Finite-dimensional predictor-based feedback stabilization of a 1D linear reaction-diffusion equation with boundary input delay

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The problem

- $L > 0$
- $c \in L^\infty(0, L)$

$$
\begin{aligned}
&y_t = y_{xx} + c(x)y, \qquad 0 < x < L,\\
&y(t,0) = 0,\ y(t,L) = u_D(t) = u(t-D),\\
&y(0,\cdot) = y_0(\cdot) \in L^2(0,L),
\end{aligned}
\tag{1}
$$

- the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$
- the control is $u_D(t) = u(t-D)$, with $D > 0$ a constant delay. We assume that $u_D(t) = 0$ for every $t \in (0, D)$.

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Note that this is equivalent to a coupling with a boundary controlled transport equation :

$$
\begin{aligned}
&y_t = y_{xx} + c(x)y, \qquad 0 < x < L,\\
&y(t,0) = 0,\ y(t,L) = z(t,L),\\
&z_t = z_x, \qquad L < x < L+D,\\
&z(t,L+D) = u(t) \qquad (\text{indeed } z(t,x) = u(t+x-L-D))
\end{aligned}
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Objective : design a (predictor-based) feedback control exponentially stabilizing (1) in H^1 norm.
Moreover this feedback is wanted to be "as simple as possible" ($\to$ finite-dimensional).

Spectral reduction

Setting

$$w(t,x) = y(t,x) - \frac{x}{L} u_D(t),$$

we get

$$\begin{aligned} w_t &= w_{xx} + cw + \frac{x}{L} c u_D - \frac{x}{L} u_D', \\ w(t,0) &= w(t,L) = 0, \\ w(0,x) &= y(0,x) - \frac{x}{L} u_D(0). \end{aligned}$$

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that is

$$w_t(t,\cdot) = Aw(t,\cdot) + a(\cdot)u_D(t) + b(\cdot)u_D'(t),$$

with

$$A = \partial_{xx} + c(\cdot)\mathrm{id} \qquad \text{on} \quad D(A) = H^2(0,L) \cap H_0^1(0,L),$$

and $a(x) = \frac{x}{L} c(x)$ and $b(x) = -\frac{x}{L}$ for every $x \in (0,L)$.

Spectral reduction

Let $(e_j)_{j\geq 1}$ be a Hilbert basis of $L^2(0,L)$ consisting of eigenfunctions of A, associated with the eigenvalues $(\lambda_j)_{j\geq 1}$. Note that

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1 \quad \text{and} \quad \lambda_j \underset{j\to+\infty}{\longrightarrow} -\infty,$$

and that $e_j(\cdot) \in H_0^1(0,L) \cap C^2([0,L])$ for every $j \geq 1$. Then, writing

$$w(t,\cdot) = \sum_{j=1}^{\infty} w_j(t) e_j(\cdot),$$

and one gets the infinite-dimensional control system

$$w_j'(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j u_D'(t), \qquad \forall j \in \mathbb{N}^*,$$

with

$$a_j = \langle a(\cdot), e_j(\cdot)\rangle_{L^2(0,L)} = \frac{1}{L}\int_0^L x c(x) e_j(x)\,dx,$$

$$b_j = \langle b(\cdot), e_j(\cdot)\rangle_{L^2(0,L)} = -\frac{1}{L}\int_0^L x e_j(x)\,dx.$$

Spectral reduction

Defining the new control

$$\alpha_D(t) = u_D'(t),$$

we have

$$\left.\begin{array}{rcl} u_D'(t) & = & \alpha_D(t) = \alpha(t-D) \\ w_1'(t) & = & \lambda_1 w_1(t) + a_1 u_D(t) + b_j \alpha_D(t) \\ & \vdots & \\ w_n'(t) & = & \lambda_n w_n(t) + a_n u_D(t) + b_n \alpha_D(t) \end{array}\right\} n+1 \text{ unstable modes}$$

$$\left.\begin{array}{rcl} & \vdots & \\ w_j'(t) & = & \lambda_j w_j(t) + a_j u_D(t) + b_j \alpha_D(t) \\ & \vdots & \end{array}\right\} \text{stable modes}$$

where n is the number of nonnegative eigenvalues.

Spectral reduction

With the matrix notations

$$X_1(t) = \begin{pmatrix} u_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix},$$

the $n+1$ first equations are written as the finite-dimensional control system with input delay

$$\dot{X}_1(t) = A_1 X_1(t) + B_1 \alpha_D(t) = A_1 X_1(t) + B_1 \alpha(t-D). \tag{2}$$

Spectral reduction

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Objective :

1. design a feedback stabilizing (2) and design a Lyapunov function ;
2. show that this feedback stabilizes as well the whole infinite-dimensional system.

Stabilization of the unstable finite-dimensional part

Artstein model reduction

$$Z_1(t) = X_1(t) + \int_{t-D}^{t} e^{(t-s-D)A_1} B_1 \alpha(s)\, ds$$

(Osipov, Manitius Olbrot, Artstein, Kwon Pearson, ...)
Then

$$\dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t)$$

$\rightarrow$ usual linear control system, without input delay, in $\mathbb{R}^{n+1}$.

Stabilization of the unstable finite-dimensional part

$$\dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t)$$

Lemma

The Kalman condition is satisfied.

Corollary

$\forall D \geq 0$ there exists $K_1(D) = \big(k_0(D), k_1(D), \ldots, k_n(D)\big)$ such that $A_1 + B_1 e^{-DA_1} K_1(D)$ admits -1 as an eigenvalue with order $n+1$. Moreover there exists a $(n+1) \times (n+1)$ symmetric positive definite matrix $P(D)$ such that

$$P(D)\left(A_1 + B_1 e^{-DA_1} K_1(D)\right) + \left(A_1 + e^{-DA_1} B_1 K_1(D)\right)^\top P(D) = -I_{n+1}. \tag{3}$$

In particular, the function

$$V_1(Z_1) = \frac{1}{2} Z_1^\top P(D) Z_1 \tag{4}$$

is a Lyapunov function for the closed-loop system $\dot{Z}_1(t) = (A_1 + e^{-DA_1} B_1 K_1(D)) Z_1(t)$.

Stabilization of the unstable finite-dimensional part

In other words, we set

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ K_1(D)Z_1(t) & \text{if } t \geq D, \end{cases} \tag{5}$$

and the resulting closed-loop system is

$$\begin{aligned} \dot{X}_1(t) &= A_1 X_1(t) + \chi_{(D,+\infty)}(t) B_1 K_1(D) Z_1(t-D) \\ \dot{Z}_1(t) &= \left(A_1 + \chi_{(D,+\infty)}(t) e^{-DA_1} B_1 K_1(D)\right) Z_1(t) \end{aligned}$$

Stabilization of the unstable finite-dimensional part

Inversion of the Artstein transform.

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ K_1(D)X_1(t) + K_1(D)\int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 \alpha(s)\, ds & \text{if } t \geq D. \end{cases}$$

yields

$$\begin{aligned} \alpha(t) = {} & K_1(D)X_1(t) + K_1(D)\int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 K_1(D) X_1(s)\, ds \\ & + K_1(D)\int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 K_1(D) \int_{\max(s-D,D)}^{s} e^{(s-D-\tau)A_1} B_1 K_1(D) X_1(\tau)\, d\tau\, ds \\ & + \cdots \end{aligned}$$

which depends on the past values of X_1 over the time interval (D, t).

Stabilization of the unstable finite-dimensional part

Similarly :

$$Z_1(t) = X_1(t) + \int_{(t-D,t)\cap(D,+\infty)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s)\, ds$$

if and only if

$$X_1(t) = Z_1(t) - \int_{(t-D,t)\cap(D,+\infty)} f(t-s) X_1(s)\, ds,$$

where

$$\begin{aligned} f(r) = {} & e^{(r-D)A_1} B_1 K_1(D) \\ & + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) e^{(\tau-D)A_1} B_1 K_1(D)\, d\tau \\ & + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) \int_0^\tau e^{(\tau-s-D)A_1} B_1 K_1(D) e^{(s-D)A_1} B_1 K_1(D)\, ds\, d\tau \\ & + \cdots \end{aligned}$$

and the series is convergent, whatever the value of the delay $D \geq 0$ may be.

Stabilization of the unstable finite-dimensional part

Consequence

The closed-loop system is

$$\begin{aligned} X_1'(t) &= A_1 X_1(t) + B_1 \alpha(t-D) \\ &= A_1 X_1(t) + B_1 K_1(D) X_1(t-D) + B_1 K_1(D) \int_{(t-D,t)\cap(D,+\infty)} f(t-s) X_1(s)\, ds, \end{aligned}$$

and is exponentially stable. Moreover, we have the Lyapunov function

$$V_1(t) = \frac{1}{2}\left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s)\, ds\right)^\top P(D) \left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s)\, ds\right).$$

with $I_t(D) = (t-D, t) \cap (D, +\infty)$.

Stabilization of the unstable finite-dimensional part

Brief comparison with existing results. For example :

Considering the functional

$$V_1(t) = X_1(t)^\top P X_1(t) + \int_{-D}^{0} X_1(t+\theta)^\top S X_1(t+\theta) d\theta,$$

one has sufficient conditions for asymptotic stability in the form of Riccati equations : there exist symmetric positive matrices P, S, R such that

$$A_1^\top P + PA_1 + PB_1 K_1 S^{-1} K_1^\top B_1^\top P + S + R = 0,$$

or, written in an equivalent way an the LMI inequality

$$\begin{pmatrix} A_1^\top P + PA_1 + S & PB_1 K_1 \\ K_1^\top B_1^\top P & -S \end{pmatrix} < 0.$$

This condition is however far from being necessary.

Stabilization of the whole system

$$
\begin{aligned}
&u_D' = \alpha, \quad w' = Aw + au_D + b\alpha,\\
&w(t,0) = w(t,L) = 0,\\
&u_D(0) = 0, \quad w(0,\cdot) = 0.
\end{aligned}
$$

Lyapunov function

$$
V_D(t) = M(D)\, V_1(t) + M(D) \int_{(t-D,t)\cap(D,+\infty)} V_1(s)\, ds - \frac{1}{2}\langle w(t), Aw(t)\rangle_{L^2(0,L)}
$$

$$
= \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t)\cap(D,+\infty)} Z_1(s)^\top P(D) Z_1(s)\, ds
$$

$$
- \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2.
$$

Stabilization of the whole system

Lyapunov function

$$V_D(t) = \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t)\cap(D,+\infty)} Z_1(s)^\top P(D) Z_1(s)\, ds - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2.$$

First fact

$$\forall t \geq 0 \qquad V_D(t) \geq Cst \left(u_D(t)^2 + \|w(t)\|^2_{H_0^1(0,L)} \right)$$

$$\forall t < D \qquad V_D(t) \leq Cst \left(u_D(t)^2 + \|w(t)\|^2_{H_0^1(0,L)} \right)$$

Stabilization of the whole system

Lyapunov function

$$V_D(t) = \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t)\cap(D,+\infty)} Z_1(s)^\top P(D) Z_1(s)\, ds - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2.$$

Second fact

V_D decreases exponentially to 0.

The theorem on H^1 stability follows.

Conclusion :

$$
\begin{aligned}
&y_t = y_{xx} + c(x)y, \qquad 0 < x < L,\\
&y(t,0) = 0,\ y(t,L) = u_D(t) = u(t-D),\\
&y(0,\cdot) = y_0(\cdot) \in L^2(0,L),
\end{aligned}
$$

The "finite-dimensional" feedback

$$
u(t-D) = K_1(D)X_1(t-D) + K_1(D)\int_{(t-D,t)\cap(D,+\infty)} f(t-s)X_1(s)\,ds
$$

stabilizes exponentially the system to 0 in H^1 norm, whatever the value of the delay may be. We have a Lyapunov function.

Perspectives

- Compare with the backstepping approach recently applied by Krstic to the heat equation with boundary input delay.
- Investigate more general parabolic systems coupled with a transport equation :

$$\begin{array}{rcl} X_t + \Lambda X_z & = & K_1 X + K_2 Y \\ Y_t - Y_{zz} & = & L_1 X + L_2 Y \end{array}$$

with Λ diagonal positive, K_1, K_2, L_1 and L_2 constant matrices, and

$$\begin{array}{c} X(0,t) = K\,X(1,t) \\ Y(0,t) = Y(0,t) = 0. \end{array}$$

- Robustness with respect to the delay.
- Nonconstant delays.