

where the infimum is with respect to $v \in \mathbb{R}_{2k}[x]$, $w \in \mathbb{R}_{2k}[x]$, $q_i, p_i, s_i, t_i^+, t_i^- \in \Sigma_{2(k-k_i)}[x]$, $i = 1, \ldots, n_X$, and q_0, p_0, s_0, t_0 SOS polynomials with appropriate degree. Vector $\mathbf{l}$ denotes the Lebesgue moments over X indexed in the same basis in which the polynomial $w(x)$ with vector of coefficients $\mathbf{w}$ is expressed.

SOS problem (3) is a tightening of problem (2) in the sense that any feasible solution in (3) gives a pair (v, w) feasible in (2).

Theorem 2 *Problem (3) is an LMI problem and any feasible solution gives an inner approximation $\hat{X}_\infty^k := \{x \in int(X), v(x) < 0\}$ of the MPI set.*

Proof: For the equivalence between SOS and LMI, see e.g. [8] and references therein. The inner approximation result is a direct consequence of Theorem 1. $\square$.

4 Convergence of the inner approximations

Besides providing a convex formulation to the problem of inner approximation of the MPI set X_∞, the Lasserre hierarchy framework allows to prove the convergence of our approximations $\hat{X}_\infty^k$ to the actual X_∞ in the sense of the Lebesgue measure, which has not been done so far in the Lyapunov framework.

However, due to the infinite time horizon, such a strong result is available only under some assumptions. It is based on the primal formulation of the MPI set computation problem.

For a given $x_0 \in X_\infty^c$, we define the exit time as

$$
\tau(x_0) := \inf\{t \geq 0 : x(t|x_0) \notin int(X)\}.
$$

In the rest of this paper we make the assumption that the average exit time of trajectories leaving $int(X)$ is finite:

Assumption 1 $\overline{\tau} := \frac{1}{\lambda(X)} \int_{X_\infty^c} \tau(x)dx < +\infty$.

4.1 Primal LP

For a given $T \in \mathbb{R}_+$, we define the following infinite-dimensional LP

$$
\begin{aligned}
p^T = \quad & \sup \quad \mu_0(X) \\
& \text{s.t.} \quad \mathrm{div}(f\mu) + \mu_\partial = \mu_0 \\
& \qquad \mu_0 + \hat{\mu}_0 = \lambda \\
& \qquad \mu(X) \leq T\,\lambda(X)
\end{aligned}
\tag{4}
$$

where the supremum is with respect to measures $\mu_0 \in \mathscr{M}^+(X)$, $\hat{\mu}_0 \in \mathscr{M}^+(X)$, $\mu_\partial \in \mathscr{M}^+(\partial X)$ and $\mu \in \mathscr{M}^+(X)$ with $\mathscr{M}^+(A)$ denoting the cone of non-negative Borel measures supported on the set A. The symbol λ denotes the n-dimensional Lebesgue measure on X.

Remark 2 *Here, T is introduced to ensure that all the feasible measures have a finite norm in total variation $|\mu| := \mu(X) < +\infty$. Otherwise, the optimization problem would be ill-posed.*

The two following lemmas link the infinite-dimensional LP (4) and the MPI set X_∞.

Lemma 3 *Assuming that $T \geq \bar{\tau}$, we have $p^T \geq \lambda(X_\infty^c)$.*

Proof: The quadruplet $(\mu_0 := \lambda_{X_\infty^c}, \hat{\mu}_0 := \lambda_{X_\infty}, \mu := A \mapsto \int_{X_\infty^c} \int_0^{\tau(x_0)} \mathbb{1}_A(x(t|x_0))\, dt\, dx_0, \mu_\partial := A \mapsto \int_{X_\infty^c} \mathbb{1}_A(x(\tau(x_0)|x_0))dx_0)$ is feasible. Indeed, one has: $\mu(X) = \int_{X_\infty^c} \int_0^{\tau(x_0)} dx_0 = \bar{\tau}\,\lambda(X) \leq T\,\lambda(X)$, $\mu_0 + \hat{\mu}_0 = \lambda$, and the first constraint in (4) is satisfied, since $\forall v \in C^1(X)$ it holds $\langle \mathrm{div}(f\mu), v \rangle = \langle \mu, -\nabla v \cdot f \rangle = -\int_{X_\infty^c} \int_0^{\tau(x_0)} \nabla v(x(t|x_0)) \cdot f(x(t|x_0)) dt dx_0 = -\int_{X_\infty^c} \left(\int_0^{\tau(x_0)} \frac{\mathrm{d}}{\mathrm{d}t}(v(x(t|x_0)))\, dt \right) dx_0 = -\int_{X_\infty^c} (v(x(\tau(x_0)|x_0)) - v(x_0))\, dx_0 = \langle \mu_0 - \mu_\partial, v \rangle$ where the braces $\langle, \rangle$ denote integration. Then $p^T \geq \mu_0(X) = \lambda_{X_\infty^c}(X) = \lambda(X_\infty^c)$. $\square$

Lemma 4 *For any quadruplet $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$ feasible in (4), μ_0 is supported on X_∞^c, i.e. $\mu_0(X_\infty) = 0$.*

The proof of this lemma uses the following assumption on the MPI set:

Assumption 2 *For all $x \in X_\infty \cap \partial X$ it holds $f(x) \cdot n(x) < 0$, where $n(x)$ stands for the unit normal vector to ∂X pointing towards $\mathbb{R}^n \setminus X$.*

In words, Assumption 2 means that at all points where X_∞ is tangent to X, the trajectories strictly enter X. Up to the choice of X, this seems to be reasonable for any physical system.

Proof: Let $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$ be a feasible quadruplet for (4). Let $\nu := \operatorname{div}(f\mu) = \mu_0 - \mu_\partial \in \mathcal{M}(X)$. For $x \in \mathbb{R}^n$, let

$$
\varphi(x) := \begin{cases} K \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}
$$

where $K > 0$ is such that $\int \varphi \, d\lambda = 1$. Then, for $\varepsilon > 0$ and $x \in \mathbb{R}^n$, let

- $\varphi_\varepsilon(x) := \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) \geq 0$,

- $\mu_\varepsilon(x) := \int_X \varphi_\varepsilon(y - x) \, d\mu(y) \geq 0$,

- $\nu_\varepsilon(x) := \operatorname{div}(f\mu_\varepsilon)(x)$.

According to the theory of mollifiers, φ, φ_ε, μ_ε and ν_ε are smooth compactly supported functions, and for any $w \in C^0(\mathbb{R}^n)$ compactly supported,

$$
\int_{\mathbb{R}^n} w(x) \, \mu_\varepsilon(x) \, dx \xrightarrow[\varepsilon \to 0]{} \int_X w(x) \, d\mu(x)
$$

from which it directly follows that for $v \in C^1(\mathbb{R}^n)$ compactly supported, it holds $\int_{\mathbb{R}^n} v(x) \, \nu_\varepsilon(x) \, dx = \int_{\mathbb{R}^n} v(x) \operatorname{div}(f\mu_\varepsilon)(x) \, dx = -\int_{\mathbb{R}^n} \nabla v(x) \cdot f(x) \, \mu_\varepsilon(x) \, dx \xrightarrow[\varepsilon \to 0]{} -\int_{\mathbb{R}^n} \nabla v(x) \cdot f(x) \, d\mu(x) = \int_{\mathbb{R}^n} v(x) \, d\nu(x)$. By uniform density of $C^1_c(\mathbb{R}^n)$ in $C^0_c(\mathbb{R})$, this implies that $\nu_\varepsilon \lambda$ weakly converges (in the sense of measures) to ν.

Then, let $\delta > 0$. We consider the set $X_\delta := \left\{ x \in X_\infty, \inf_{y \in \partial X} |x - y| > \delta \right\}$. By definition, $X_\delta \cap \partial X = \emptyset$, and then for any Borel set $A \subset X_\delta$, one has $\nu(A) = \mu_0(A)$. In particular, $\nu(\partial X_\delta) = \mu_0(\partial X_\delta) = 0$ since $\mu_0 \leq \lambda$. Then, we can apply the Portmanteau lemma (equality marked with a $*$) to $\nu(X_\delta)$, we get $\mu_0(X_\delta) = \nu(X_\delta) \overset{*}{=} \lim_{\varepsilon \to 0} \int_{X_\delta} \nu_\varepsilon(x) \, dx \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \int_{X_\delta} \operatorname{div}(f\mu_\varepsilon)(x) \, dx = \lim_{\varepsilon \to 0} \int_{\partial X_\delta} f(x) \cdot n_\delta(x) \, \mu_\varepsilon(x) \, dx$ where n_δ stands for the unit normal vector to ∂X_δ pointing towards X_δ^c, according to Stokes' theorem. Now, let Δ be the function

$$
\begin{aligned}
\partial X_\infty \cap \partial X &\longrightarrow \mathbb{R}_+ \\
x &\longmapsto \sup \left\{ \begin{array}{c} \Delta > 0, \forall \delta \in (0, \Delta), \forall y \in \partial X_\delta \\ |x - y| < \Delta \implies f(y) \cdot n_\delta(y) \leq 0 \end{array} \right\}.
\end{aligned}
$$

In words, $\Delta(x)$ is the largest range around x within which $f \cdot n_\delta$ is non-positive. According to Assumption 2, f being continuous, Δ takes only positive values.

Moreover, due to the regularity of f, X and X_∞, Δ is continuous on the compact set $\partial X_\infty \cap \partial X$, therefore it attains a minimum $\Delta^* > 0$.

Let $\delta \in (0, \Delta^*)$, $x \in \partial X_\delta$. Then, there are two possibilities:

- either $x \in \partial X_\infty$, and then by positive invariance of X_∞, $f(x) \cdot n_\delta(x) \leq 0$;

- or $\inf_{y \in \partial X} |x - y| = \delta < \Delta^*$, and by definition of Δ^*, $f(x) \cdot n_\delta(x) \leq 0$.

It follows that for any $x \in \partial X_\delta$, $f(x) \cdot n(x) \leq 0$. Thus, one obtains $\int_{\partial X_\delta} f(x) \cdot n_\delta(x) \mu_\varepsilon(x) \, dx \leq 0$ and after letting ε tend to 0, we have $\mu_0(X_\delta) \leq 0$, which means, by non-negativity of μ_0, that $\mu_0(X_\delta) = 0$.

Eventually, we note that constraint $\mu_0 \leq \lambda$ ensures that the function $\delta \longmapsto \mu_0(X_\delta)$ is continuous, which leads to the conclusion that $\mu_0(X_\infty) = \lim_{\delta \to 0} \mu_0(X_\delta) = 0$.

$\square$.

Theorem 5 *Assuming that $T \geq \overline{\tau}$, the infinite-dimensional LP (4) has value $p^T = \lambda(X_\infty^c)$. Moreover the supremum is attained, and the μ_0 component of any solution is necessarily the measure $\lambda_{X_\infty^c}$.*

Proof: This is a straightforward consequence of Lemmas 3 and 4. $\square$.

4.2 Dual LP

For a given $T \in \mathbb{R}_+$, the dual LP of (4) reads

$$
\begin{aligned}
d^T = \quad &\inf \quad \int_X (w(x) + uT) \, d\lambda(x) \\
&\text{s.t.} \quad \nabla v \cdot f(x) \leq u, \forall x \in X \\
&\qquad\ w(x) \geq v(x) + 1, \forall x \in X \\
&\qquad\ w(x) \geq 0, \forall x \in X \\
&\qquad\ v(x) \geq 0, \forall x \in \partial X
\end{aligned}
\tag{5}
$$

where the infimum is with respect to $u \geq 0$, $v \in C^1(X)$ and $w \in C^0(X)$.

Remark 3 *Problem (5) is very similar to the initial problem (2), but with an additional slack variable u related to the constraint $\mu(X) \leq T\lambda(X)$ in the primal (4). For $u = 0$, (2) and (5) are equivalent. Otherwise, there is no guarantee that the solution of (5) yields an inner approximation of X_∞. We will see that under Assumption 1, $u = 0$ can always be enforced.*

Lemma 6 *For any triplet (u, v, w) feasible in (5) and for any $t > 0$, it holds $\{x_0 \in int(X), v(x_0) + ut < 0\} \subset X_t$. In particular, if $(0, v, w)$ is feasible in (5), then the set $\{x_0 \in int(X), v(x_0) < 0\}$ is included in X_∞ and it is positively invariant.*

Proof: Let (u, v, w) be a feasible triplet in (5) and let x_0 be a element of X_t^c for a given $t > 0$.

By definition of X_t we know that $t \geq \tau(x_0)$, where τ is the exit time, and that for any $s \in [0, \tau(x_0)], x(s|x_0) \in X$. Thanks to the first constraint in (5), we can therefore say that for any $s \in [0, \tau(x_0)], (\nabla v \cdot f)(x(s|x_0)) \leq u$. This can be written as:

$$\forall s \in [0, \tau(x_0)], \frac{d(v(x(s'|x_0)))}{ds'}\Big|_{|s'=s} \leq u.$$

Hence for any $s \in [0, \tau(x_0)], v(x(s|x_0)) \leq v(x_0) + us$. In particular, we deduce that

$$v(x(\tau(x_0)|x_0)) \leq v(x_0) + u\tau(x_0) \leq v(x_0) + ut.$$

As $x(\tau(x_0)|x_0) \in \partial X$, we know that $v(x(\tau(x_0)|x_0)) \geq 0$ and thus $v(x_0) \geq -ut$. This proves that

$$X_t^c \subset \{x_0 \in \text{int}(X), v(x_0) \geq -ut\}$$

hence

$$\{x_0 \in \text{int}(X), v(x_0) + ut < 0\} \subset X_t.$$

Let us suppose now that $(0, v, w)$ is a feasible triplet in (5). Let x_0 be an element of X such that $v(x_0) < 0$. Applying the first result with $u = 0$, we know that for all $t > 0$, $x_0 \in X_t$ thus $x_0 \in X_\infty$. This proves that $\{x_0 \in \text{int}(X), v(x_0) < 0\} \subset X_\infty$, from which we deduce the positive invariance of this set using the property that v decreases along trajectories staying in X. $\square$

Theorem 7 *There is no duality gap between primal LP problem* (4) *on measures and dual LP problem* (5) *on functions, i.e.* $p^T = d^T$.

Proof: Here we only outline the basic steps; for a detailed argument in a similar setting see [5, Theorem 2]. The feasible set of (4) is non-empty since it contains the trivial solution $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial) = (0, \lambda, 0, 0)$. Moreover, for any feasible quadruplet $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$, we have that $\mu_0(X) = \mu_\partial(X) \leq \lambda(X), \hat{\mu}_0(X) \leq \lambda(X)$ and $\mu(X) \leq T \lambda(X)$. Therefore $0 \leq p^T < \infty$ and the feasible set is weakly-* bounded. The absence of a duality gap then follows from Alaoglu's theorem and the weak-* continuity of the operator $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial) \to (\text{div}(f\mu) + \mu_\partial - \mu_0, \mu_0 + \hat{\mu}_0)$. $\square$

4.3 LMI relaxations

Throughout the rest of this section we make the following standard standing assumption:

Assumption 3 *One of the polynomials modeling the set X is equal to* $g_i(x) = R^2 - |x|^2$.

This assumption is without loss of generality since a redundant ball constraint can be always added to the description of the bounded set X.

The primal LMI or moment relaxation of order k reads

$$
\begin{aligned}
p_k^T = \;\; &\sup \;\; (y_0)_0 \\
&\text{s.t.} \;\; A_k(y, y_0, \hat{y}_0, y_\partial) = b_k \\
&\qquad (y)_0 \leq T\,\lambda(X) \\
&\qquad M_k(y) \succeq 0, M_{k-k_i}(g_i, y) \succeq 0, i = 1, 2, \ldots, n_X \\
&\qquad M_k(y_0) \succeq 0, M_{k-k_i}(g_i, y_0) \succeq 0, i = 1, 2, \ldots, n_X \\
&\qquad M_k(\hat{y}_0) \succeq 0, M_{k-k_i}(g_i, \hat{y}_0) \succeq 0, i = 1, 2, \ldots, n_X \\
&\qquad M_k(y_\partial) \succeq 0, i = 1, 2, \ldots, n_X \\
&\qquad M_{k-k_i}(g_i, y_\partial) \succeq 0, i = 1, 2, \ldots, n_X \\
&\qquad M_{k-k_i}(-g_i, y_\partial) \succeq 0, i = 1, 2, \ldots, n_X
\end{aligned}
\tag{6}
$$

where the notation $\succeq 0$ stands for positive semi-definite and the minimum is over moments sequences $(y_0, \hat{y}_0, y, y_\partial)$ truncated to degree $2k$ corresponding to measures $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$. The LMI constraints involve moment and localizing matrices not described here for the sake of brevity, see e.g. [5] in a similar context, or the comprehensive monograph [8]. The linear equality constraint captures the two linear equality constraints of (4) with $v \in \mathbb{R}_{2k}[x]$ and $w \in \mathbb{R}_{2k}[x]$ being monomials of total degree less than or equal to $2k$.

The dual LMI problem of (6) can be formulmated as an SOS problem

$$
\begin{aligned}
d_k^T = \;\; &\inf \;\; \mathbf{w}'\mathbf{l} + u\,T\,\mathbf{l}_0 \\
&\text{s.t.} \;\; u - \nabla v \cdot f = q_0 + \sum_i q_i\, g_i \\
&\qquad w - v - 1 = p_0 + \sum_i p_i\, g_i \\
&\qquad w = s_0 + \sum_i s_i\, g_i \\
&\qquad v = t_0 + \sum_i t_i^+ g_i - \sum_i t_i^- g_i
\end{aligned}
\tag{7}
$$

where the infimum is with respect to $u \geq 0$ and the same decision variables as in problem (3).

SOS problem (7) is a tightening of problem (5) in the sense that any feasible solution in (7) gives a triplet (u, v, w) feasible in (5).

Lemma 8 *Let $T \geq 0$ and $k \geq k_{min}$. Then,*

1. *$p_k^T = d_k^T$ i.e. there is no duality gap between the primal LMI (6) and the dual LMI (7).*

2. *The optimum of primal LMI (6) is attained.*

3. *For any $t > 0$ and for any feasible solution (u_k, v_k, w_k) of dual LMI (7), it holds*
$$
\hat{X}_t^k := \{x \in int(X), v_k(x) + u_k\, t < 0\} \subset X_t.
$$

In particular, if $u_k = 0$, we have then

$$\hat{X}_\infty^k := \{x \in int(X), v_k(x) < 0\} \subset X_\infty$$

.

Proof:

1. Follows by the same arguments based on standard semidefinite programming duality theory as the proof of [5, Theorem 4]. The main argument is the non-emptiness and compactness of the feasible set of the primal problem. It is non-empty since the zero moment vector is feasible. Boundedness of the even components of each moment vector follows from the structure of the localizing matrices corresponding to the functions from Assumption 3 and from the fact that the masses (zero order moments) of the measures are bounded. Boundedness of the whole moment vectors then follows since the even moments appear on the diagonal of the positive semidefinite moment matrices.

2. The second point derives also from the non-emptiness and compactness of the feasible set of the primal problem.

3. This is a straightforward consequence of Lemma 6.

□.

Theorem 9 *Let $T > \overline{\tau}$. Then,*

1. *The sequences (p_k^T) and (d_k^T) are monotonically decreasing and converging to $\lambda(X_\infty^c)$.*

For every $k \geq k_{min}$, let (u_k, v_k, w_k) denote a $\frac{1}{k}$-optimal solution of the dual tightening of order k. One has then :

2. $u_k \xrightarrow[k \to \infty]{} 0$

3. $w_k \xrightarrow[k \to \infty]{L^1(X)} \mathbb{1}_{X_\infty^c}$.

Proof:

1. The proof of this point follow exactly the same principle as the proof of [5, Theorem 5], therefore we detail only the main ideas. Using Stone-Weierstrass, one can prove that there exists a minimizing sequence of (5) where the w and v components are polynomials. This step exploits the fact that the variable u is free and is not constrained to be zero. One can conclude using the classical Positivstellensatz by Putinar, as in e.g. [8].

2. We define the quadruplet $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$ as follows:

   - $\mu_0 := \lambda_{X_\infty^c}$, $\hat{\mu}_0 := \lambda_{X_\infty}$,
   - $\mu := A \mapsto \int_{X_\infty^c} \int_0^{\tau(x_0)} \mathbb{1}_A(x(t|x_0)) \, dt \, dx_0$,
   - $\mu_\partial =: A \mapsto \int_{X_\infty^c} \mathbb{1}_A(x(\tau(x_0)|x_0)) dx_0$.

   Quadruplet $(\mu_0, \hat{\mu}_0, \mu, \mu_\partial)$ has already been proven to be an optimal solution of (6). Using standard duality method, one can prove that :

   $$< w_k, \lambda > + u_k T\lambda(X) \geq < w_k, \lambda > + u_k \mu(X) \geq \mu_0(X).$$

   Since $< w_k, \lambda > + u_k T\lambda(X) \longrightarrow d^T = p^T = \mu_0(X)$ we can deduce that for any accumulation point u^* of the sequence u_k, we have $u^* T\lambda(X) = u^* \mu(X)$. Since $T\lambda(X) > \mu(X)$ it proves that any accumulation point of the sequence u_k is zero. Noticing moreover that the sequence is included in the compact interval $[0, \frac{p_{k_{min}}^T}{T\lambda(X)}]$, this proves that $u_k \longrightarrow 0$.

3. Let $\varepsilon > 0$. Let $t > 0$ such that $\lambda(X_t \setminus X_\infty) \leq \varepsilon$. Let $\bar{k} \geq k_{min}$ such that for all $k \geq \bar{k}$ one has that $\|u_k t\|_{L^1(X)} \leq \varepsilon$ and $|\int_X w_k d\lambda - \lambda(X_\infty^c)| \leq \varepsilon$. Such an integer exists from points 1 and 2. Using the triangular inequality and the fact that $\|u_k t\|_{L^1(X)} \leq \varepsilon$ one has

   $$\|w_k - \mathbb{1}_{X_\infty^c}\|_{L^1(X)} \leq \|w_k + u_k t - \mathbb{1}_{X_\infty^c}\|_{L^1(X)} + \varepsilon. \qquad (8)$$

   With the notation $\Delta = \|w_k + u_k t - \mathbb{1}_{X_\infty^c}\|_{L^1(X)}$, one has that

   $$\Delta = \int_{X_t^c} |w_k + u_k t - \mathbb{1}_{X_\infty^c}| d\lambda + \int_{X_t} |w_k + u_k t - \mathbb{1}_{X_\infty^c}| d\lambda.$$

   We denote Δ_1 and Δ_2 these two terms, respectively. Using that $X_t^c \subset X_\infty^c$ and that $w_k(x) + u_k t \geq 1 + v_k(x) + u_k t \geq 1, \forall x \in X_t^c$ (from point 3 of Lemma 8) we have then that

   $$\Delta_1 = \int_{X_t^c} w_k + u_k t - 1 \, d\lambda = \int_{X_t^c} w_k d\lambda - \lambda(X_t^c) + \lambda(X_t^c) u_k t$$

   and since $\lambda(X_t^c) u_k t \leq \|u_k t\|_{L^1(X)} \leq \varepsilon$,

   $$\Delta_1 \leq \int_{X_t^c} w_k d\lambda - \lambda(X_t^c) + \varepsilon. \qquad (9)$$

   Moreover, we have that

   $$\Delta_2 \leq \int_{X_t} |w_k| + |u_k t| + |\mathbb{1}_{X_\infty^c}| d\lambda$$

and therefore using that $w_k \geq 0$ and that $\|u_k t\|_{L^1(X)} \leq \varepsilon$:

$$\Delta_2 \leq \int_{X_t} w_k d\lambda + \varepsilon + \lambda(X_t \setminus X_\infty).$$

Since we have $\lambda(X_t \setminus X_\infty) \leq \varepsilon$ by choice of t, we deduce that $\Delta_2 \leq \int_{X_t} w_k d\lambda + 2\varepsilon$. Combining this inequality with (9), we have :

$$\Delta = \Delta_1 + \Delta_2 \leq \int_X w_k d\lambda - \lambda(X_t^c) + 3\varepsilon$$

from which we deduce that $\Delta \leq 5\varepsilon$, using that $|\int_X w_k d\lambda - \lambda(X_\infty^c)| \leq \varepsilon$ and $\lambda(X_\infty^c \setminus X_t^c) \leq \varepsilon$. Combining this with (8), we have that $\|w_k - \mathbb{1}_{X_\infty^c}\|_{L^1(X)} \leq 6\varepsilon$.

$\square$.

5 Numerical examples

For this paper, we chose to focus on the simple example of the Van der Pol oscillator, as was done in [5]. Thus, we consider the two-dimensional ODE

$$\dot{x}_1 = -2 \, x_2 \tag{10a}$$

$$\dot{x}_2 = 0.8 \, x_1 + 10 \, (\alpha^2 x_1^2 - 0.2) \, x_2 \tag{10b}$$

with $\alpha = 1.02$. Let $X = \{x \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1\}$ and $T = \frac{100}{\pi}$.

We implemented the hierarchy of SOS problems (7) in Matlab using the toolbox YALMIP interfaced with the SDP solver MOSEK. For the 6th and 7th tightenings (SOS degrees 12 and 14 respectively), we compared the obtained regions to the outer approximations computed using the framework presented in [7], see Figure 1.

Here the MPI set is tangent to the unit circle at some points; as a consequence, the inner approximations are tangent to the unit circle and the outer approximations (which are identical for $k = 6$ and 7) exceed the unit disk. In this implementation, we checked at each relaxation whether u was near to zero: for $k = 6$, we get $u \sim 10^{-7}$, and for $k = 7$ we get $u \sim 10^{-6}$, which is satisfactory. Moreover, we compared our results with those obtained by applying the SOS tightenings (3) of problem (2) (i.e. by forcing $u = 0$ in the hierarchy), and we obtained the same inner approximations.
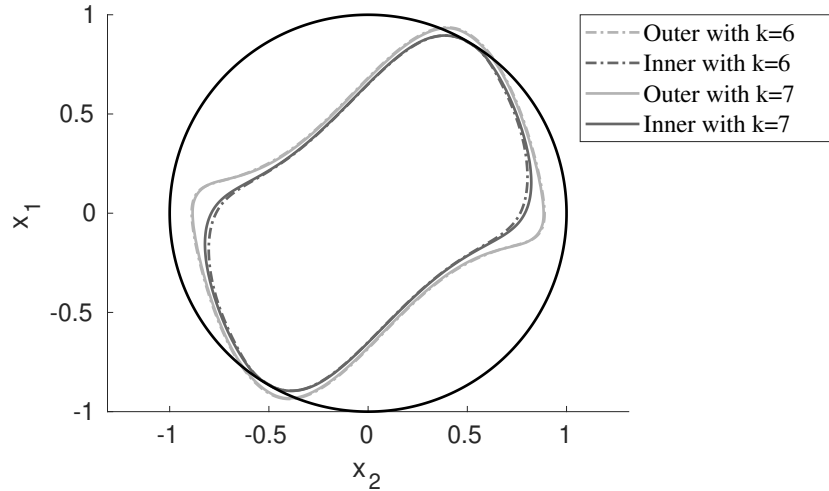
However, we observed some difficulties:

Figure 1: Outer and inner approximations of the Van der Pol MPI set in the unit disk.

- For low degrees, the only solution found by the solver is very close to the zero polynomial: the coefficients are of the order 10^{-5}, therefore the plots are irrelevant; one loses conservativeness and several constraints are violated (namely the positivity constraint on v on ∂X).

- For higher degrees, the basis of monomials is not adapted since x^α is close to the indicator of the unit circle. As a result, the coefficients are of the order 10^5 or more, and again the plots make little sense.

One can also find numerical applications of this method to actual eletrical engineering problems in [12] with very promising results.

6 CONCLUSIONS

The original motivation behind our current work is the study of transient phenomena in large-scale electrical power systems, see [4] and references therein. Our objective is to design a hierarchy of approximations of the MPI set for large-scale systems described by non-linear differential equations. A first step towards non-polynomial dynamics can be found in [12]. Since the initial work [5] relied on the mathematical technology behind the approximation of the volume of semi-algebraic sets, we already studied in [14] the problem of approximating the volume of a large-scale sparse semi-algebraic set. We are now investigating extensions of the techniques for approximating the MPI set of large-scale sparse

dynamical systems, and the current paper contributes to a better understanding of its inner approximations, in the small-scale non-sparse case. Our next step consists of combining the ideas of [14] with those of the current paper, so as to design a Lasserre hierarchy of inner approximations of the MPI set in the large-scale case, and apply it to electrical power system models.

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