# Analysis of dissipative nonlinear systems using the eigenfunctions of the Koopman operator

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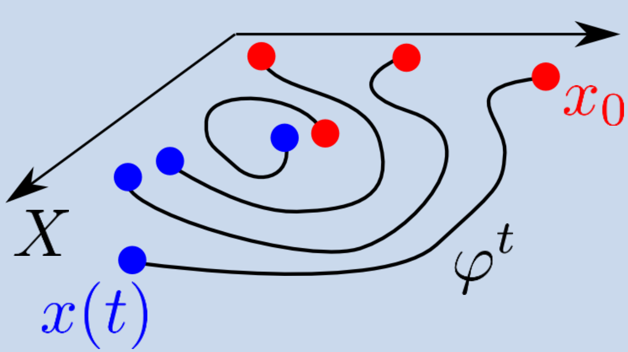
# Two descriptions of dynamical systems

**Trajectory-oriented approach**

➔ pointwise description

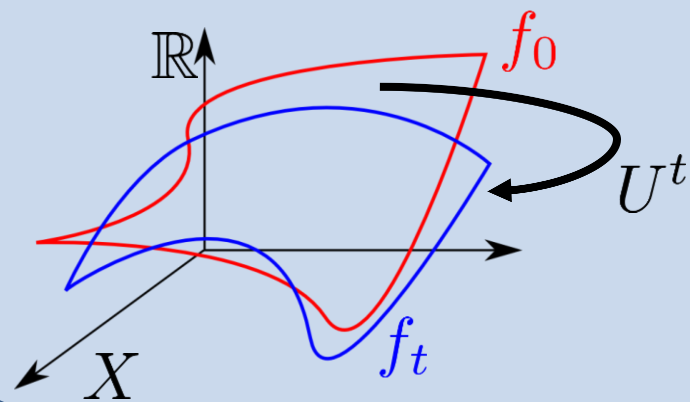
SYSTEM ≡ flow $\varphi^t : X \to X$
acting on the state space




**Operator-theoretic approach**

➔ global description

SYSTEM ≡ operator $U^t : \mathcal{G} \to \mathcal{G}$
acting on a functional space



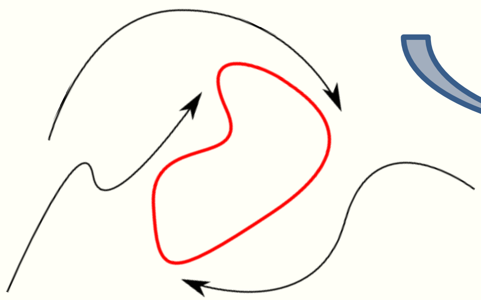

**nonlinear**

lifting

**linear**

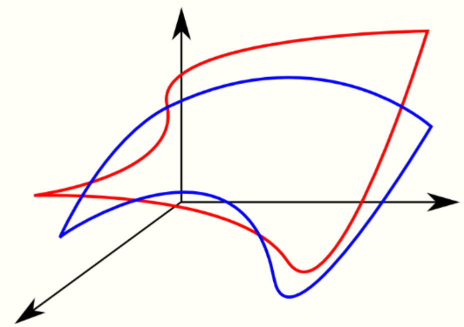
# The spectral properties of the Koopman operator capture the global stability properties of the system

**Nonlinear dissipative system**



lifting

**Koopman operator**



**Geometric properties**
**➔ Stability**

**Spectral properties**

➔ systematic spectral analysis of nonlinear systems

# Outline

Operator-theoretic approach and first stability results

From Koopman eigenfunctions to global stability

Systematic numerical methods for global stability analysis

Differential framework: contraction and positivity

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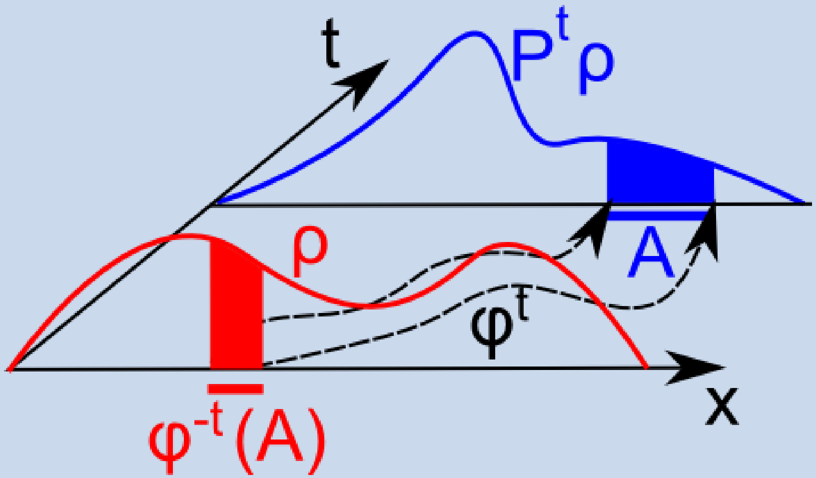
# A dynamical system is described by two dual operators

**Perron-Frobenius operator** $P^t$

*[Liouville 1838, Poincaré, Ulam 1960]*

$\rho \in \mathcal{F}^\dagger$ is a density

$$\int_A P^t \rho(x)\, dx = \int_{\varphi^{-t}(A)} \rho(x)\, dx$$

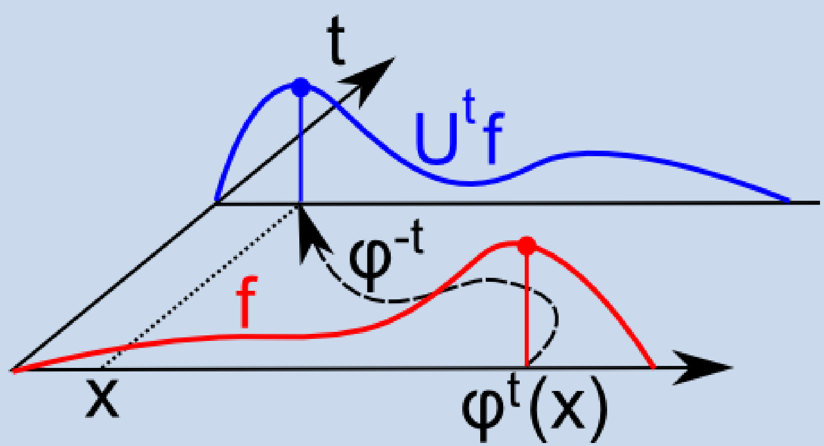



**Koopman operator** $U^t$

*[Koopman 1930]*

$f \in \mathcal{F}$ is an observable

$$U^t f(x) = f \circ \varphi^t(x)$$




Duality: $\displaystyle\int_{\mathcal{S}} (U^t f(x))\, \rho(x)\, dx = \int_{\mathcal{S}} f(x)\, (P^t \rho(x))\, dx$ $\qquad \langle U^t f, \rho \rangle = \langle f, P^t \rho \rangle$

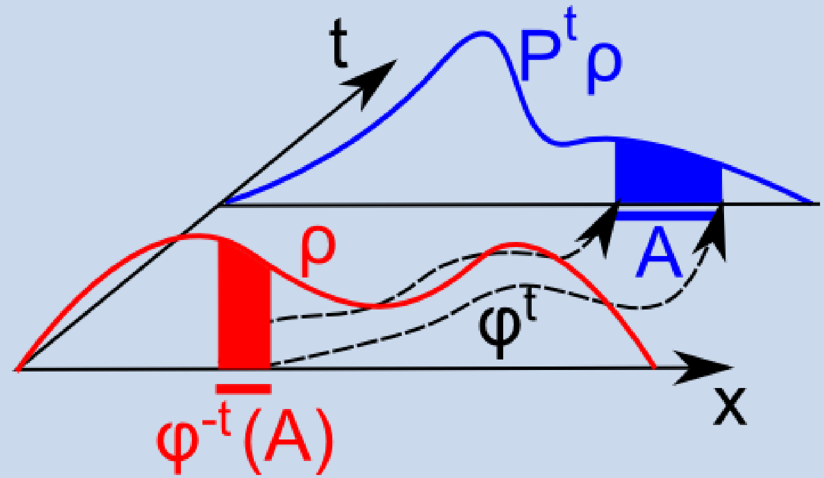
# A dynamical system is described by two dual operators

**Perron-Frobenius operator** $P^t$

[Liouville 1838, Poincaré, Ulam 1960]

$\mu \in \mathcal{M}$ is a measure

$$\int_A P^t \mu(dx) = \int_{\varphi^{-t}(A)} \mu(dx)$$




**Koopman operator** $U^t$

[Koopman 1930]

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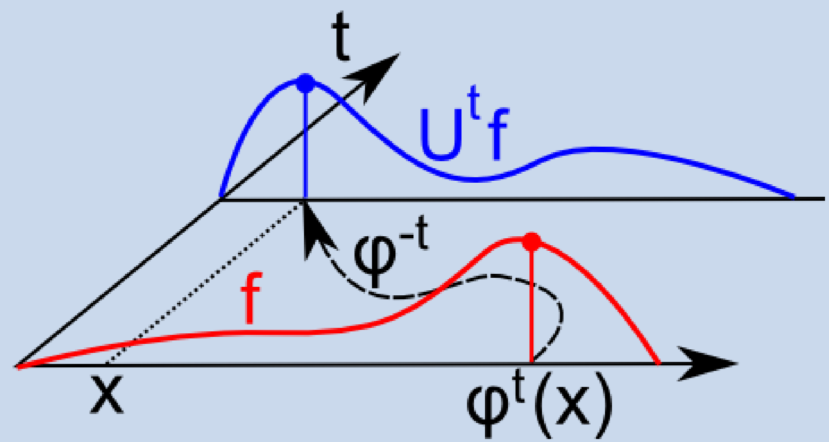



Duality: $\int_{\mathcal{S}} (U^t f(x))\, \mu(dx) = \int_{\mathcal{S}} f(x)\, P^t \mu(dx)$ $\qquad \langle U^t f, \mu \rangle = \langle f, P^t \mu \rangle$

# Operator theory provides a powerful insight into stability analysis

**Stability analysis**

Lyapunov function $\mathcal{V}$
c. 1890

≈ 100 years

Lyapunov density $\rho$
*[Rantzer 2001]*

$$\dot{\mathcal{V}} = \frac{d}{dt} U^t \mathcal{V} < 0$$

$$\dot{\rho} = \frac{d}{dt} P^t \rho < 0$$

*[Vaidya et al. 2008]*

**Operator theory**

Koopman operator
*[Koopman 1930]*

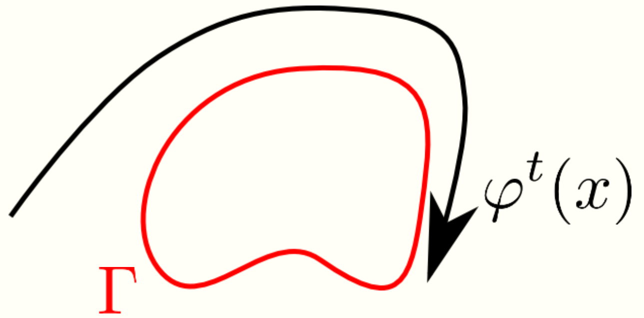
duality known
for decades!

Perron-Frobenius operator
< 1960 *[Ulam]*

# Stability properties are related to the properties of the Koopman operator

The attractor $X$ is globally attractive in $\Gamma$ if

$$\omega(x) = \bigcap_{T\in\mathbb{R}} \overline{\{\phi(t), t > T\}} \subseteq \Gamma \quad \forall x \in X$$




Let $\mathcal{F}_{X\setminus\Gamma} \subset \mathcal{F}$ be the subspace of functions with support on $X \setminus \Gamma$ , i.e.

$$\mathcal{F}_{X\setminus\Gamma} = \{f \in \mathcal{F} | f(x) = 0 \ \forall x \in \Gamma\}$$

Proposition: The attractor $\Gamma$ is globally attractive in $X$ iff

$$\lim_{t\to\infty} U^t f = 0 \qquad \forall f \in \mathcal{F}_{X\setminus\Gamma}$$

with $\mathcal{F} \subseteq C^0(X)$ .

☺ no assumption on the nature of the attractor(s) or on the flow

☹ but difficult to use in practise…

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Operator-theoretic approach and first stability results

From Koopman eigenfunctions to global stability

Systematic numerical methods for global stability analysis

Differential framework: contraction and positivity

# It is natural to consider the spectral properties of the Koopman operator

Dissipative systems

➔ eigenfunctions of the Perron-Frobenius operator ≡ Dirac measures

➔ consider Koopman eigenfunctions

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$

Koopman eigenvalue $\lambda \in \sigma(U)$

$\Longleftrightarrow$

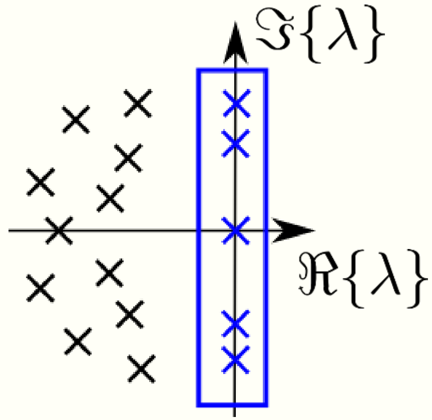
$$\frac{d}{dt} U^t \phi_\lambda = \lambda \phi_\lambda$$

$$U^t \phi_\lambda = e^{\lambda t} \phi_\lambda$$

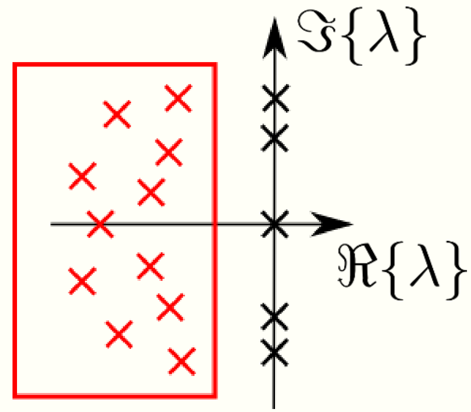
For the dynamics $\dot{x} = F(x)$:

eigenvalue equation $\underbrace{\boxed{F \cdot \nabla}}_{\equiv \frac{d}{dt} U^t} \phi_\lambda = \lambda \, \phi_\lambda$ if $\phi_\lambda \in C^1(X)$

# The spectrum of the Koopman operator can be decomposed in two parts

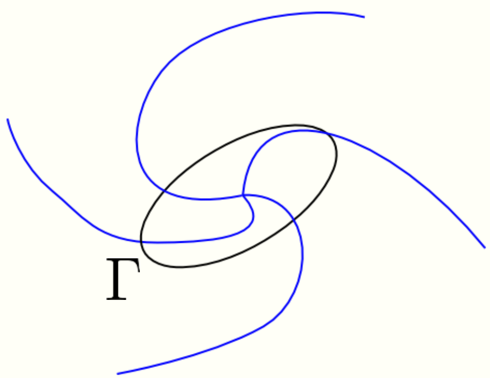



Spectrum of the Koopman operator acting on $\mathcal{F}|_\Gamma = \{f|_\Gamma : \Gamma \to \mathbb{C} | f \in \mathcal{F}\}$

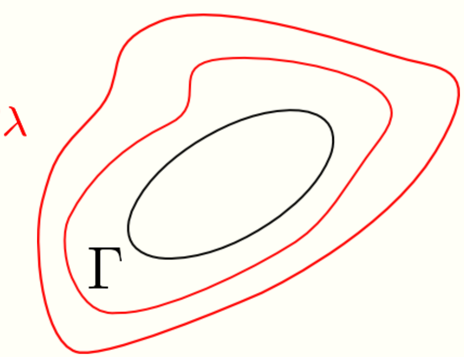



Spectrum of $U^t$ restricted to $\mathcal{F}_{X\backslash\Gamma} = \{f \in \mathcal{F} : f(x) = 0 \, \forall x \in \Gamma\}$

Level sets of $\phi_\lambda$ ≡ isochrons




Level sets of $\phi_\lambda$ ≡ isostables




➔ ergodic motion on $\Gamma$

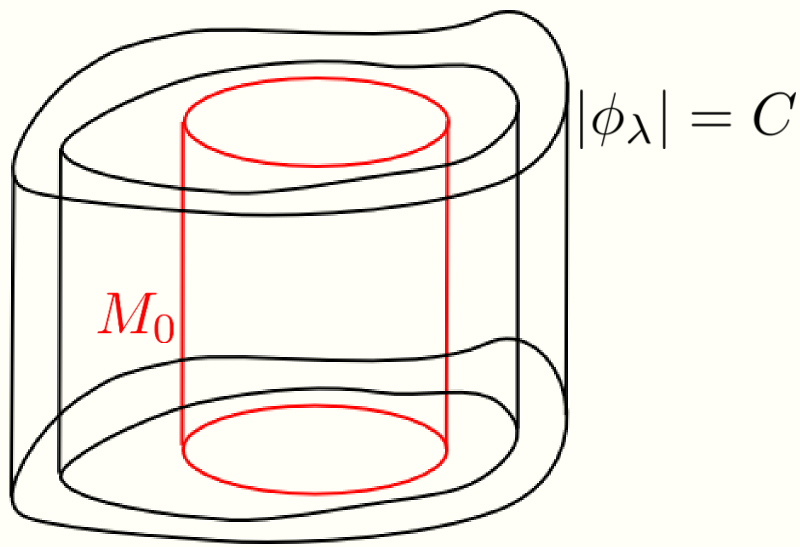
➔ asymptotic convergence to $\Gamma$

# Stability is captured by eigenfunctions that are zero on the attractor

**Theorem:** Assume that $X \subset \mathbb{R}^n$ is compact and forward invariant. If there exists an eigenfunction $\phi_\lambda \in C^0(X)$ with an eigenvalue $\lambda$ such that $\Re\{\lambda\} < 0$ , then the set

$$M_0 = \{x \in X | \phi_\lambda(x) = 0\}$$

is globally asymptotically stable in $X$ .



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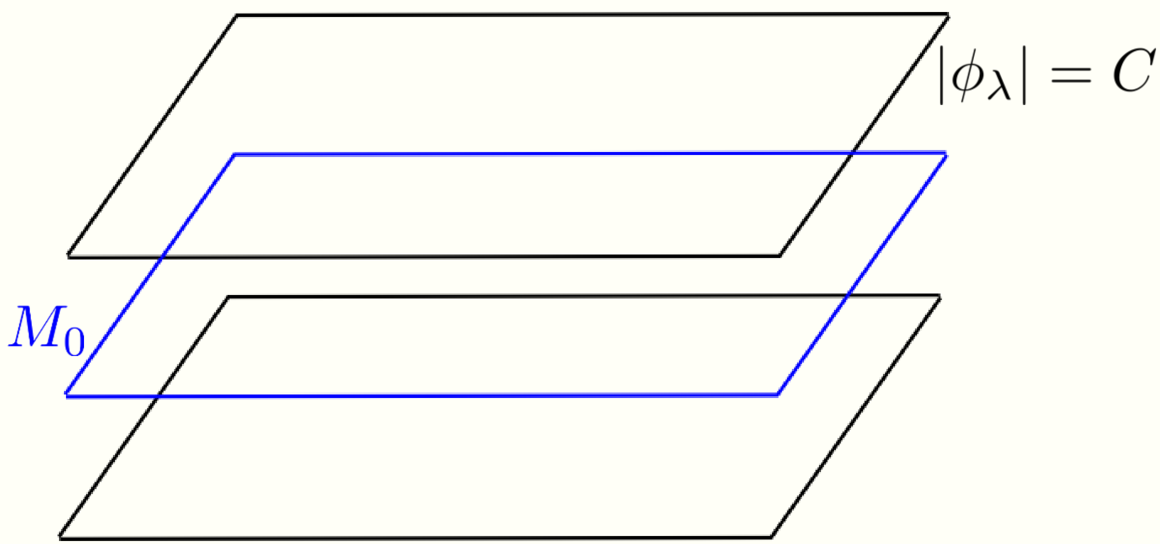
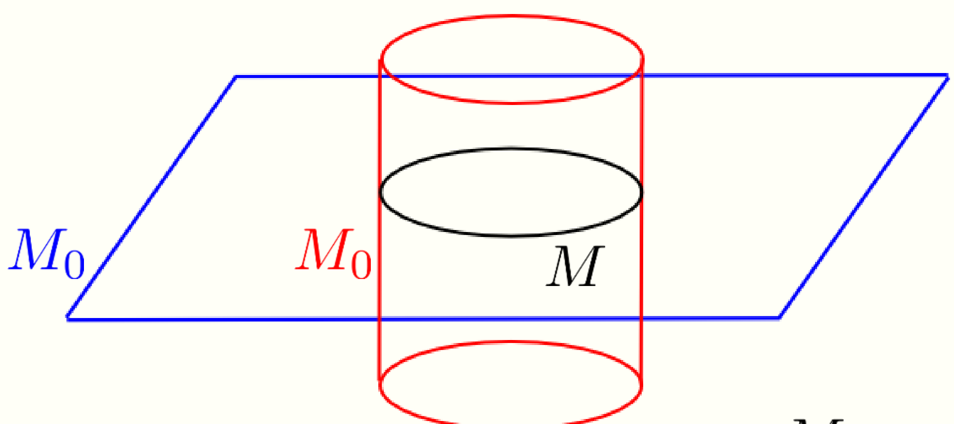


# Stability is captured by eigenfunctions that are zero on the attractor

**Theorem:** Assume that $X \subset \mathbb{R}^n$ is compact and forward invariant. If there exist eigenfunctions $\phi_{\lambda_i} \in C^0(X)$ with eigenvalues $\lambda_i$ such that $\Re\{\lambda_i\} < 0$, $i = 1, \ldots, m$, then the set

$$M = \bigcap_{i=1}^{m} \{x \in X | \phi_{\lambda_i}(x) = 0\}$$

is globally asymptotically stable in $X$.




e.g. $M$ is an unstable manifold

# In the case of a fixed point, we obtain a global equivalent of the well-known local stability result

The attractor is a hyperbolic fixed point $x^* \in X$

Jacobian matrix $\dfrac{\partial F}{\partial x}(x^*)$ has eigenvalues $\lambda_j$ and independent eigenvectors

$$\Longrightarrow \quad \lambda_j \in \sigma(U)$$

**Theorem:** Assume that $X \subset \mathbb{R}^n$ is compact, forward invariant and connected.
The fixed point $x^*$ is globally asymptotically stable in $X$ iff
(i) $\Re\{\lambda_j\} < 0$
(ii) there exist $n$ eigenfunctions $\phi_{\lambda_j} \in C^1(X)$ with $\nabla \phi_{\lambda_j}(x^*) \neq 0$

Linear system
$\dot{x} = Ax$

(i) $\boxed{\Re\{\lambda_j\} < 0}$

✓ (ii) $\phi_{\lambda_j} = w_j^T x \quad \in C^1(\mathbb{R}^n)$

left eigenvector of $A$

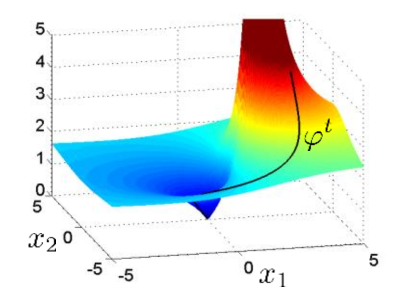
$$\nabla \phi_{\lambda_j}(x^*) = w_j \neq 0$$

*[AM and Mezic, IEEE TAC, in press]*

# We recover classical concepts

$$|\phi_\lambda \circ \varphi^t| = |U^t \phi_\lambda| \leq \exp(\Re\{\lambda_1 t\})|\phi_\lambda| \qquad \lambda_1 = \mathrm{argmax}_{\lambda \in \sigma(U)} \Re\{\lambda\}$$

**Lyapunov functions**



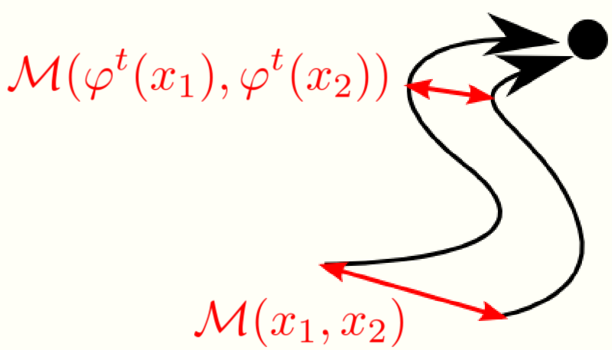

$$\mathcal{V}(x) = \left( \sum_{j=1}^{n} |\phi_{\lambda_j}(x)|^p \right)^{1/p}$$

$$\mathcal{V}(x^*) = 0$$

$$\mathcal{V}(\varphi^t(x)) \leq \exp(\Re\{\lambda_1\}t)\, \mathcal{V}(x)$$

**Contraction**

$$\mathcal{M}(x, y) = \left( \sum_{j=1}^{N} |\phi_{\lambda_j}(x) - \phi_{\lambda_j}(y)|^p \right)^{1/p}$$




$$\mathcal{M}(\varphi^t(x_1), \varphi^t(x_2)) \leq \exp(\Re\{\lambda_1\}t)\, \mathcal{M}(x_1, x_2)$$

# An equivalent result is obtained for global stability of the limit cycle

The attractor is a hyperbolic limit cycle $\Gamma \subset X$

Floquet matrix has eigenvalues $\lambda_j$ and independent eigenvectors $v_j$

$$\Longrightarrow \quad \lambda_j \in \sigma(U)$$

**Theorem:** Assume that $X \subset \mathbb{R}^n$ is compact, forward invariant and connected.
The limit cycle $\Gamma$ is globally asymptotically stable in $X$ iff
(i) $\Re\{\lambda_j\} < 0$ for $j = 1, \ldots, n-1$
(ii) there exist $n-1$ eigenfunctions $\phi_{\lambda_j} \in C^1(X)$
with $\nabla \phi_{\lambda_j}(x^\gamma) \cdot v_j \neq 0$ for all $x^\gamma \in \Gamma$

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# Koopman eigenfunctions can be obtained through Laplace averages

The Laplace average

$$f_\lambda^*(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\circ\varphi^t(x)\, e^{-\lambda t}\, dt$$

is a projection of $f$ on $\phi_\lambda(x)$

$\Rightarrow$ $\phi_\lambda(x) \triangleq f_\lambda^*(x)$ if it exists and is nonzero

☹ Not efficient for non dominant eigenfunctions

☹ Not efficient for limit cycle

☹ Need to compute trajectories

# The eigenfunctions can also be computed with Taylor expansions

Eigenvalue equation: $F \cdot \nabla \phi_\lambda = \lambda \, \phi_\lambda$ Assumption: $\phi_\lambda, \; F$ analytic

Taylor expansion of $\phi_\lambda$ (and $F$):

$$\phi_\lambda(x) = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \phi_\lambda^{(k_1,\ldots,k_n)} \, (x_1 - x_1^*)^{k_1} \cdots (x_n - x_n^*)^{k_n}$$

Basis of monomials : related to moments in the dual

We obtain algebraic equations that we solve by induction:

vector of $\phi_\lambda^{k_1,\ldots,k_n}$ with $\sum_j k_j = s$

$$(A^{(s)} - \lambda I) \boxed{\Phi_\lambda^{(s)}} = \boxed{b^{(s)}}$$

depends on $\Phi_\lambda^{(1)}, \ldots, \Phi_\lambda^{(s-1)}$

For $\sum_j k_j = 1$ $\qquad J^T(x^*) \nabla \phi_\lambda(x^*) = \lambda \nabla \phi_\lambda(x^*)$

# We can use the methods to estimate the basin of attraction of a fixed point

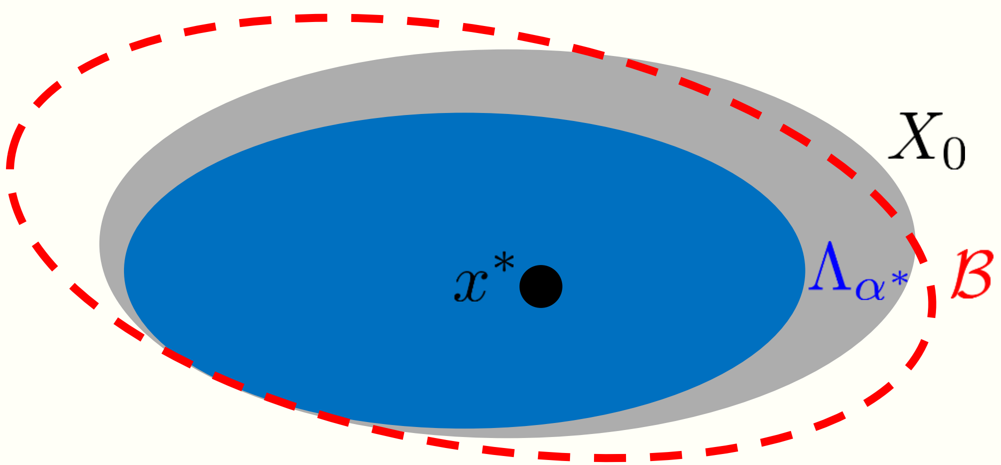
1. Compute the Koopman eigenfunctions $\phi_{\lambda_j}$

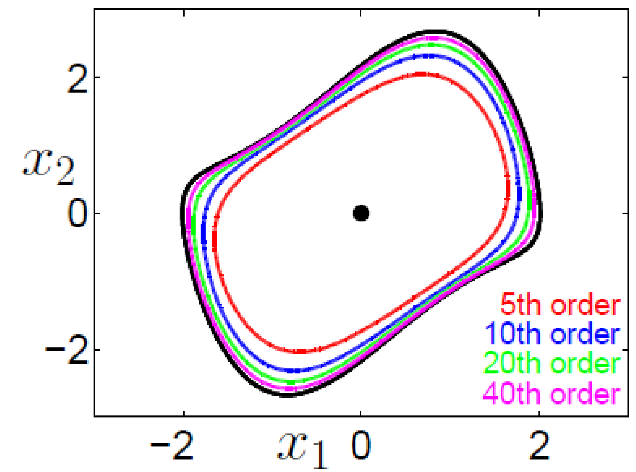
2. Construct a candidate Lyapunov function, e.g. $\mathcal{V} = \sqrt{\phi^2_{\lambda_1} + \cdots + \phi^2_{\lambda_n}}$

3. Identify the set $X_0 \subseteq X$ such that $\dot{\mathcal{V}}(x) = \nabla F(x) \cdot \mathcal{V}(x) < 0 \ \forall x \in X_0$

4. Consider the sets $\Lambda_\alpha = \{x \in X | V(x) \leq \alpha\}$ and find the largest set $\Lambda_{\alpha^*}$ such that $\Lambda_{\alpha^*} \subseteq X_0$

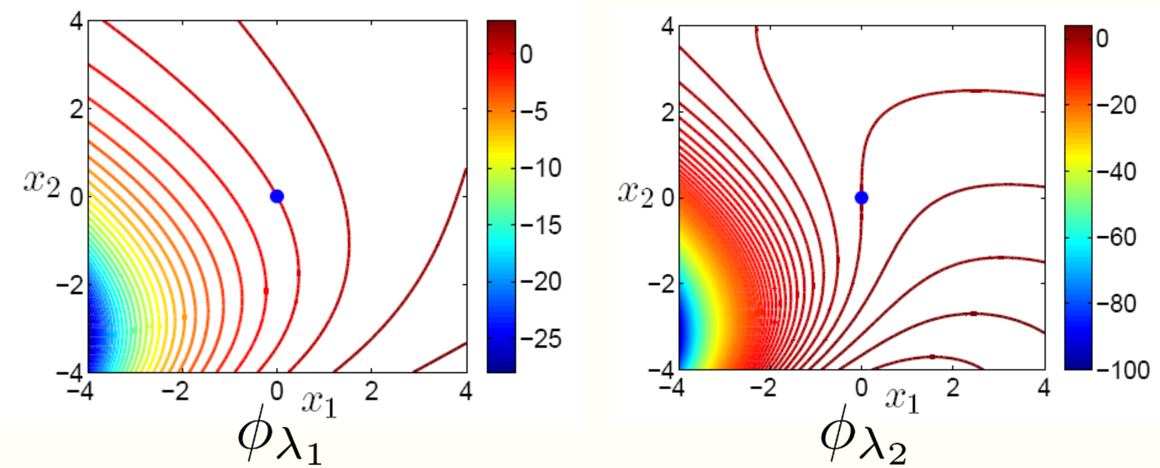
➔ $\Lambda_{\alpha^*}$ is an inner approximation of the basin of attraction $\mathcal{B}$



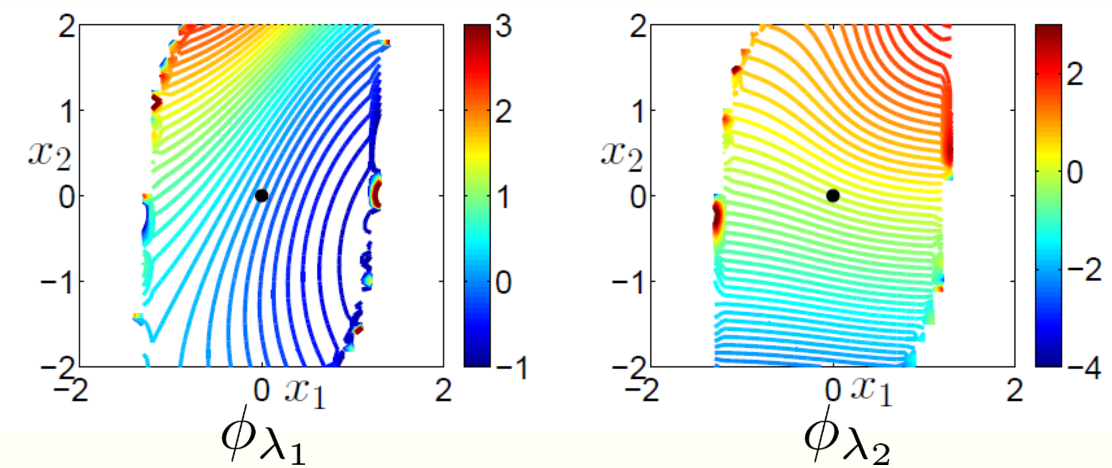



# The method fails when the eigenfunctions are not analytic

Global stability on $[-4,4]\times[-4,4]$




Global stability on $[-2,2]\times[-2,2]$




Analytic vector field,
but complex zeros (fixed points)

➔ non-analytic eigenfunctions!

# The Koopman eigenfunctions can be computed in a finite basis

We consider a finite (polynomial) basis $\{p_k(x), k = 1, \ldots, m\}$

e.g. Bernstein polynomials $b_k^n(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$

$$\phi_\lambda(x) = \sum_{k=1}^{m} \phi_\lambda^{(k)} p_k(x) = \Phi_\lambda^T P(x)$$ ➔ unknown: $\Phi_\lambda$

$$F \cdot \nabla \phi_\lambda = \lambda \phi_\lambda$$

$$\phi_\lambda(x^*) = 0$$

$$\nabla \phi_\lambda(x^*) = w$$

⇨

$$M\, D\, \Phi_\lambda = \lambda \boxed{T} \Phi_\lambda$$

$$P(x^*)^T \Phi_\lambda = 0$$

$$\nabla P(x^*)^T \Phi_\lambda = w$$

expansion in larger basis

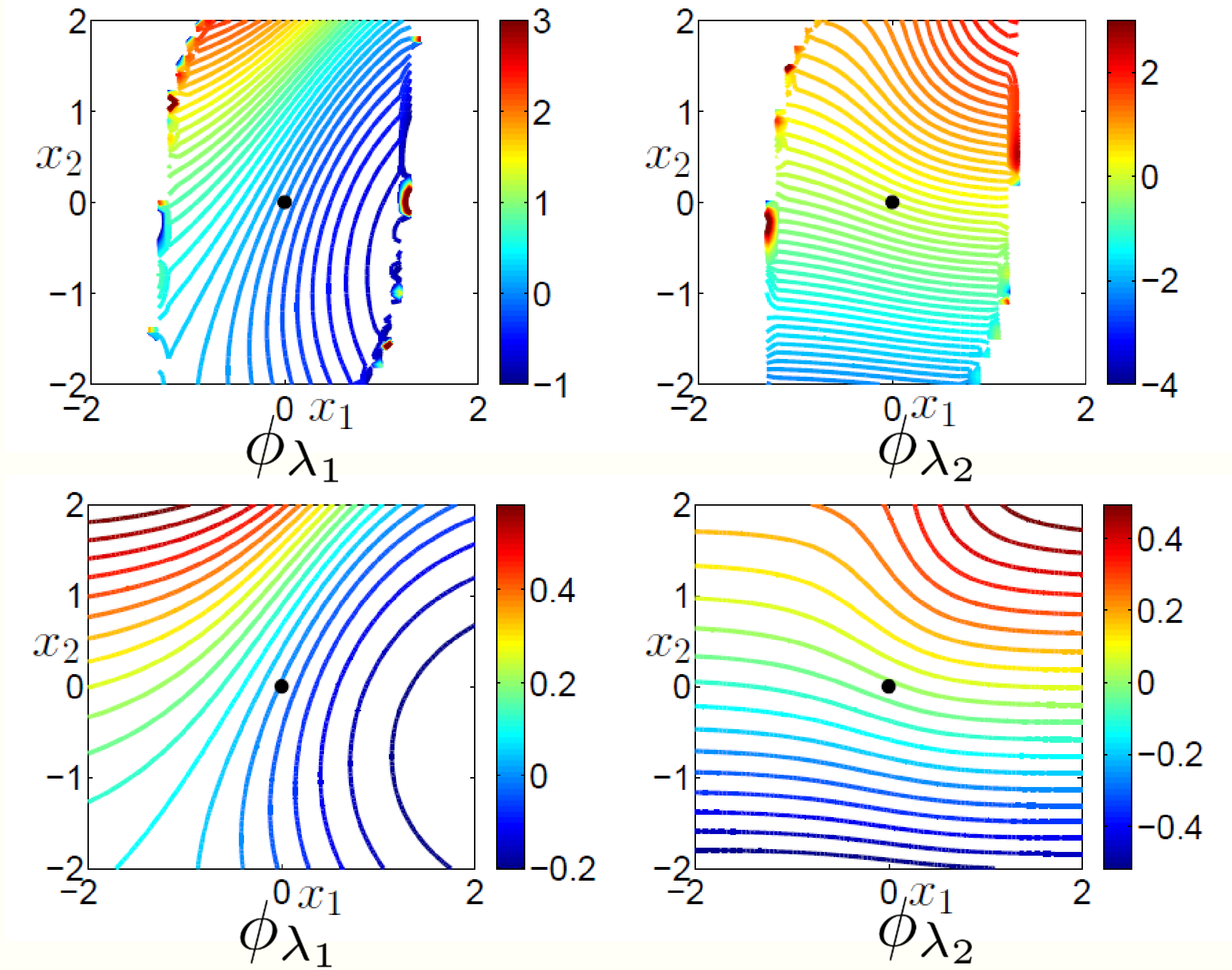
Over-determined system

➔ least-squares solution $\Phi_\lambda$

*[AM and Mezic, IEEE TAC, in press]*

# The method works when the eigenfunctions are not analytic

Taylor approximation




basis of Bernstein polynomials

Global stability on $[-2, 2] \times [-2, 2]$

But the method only proves or disproves global stability in the entire set

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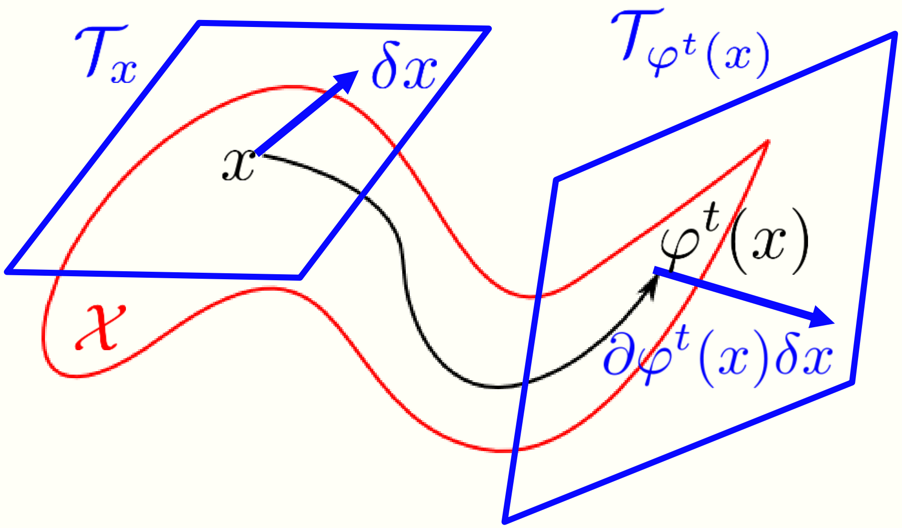
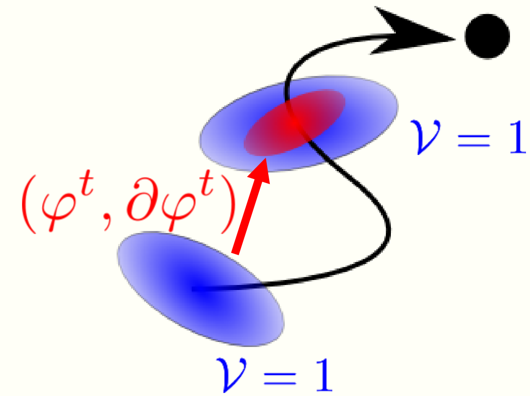
# We consider dynamical systems on manifolds




| | **Prolonged system** | **Flow** |
|---|---|---|
| $x \in \mathcal{X}$ | $\dot{x} = F(x)$ | $x \mapsto \varphi^t(x)$ |
| $\delta x \in T_x\mathcal{X}$ | $\dot{\delta x} = \partial F(x)\delta x$ | $\delta x \mapsto \partial\varphi^t(x)\delta x$ |

# Differential contraction of the system implies contraction of the Koopman operator

**Differential contraction**




Finsler-Lyapunov function $\mathcal{V}(x, \delta x)$

$$\mathcal{V}(\varphi^t(x), \partial\varphi^t(x)\delta x) < \mathcal{V}(x, \delta x) \qquad \forall x \in \mathcal{X},\ \forall t > 0$$

*[Forni and Sepulchre, IEEE TAC, 2014]*

**Contraction of the Koopman operator**

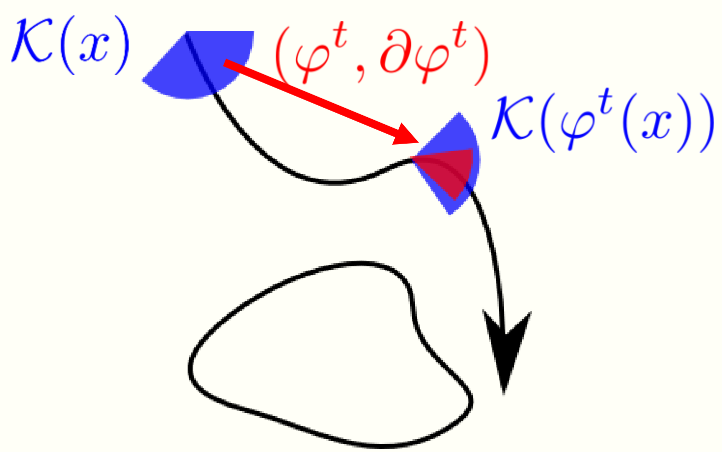
semi-norm $$\|f\| = \max_{x \in \mathcal{X}} \mathcal{V}^*(x, \partial f)$$

differential contraction with respect to $\mathcal{V}(x, \delta x)$ $\Rightarrow$ $\|U^t f\| < \|f\| \quad \forall t > 0$

**In terms of Koopman eigenfunctions:** $$\mathcal{V}(x, \delta x) = \left( \sum_{j=1}^{N} |\partial \phi_{\lambda_j}(x)\delta x|^p \right)^{1/p}$$

# Differential positivity of the system is equivalent to positivity of the Koopman operator

**Differential positivity**




cone field $\mathcal{K}(x) \in T_x\mathcal{X}$

$$\partial\varphi^t(x)\mathcal{K}(x) \subseteq \mathcal{K}(\varphi^t(x)) \qquad \forall x \in \mathcal{X},\ \forall t > 0$$

*[Forni and Sepulchre, IEEE TAC, 2015]*

**Positivity of the Koopman operator**

cone of functions $\mathcal{H}_\mathcal{K} \subset \mathcal{F}$ induced by $\mathcal{K}(x)$

$$\mathcal{H}_\mathcal{K} = \{f \in C^1(\mathcal{X}) \,|\, \partial f(x) \in \mathcal{K}^*(x)\}$$

differential positivity with respect to $\mathcal{K}(x)$ $\Leftrightarrow$ $U^t\mathcal{H}_\mathcal{K} \subseteq \mathcal{H}_\mathcal{K} \quad \forall t > 0$

*[Sootla & AM, subm. to MTNS 2016]*

Monotone system (i.e. $\mathcal{K}(x) = \mathbb{R}^n_{\geq 0}$) ➔ $\mathcal{H}_\mathcal{K}$ = cone of increasing functions

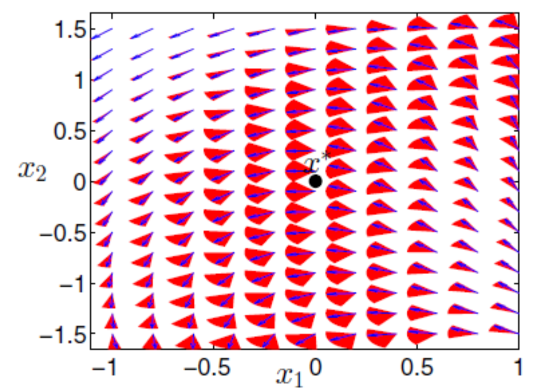
# Properties of positivity are studied through Koopman eigenfunctions

**Converse results for differential positivity**

$$\mathrm{argmax}_{\lambda\in\sigma(U)}\Re\{\lambda\} = \lambda_1 \in \mathbb{R}$$

$$\Rightarrow \mathcal{K}(x) = \left\{\delta x \in \mathcal{T}_x\mathcal{X} \,|\, \partial\phi_{\lambda_1}(x)\delta x - |\partial\phi_{\lambda_j}(x)\delta x| \geq 0,\ \forall j \geq 2\right\}$$

Systems with a hyperbolic stable node are differentially positive




*[AM, Forni and Sepulchre, CDC 2015]*

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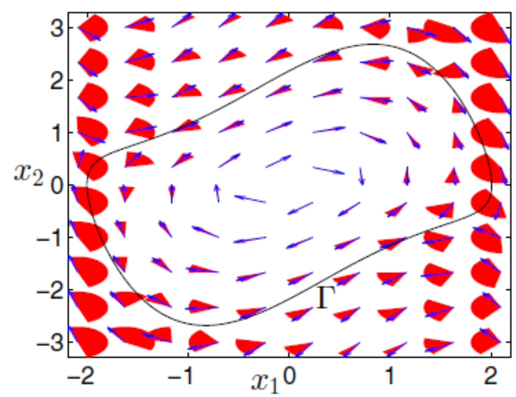
Systems with a hyperbolic stable limit cycle are differentially positive

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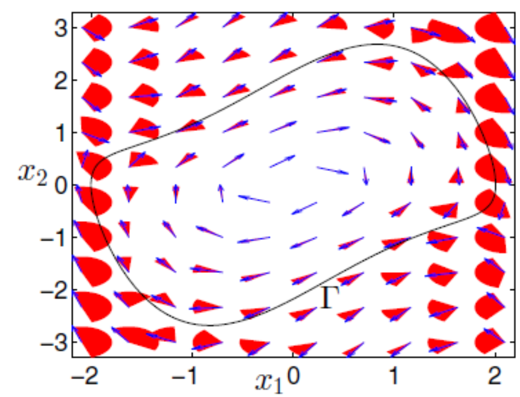



*[AM, Forni and Sepulchre, CDC 2015]*

**Eventual monotonicity**

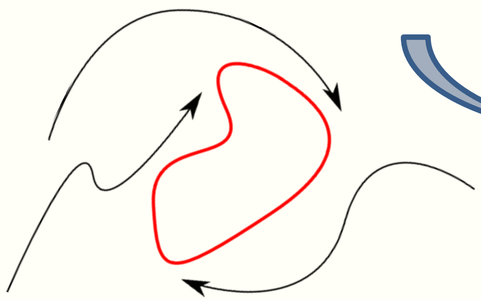
$$\partial\varphi^t\mathcal{K} \subseteq \mathcal{K} \quad \forall t > T \iff U^t\mathcal{H}_\mathcal{K} \subseteq \mathcal{H}_\mathcal{K} \quad \forall t > T$$

$$\Rightarrow \phi_{\lambda_1} \in \mathcal{H}_\mathcal{K} \Rightarrow \partial\phi_{\lambda_1}(x) \in \mathcal{K}^* \quad \forall x \in \mathcal{X}$$

*[Sootla and AM, http://arxiv.org/abs/1510.01149 ]*

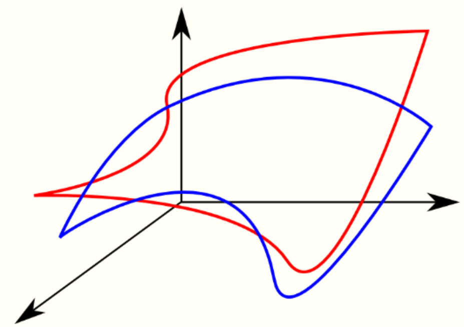
# The spectral properties of the Koopman operator capture the global stability properties of the system

**Nonlinear dissipative system**



lifting

**Koopman operator**



**Geometric properties**
**Stability, positivity, contraction**

**Spectral properties**

➔ systematic spectral analysis of nonlinear systems

# Open questions

Spectral stability conditions
for **non hyperbolic** attractors (continuous spectrum)

Complete characterization of **contraction and positivity**
of the Koopman operator

Spectral properties of the Koopman operator
for nonlinear **switched systems** (➔ joint spectral radius)

Koopman operator for **input-output systems**

Connection to / computations with **optimization-based methods**

# Analysis of dissipative nonlinear systems using the eigenfunctions of the Koopman operator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