Operator theoretical methods for dynamical systems control and optimization

European Control Conference workshop of July 14th, 2015 organized by
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1 Abstract

Nonconvex control and optimization problems for nonlinear dynamical systems can be approached with numerical methods inspired by operator theory. The workshop is an opportunity to present for the first time in a unified way two major operator theoretical approaches to nonlinear dynamical systems:

- Koopman operator methods for dynamical systems, relying on Galerkin numerical discretization techniques;
- polynomial optimization and optimal control formulated as generalized problems of moments, discretized by hierarchies of convex linear matrix inequalities, and solved numerically with semidefinite programming.

These techniques are applied to compute regions of attraction and invariant sets, and to solve dynamical systems problems arising in neuroscience.

2 Duration

This is a full-day workshop.

3 Presenters

3.1 Didier Henrion

D. Henrion is a CNRS Researcher at LAAS in Toulouse, France. He is also a Professor at the Faculty of Electrical Engineering of the Czech Technical University in Prague, Czech Republic. He is interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory. See `homepages.laas.fr/henrion`

3.2 Milan Korda

M. Korda is a PhD Student at Ecole Polytechique Fédérale de Lausanne, Switzerland, under the supervision of C. N. Jones. He has collaborated with D. Henrion on the topic of LMI relaxations for computing regions of attractions and invariant sets of nonlinear systems control. See `people.epfl.ch/milan.korda`

3.3 Alexandre Mauroy

A. Mauroy is a Postdoctoral Researcher at the University of Liège, Belgium. His research lies at the intersection of dynamical systems theory, control theory, and operator theory. After a PhD in Applied Mathematics supervised by R. Sepulchre at the University of Liège, A. Mauroy spent two years as a Postdoctoral Researcher at the University of California at Santa Barbara, supervised by I. Mezić. See `www.montefiore.ulg.ac.be/~mauroy`

3.4 Igor Mezić

I. Mezić is a Professor at the Department of Mechanical Engineering of the University of California at Santa Barbara, USA. He is interested in reformulation of dynamical systems theory utilizing spectral properties of the Koopman operator and its relationship with geometrical properties of dynamical systems in high dimensions, and under uncertainty. See `www.engr.ucsb.edu/~mgroup/joomla`

4 Goals

The objective of the workshop is to explain how control and optimization problems for nonlinear systems can be solved with numerical methods inspired by operator theory.

The workshop is also an opportunity to present for the first time in a unified way two major operator theoretical approaches to dynamical systems:

- Koopman operator methods for dynamical systems, studied by I. Mezić's group at UCSB, see e.g. [3];
- polynomial optimization and optimal control formulated as generalized problems of moments, studied by J. B. Lasserre and D. Henrion at LAAS-CNRS, see e.g. [15, 6].

5 Contents and topics

The key idea between the two approaches mentioned above is the observation that a nonlinear ordinary differential equations (ODE) can be interpreted as a linear partial differential equations (PDE) in the space of probability measures or in the dual space of observables. Historically, this idea can be tracked back to the early 19th century. It was Joseph Liouville in 1838 who first introduced the linear PDE involving the Jacobian of the transformation exerted by the solution of an ODE on its initial condition [17]. The idea was then largely expanded in Henri Poincaré's work on dynamical systems at the end of the 19th century, see in particular [27, Chapitre XII (Invariants intégraux)].

This work was pursued in the 20th century in [11], [26, Chapter VI (Systems with an integral invariant)] and more recently in the context of optimal transport by e.g. [33] or [1]. Poincaré himself in [28, Section IV] mentions the potential of formulating nonlinear ODEs as linear PDEs, and this programme has been carried out to some extent by [4], see also [12], [10], [8].

In the context of dynamical systems, the idea of turning a nonlinear system into a linear but infinite-dimensional system is supported by operator theory. In particular, nonlinear systems can be described by the so-called Perron-Frobenius operator on the space of probability measures, or by its adjoint Koopman operator on the space of observable functions [12]. Since these two operators are linear, they are amenable to spectral analysis, thereby providing a powerful alternative approach to the systems. For instance, it has been shown in [24, 25] that the spectral properties of the Koopman operator are directly related to geometric properties of ergodic systems. This interplay was also exploited to study dissipative systems. In this case, the spectral properties of the operator can be used for sensitivity analysis [19, 21] and global stability analysis [20]. The approach is also conducive to data analysis [31, 32, 5].

In the context of polynomial optimization and optimal control, the approach consists of formulating a nonlinear nonconvex problem as a linear programming (LP) problem in the cone of nonnegative measures, with a convex dual in the cone of nonnegative continuous functions. The infinite-dimensional LPs are then truncated to finite-dimensional linear matrix inequalities (LMI) solved numerically with interior-point algorithms for semidefinite programming (SDP) [2]. The approach is described in [14] to address polynomial optimal control problems, and in [7, 9] to compute regions of attractions and invariant sets of polynomial dynamical systems, see also [18] for applications in robotics. The use of LMI and measures (with densities) was also investigated in [29] for building Lyapunov barrier certificates, and based on a dual to Lyapunov's theorem described in [30].

6 Intended audience and prerequisites

The workshop should be of interest to PhD students or more experienced scholars having some background either in a) systems control theory, b) operator theory, c) convex optimization, or d) dynamical systems, and who are willing to investigate connections between these topics.

7 Schedule

The proposed workshop schedule (9:00-18:00) is as follows:

- Morning session focusing more on theoretical background
  - 9:00-9:45: General background on convex optimization and polynomial optimization (D. Henrion): We survey linear programming in finite-dimensional cones, with a focus on the semidefinite cone, or cone of non-negative quadratic

forms, see e.g. [2, 6]. We then leverage this knowledge to infinite-dimensional cones of non-negative continuous functions and non-negative Borel measures on compact supports, and show its relevance in the context of polynomial optimization [13, 15, 6].

  - 10:00-10:45: General background on dynamical systems (I. Mezić): We introduce the representation of a dynamical system in terms of a linear composition operator (the Koopman operator). We present the formalism in the context of linear systems first, to recover and reinterpret the classical spectral expansion. Then we derive the so-called Koopman Mode Expansion for general nonlinear (and potentially nonsmooth) systems [25, 3].
  - 11:15-12:00: LMI relaxations for generalized problems of moments and their applications (D. Henrion): Using infinite-dimensional linear programming duality, we build Lasserre's hierarchy of moment-sum-of-squares relaxations for polynomial optimization [13, 15, 6] and motivate its extension to generalized problems of moments in polynomial optimal control [14, 15].
  - 12:15-13:00: Numerical methods for ergodic partition theory (I. Mezić): We concentrate on the first term in the Koopman model analysis, that contains system invariants. We describe all the possible invariants of the dynamical system. We discuss a connection between ergodic sets in dynamical systems and reachable sets in control theory. Then, we introduce numerical methods for computation of ergodic sets and ergodic partition [23, 16, 3].

- Afternoon session focusing more on applications

  - 14:00-14:45: Region of attraction computations for polynomial dynamical systems (M. Korda): This part of the workshop describes how to use the concept of occupation measures to characterize the region of attraction as an infinite dimensional linear program in the cone of nonnegative measures. The infinite dimensional linear program is then relaxed to a finite-dimensional semidefinite program providing outer approximations to the region of attraction [7]. We discuss theoretical issues (such as the existence and uniqueness of the solutions to the Liouville's equation) as well as practical issues such as numerical issues arising when solving the SDP relaxations.
  - 15:00-15:45: Global stability analysis for nonlinear systems using the eigenfunctions of the Koopman operator (A. Mauroy): The operator-theoretic framework yields a novel approach to stability analysis, which mirrors the spectral stability analysis of linear systems. In this context, we will show that the existence of specific eigenfunctions of the Koopman operator implies global stability of the attractor. Several systematic methods for computing these eigenfunctions will be presented, which are related to the use of moments in the dual space.
  - 16:15-17:00: Invariant set computations for polynomial dynamical systems (M. Korda): This part of the workshop describes how to handle problems on infinite horizon using the previously developed techniques of occupation measures. We describe the discounted infinite-time Liouville's equation and discuss the relation of its solutions to the trajectories of the controlled system. Then we

apply these techniques to characterize the maximum controlled invariant set as well as the value function of an infinite-time discounted optimal control problem as an infinite-dimensional LP in the cone of nonnegative measures. SDP relaxations of this LP provide converging hierarchy of outer approximations to the MCI set and approximations from below to the value function [9].

– 17:15-18:00: Neuroscience applications: isochrons and isostables (A. Mauroy): The spectral analysis of the Koopman operator reveals geometric properties of the system that play a central role for phase reduction and sensitivity analysis. We will show that these results are of great interest in computational neuroscience: they lead to a novel interpretation of the concept of isochron and enable to extend this notion to excitable neuron models. The results are also instrumental in obtaining new numerical schemes that can be used to compute the highly complex isochrons of bursting neuron models [22].

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