The problem

- \( L > 0 \)
- \( c \in L^\infty(0, L) \)

\[
y_t = y_{xx} + c(x)y, \quad 0 < x < L, \\
y(t, 0) = 0, \ y(t, L) = u_D(t) = u(t - D), \\
y(0, \cdot) = y_0(\cdot) \in L^2(0, L),
\]

(1)

- the state is \( y(t, \cdot) : [0, L] \rightarrow \mathbb{R} \)
- the control is \( u_D(t) = u(t - D) \), with \( D > 0 \) a constant delay. We assume that \( u_D(t) = 0 \) for every \( t \in (0, D) \).
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Note that this is equivalent to a coupling with a boundary controlled transport equation :

\[ y_t = y_{xx} + c(x)y, \quad 0 < x < L, \]
\[ y(t, 0) = 0, \quad y(t, L) = z(t, L), \]
\[ z_t = z_x, \quad L < x < L + D, \]
\[ z(t, L + D) = u(t) \quad \text{(indeed } z(t, x) = u(t + x - L - D)) \]
The problem

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\]

Objective: design a (predictor-based) feedback control exponentially stabilizing (1) in \( H^1 \) norm.
Moreover this feedback is wanted to be "as simple as possible" (\( \rightarrow \) finite-dimensional).
Setting

\[ w(t, x) = y(t, x) - \frac{x}{L} u_D(t), \]

we get

\[ w_t = w_{xx} + cw + \frac{x}{L} cu_D - \frac{x}{L} u'_D, \]

\[ w(t, 0) = w(t, L) = 0, \]

\[ w(0, x) = y(0, x) - \frac{x}{L} u_D(0). \]
Setting

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$$w(t, 0) = w(t, L) = 0,$$

$$w(0, x) = y(0, x) - \frac{x}{L} u_D(0).$$

that is

$$w_t(t, \cdot) = Aw(t, \cdot) + a(\cdot)u_D(t) + b(\cdot)u'_D(t),$$

with

$$A = \partial_{xx} + c(\cdot)\text{id} \quad \text{on} \quad D(A) = H^2(0, L) \cap H^1_0(0, L),$$

and $a(x) = \frac{x}{L} c(x)$ and $b(x) = -\frac{x}{L}$ for every $x \in (0, L)$. 
Let \((e_j)_{j \geq 1}\) be a Hilbert basis of \(L^2(0, L)\) consisting of eigenfunctions of \(A\), associated with the eigenvalues \((\lambda_j)_{j \geq 1}\). Note that

\[-\infty < \cdots < \lambda_j < \cdots < \lambda_1\quad \text{and} \quad \lambda_j \to +\infty \to -\infty,
\]

and that \(e_j(\cdot) \in H_0^1(0, L) \cap C^2([0, L])\) for every \(j \geq 1\). Then, writing

\[w(t, \cdot) = \sum_{j=1}^{\infty} w_j(t) e_j(\cdot),\]

and one gets the infinite-dimensional control system

\[w'_j(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j u'_D(t), \quad \forall j \in \mathbb{N}^*,\]

with

\[
a_j = \langle a(\cdot), e_j(\cdot) \rangle_{L^2(0,L)} = \frac{1}{L} \int_0^L xc(x) e_j(x) \, dx,
\]

\[
b_j = \langle b(\cdot), e_j(\cdot) \rangle_{L^2(0,L)} = -\frac{1}{L} \int_0^L xe_j(x) \, dx.
\]
Defining the new control

\[ \alpha_D(t) = u'_D(t), \]

we have

\[
\begin{align*}
    u'_D(t) &= \alpha_D(t) = \alpha(t - D) \\
    w'_1(t) &= \lambda_1 w_1(t) + a_1 u_D(t) + b_j \alpha_D(t) \\
    \vdots \\
    w'_n(t) &= \lambda_n w_n(t) + a_n u_D(t) + b_n \alpha_D(t) \\
    \vdots \\
    w'_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j \alpha_D(t) \\
    \vdots
\end{align*}
\]

\[ n + 1 \text{ unstable modes} \]

\[ \text{stable modes} \]

where \( n \) is the number of nonnegative eigenvalues.
Spectral reduction

With the matrix notations

\[ X_1(t) = \begin{pmatrix} u_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \]

the \( n + 1 \) first equations are written as the finite-dimensional control system with input delay

\[ \dot{X}_1(t) = A_1 X_1(t) + B_1 \alpha_D(t) = A_1 X_1(t) + B_1 \alpha(t - D). \quad (2) \]
Spectral reduction

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Objective:

1. design a feedback stabilizing (2) and design a Lyapunov function;
2. show that this feedback stabilizes as well the whole infinite-dimensional system.

E. Trélat

Predictor-based feedback stabilization
Stabilization of the unstable finite-dimensional part

Artstein model reduction

\[ Z_1(t) = X_1(t) + \int_{t-D}^{t} e^{(t-s-D)A_1} B_1 \alpha(s) \, ds \]

(Osipov, Manitius Olbrot, Artstein, Kwon Pearson, ...)

Then

\[ \dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t) \]

→ usual linear control system, without input delay, in \( \mathbb{R}^{n+1} \).
Stabilization of the unstable finite-dimensional part

\[
\dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t)
\]

Lemma

The Kalman condition is satisfied.

Corollary

\(\forall D \geq 0\) there exists \(K_1(D) = (k_0(D), k_1(D), \ldots, k_n(D))\) such that \(A_1 + B_1 e^{-DA_1} K_1(D)\) admits \(-1\) as an eigenvalue with order \(n + 1\). Moreover there exists a \((n + 1) \times (n + 1)\) symmetric positive definite matrix \(P(D)\) such that

\[
P(D) \left( A_1 + B_1 e^{-DA_1} K_1(D) \right) + \left( A_1 + e^{-DA_1} B_1 K_1(D) \right)^\top P(D) = -I_{n+1}.
\] (3)

In particular, the function

\[
V_1(Z_1) = \frac{1}{2} Z_1^\top P(D) Z_1
\] (4)

is a Lyapunov function for the closed-loop system \(\dot{Z}_1(t) = (A_1 + e^{-DA_1} B_1 K_1(D))Z_1(t)\).
In other words, we set

$$\alpha(t) = \begin{cases} 
0 & \text{if } t < D, \\
K_1(D)Z_1(t) & \text{if } t \geq D,
\end{cases}$$

and the resulting closed-loop system is

$$\dot{X}_1(t) = A_1 X_1(t) + \chi(D, +\infty)(t)B_1 K_1(D)Z_1(t - D)$$

$$\dot{Z}_1(t) = \left( A_1 + \chi(D, +\infty)(t)e^{-DA_1} B_1 K_1(D) \right) Z_1(t)$$
Inversion of the Artstein transform.

\[ \alpha(t) = \begin{cases} 
0 & \text{if } t < D, \\
K_1(D)X_1(t) + K_1(D) \int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 \alpha(s) \, ds & \text{if } t \geq D.
\end{cases} \]

yields

\[ \alpha(t) = K_1(D)X_1(t) + K_1(D) \int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 K_1(D)X_1(s) \, ds \]

\[ + K_1(D) \int_{\max(t-D,D)}^{t} e^{(t-D-s)A_1} B_1 K_1(D) \int_{\max(s-D,D)}^{s} e^{(s-D-\tau)A_1} B_1 K_1(D)X_1(\tau) \, d\tau \, ds \]

\[ + \ldots \]

which depends on the past values of \( X_1 \) over the time interval \( (D, t) \).
Stabilization of the unstable finite-dimensional part

Similarly:

\[
Z_1(t) = X_1(t) + \int_{(t-D,t) \cap (D,+\infty)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s) \, ds
\]

if and only if

\[
X_1(t) = Z_1(t) - \int_{(t-D,t) \cap (D,+\infty)} f(t-s) X_1(s) \, ds,
\]

where

\[
f(r) = e^{(r-D)A_1} B_1 K_1(D)
\]

\[
+ \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) e^{(\tau-D)A_1} B_1 K_1(D) \, d\tau
\]

\[
+ \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) \int_0^\tau e^{(\tau-s-D)A_1} B_1 K_1(D) e^{(s-D)A_1} B_1 K_1(D) \, ds \, d\tau
\]

\[
+ \cdots
\]

and the series is convergent, whatever the value of the delay \(D \geq 0\) may be.
Stabilization of the unstable finite-dimensional part

Consequence

The closed-loop system is

\[ X_1'(t) = A_1 X_1(t) + B_1 \alpha(t - D) \]

\[ = A_1 X_1(t) + B_1 K_1(D) X_1(t - D) + B_1 K_1(D) \int_{(t-D,t) \cap (D, +\infty)} f(t - s) X_1(s) \, ds, \]

and is exponentially stable. Moreover, we have the Lyapunov function

\[ V_1(t) = \frac{1}{2} \left( X_1(t) + \int_{l_t(D)} f(t - s) X_1(s) \, ds \right)^\top P(D) \left( X_1(t) + \int_{l_t(D)} f(t - s) X_1(s) \, ds \right), \]

with \( l_t(D) = (t - D, t) \cap (D, +\infty) \).
Brief comparison with existing results. For example:

Considering the functional

\[ V_1(t) = X_1(t)^\top PX_1(t) + \int_{-D}^{0} X_1(t + \theta)^\top SX_1(t + \theta)d\theta, \]

one has **sufficient** conditions for asymptotic stability in the form of Riccati equations: there exist symmetric positive matrices \( P, S, R \) such that

\[ A_1^\top P + PA_1 + PB_1 K_1 S^{-1} K_1^\top B_1^\top P + S + R = 0, \]

or, written in an equivalent way an the LMI inequality

\[ \begin{pmatrix} A_1^\top P + PA_1 + S & PB_1 K_1 \\ K_1^\top B_1^\top P & -S \end{pmatrix} < 0. \]

This condition is however far from being necessary.
Stabilization of the whole system

\[ u_D' = \alpha, \quad w' = Aw + au_D + b\alpha, \]
\[ w(t, 0) = w(t, L) = 0, \]
\[ u_D(0) = 0, \quad w(0, \cdot) = 0. \]

Lyapunov function

\[ V_D(t) = M(D) V_1(t) + M(D) \int_{(t-D,t) \cap (D,+\infty)} V_1(s) \, ds - \frac{1}{2} \langle w(t), Aw(t) \rangle_{L^2(0,L)} \]
\[ = \frac{M(D)}{2} Z_1(t)^T P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t) \cap (D,+\infty)} Z_1(s)^T P(D) Z_1(s) \, ds \]
\[- \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2.\]
Stabilization of the whole system

**Lyapunov function**

\[ V_D(t) = \frac{M(D)}{2} Z_1(t)^T P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t) \cap (D, +\infty)} Z_1(s)^T P(D) Z_1(s) \, ds \]
\[ - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2. \]

**First fact**

\[ \forall t \geq 0 \quad V_D(t) \geq Cst \left( u_D(t)^2 + \| w(t) \|^2_{H_0^1(0,L)} \right) \]
\[ \forall t < D \quad V_D(t) \leq Cst \left( u_D(t)^2 + \| w(t) \|^2_{H_0^1(0,L)} \right) \]
Stabilization of the whole system

Lyapunov function

\[ V_D(t) = \frac{M(D)}{2} Z_1(t)^T P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t) \cap (D, +\infty)} Z_1(s)^T P(D) Z_1(s) \, ds \]
\[ - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2. \]

Second fact

\( V_D \) decreases exponentially to 0.

The theorem on \( H^1 \) stability follows.
Conclusion:

\[ y_t = y_{xx} + c(x)y, \quad 0 < x < L, \]
\[ y(t, 0) = 0, \quad y(t, L) = u_D(t) = u(t - D), \]
\[ y(0, \cdot) = y_0(\cdot) \in L^2(0, L), \]

The "finite-dimensional" feedback

\[ u(t - D) = K_1(D)X_1(t - D) + K_1(D) \int_{(t-D,t)\cap(D,\infty)} f(t - s)X_1(s) \, ds \]

stabilizes exponentially the system to 0 in $H^1$ norm, whatever the value of the delay may be. We have a Lyapunov function.
Perspectives

- Compare with the backstepping approach recently applied by Krstic to the heat equation with boundary input delay.
- Investigate more general parabolic systems coupled with a transport equation:

\[
\begin{align*}
X_t + \Lambda X_z &= K_1 X + K_2 Y \\
Y_t - Y_{zz} &= L_1 X + L_2 Y
\end{align*}
\]

with $\Lambda$ diagonal positive, $K_1$, $K_2$, $L_1$ and $L_2$ constant matrices, and

\[
\begin{align*}
X(0, t) &= K X(1, t) \\
Y(0, t) &= Y(0, t) = 0.
\end{align*}
\]

- Robustness with respect to the delay.
- Nonconstant delays.