A semidefinite hierarchy for containment of spectrahedra

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(joint work with Kai Kellner and Christian Trabandt, arXiv: 1308.5076)

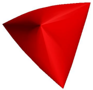
Spectrahedra

Given a linear pencil $A(x) = A_0 + \sum_{i=1}^{n} x_i A_i$, the set

$$S_A \ := \ \{x \in \mathbb{R}^n \ : \ A(x) \succeq 0\}$$

is called a spectrahedron.

- Feasible sets of semidefinite programming



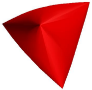
$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \succeq 0$$

Usual assumption: $0 \in \operatorname{int} S_A$. ($\rightsquigarrow$ W.l.o.g., pencil monic: $A_0 = I_k$).

Containment problems for polyhedra and spectrahedra

Uniform general setup: Given two linear pencils $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$, is $S_A \subseteq S_B$?

$\subseteq$



?

- Primary containment problem; larger classes available (up to homotheties, rotations, ...)
- Decision version; optimization versions as well (e.g., smallest enclosing (centered) sphere, geometric radii, ...)

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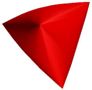

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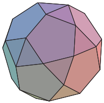
- Decision version; optimization versions as well (e.g., smallest enclosing (centered) sphere, geometric radii, . . .)

Containment problems for polytopes

For polytopes much work has been done (Freund, Orlin ('86), Gritzmann, Klee ('93), ...):

- computational complexity strongly depends on type of the input:
  - $\mathcal{V}$-polytopes: given by the vertices
  - $\mathcal{H}$-polytopes: given as an intersection of halfspaces $\{x \in \mathbb{R}^n : b + Ax \geq 0\}$



P \ Q	$\mathcal{H}$	$\mathcal{V}$
$\mathcal{H}$	P	co-NPC
$\mathcal{V}$	P	P

$P \subseteq Q$?

Hardness for polytopes and spectrahedra

	$\mathcal{H}$	$\mathcal{V}$	$\mathcal{S}$
$\mathcal{H}$	P	co-NPC	co-NP-hard
$\mathcal{V}$	P	P	P
$\mathcal{S}$	"SDP"	co-NP-hard	co-NP-hard

The following problems are co-NP-hard:

1. Deciding whether a spectrahedron is contained in a $\mathcal{V}$-polytope.
2. Deciding whether an $\mathcal{H}$-polytope or a spectrahedron is contained in a spectrahedron. This persists if the $\mathcal{H}$-polytope is a standard cube or if the outer spectrahedron is a ball.

(Cubes in spectrahedra already by Ben-Tal and Nemirovski.)

Constructive approaches

- Sufficient semidefinite criterion
  (Helton, Klep, McCullough 2013)
- Simplified proofs; exactness in several cases
  (Kellner, Trabandt, T.T. 2013)

Recently:

- Broader picture: Hierarchy of PMIs (Polynomial Matrix Inequalities),
  PMI-relaxation and positivity of maps

Relaxations: The point of departure

Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils (with $A_0 = I_k$ and $B_0 = I_l$).

In the following, the indeterminate matrix $C = \left(C_{ij}\right)_{i,j=1}^{k}$ ("*Choi matrix*") is a symmetric $kl \times kl$-matrix where the C_{ij} are $l \times l$-blocks.

Consider:

$$C = \left(C_{ij}\right)_{i,j=1}^{k} \succeq 0, \quad I_l = \sum_{i=1}^{k} C_{ii}, \quad \forall p = 1, \ldots, n: \; B_p = \sum_{i,j=1}^{k} a_{ij}^{p} C_{ij}$$

Theorem (Helton, Klep, McCullough '13)

Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils. If this system is feasible then $S_A \subseteq S_B$.

Sufficient criterion: Background

Let $\mathcal{A} \subseteq \mathcal{S}_k$ and $\mathcal{B} \subseteq \mathcal{S}_l$ be linear subspaces.

A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called

- positive if every positive semidefinite matrix in $\mathcal{A}$ is mapped to a positive semidefinite matrix in $\mathcal{B}$, i.e., $\Phi(\mathcal{A} \cap \mathcal{S}_k^+) \subseteq \mathcal{B} \cap \mathcal{S}_l^+$.
- d-positive if the map

$$\Phi_d : \mathbb{R}^{d \times d} \otimes \mathcal{A} \to \mathbb{R}^{d \times d} \otimes \mathcal{B}, \; M \otimes A \mapsto M \otimes \Phi(A)$$

  is positive.
- completely positive if Φ_d is positive for all $d > 0$.

Sufficient criterion: Once more the statement

Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils (with $A_0 = I_k$ and $B_0 = I_l$).

In the following, the indeterminate matrix $C = (C_{ij})_{i,j=1}^{k}$ ("*Choi matrix*") is a symmetric $kl \times kl$-matrix where the C_{ij} are $l \times l$-blocks.

Generalized criterion:

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0, \quad I_l = \sum_{i=1}^{k} C_{ii}, \quad \forall p = 1, \ldots, n: \; B_p = \sum_{i,j=1}^{k} a_{ij}^{p} C_{ij} \quad (1)$$

Theorem (Helton, Klep, McCullough '13)

Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils. If the system (1) is feasible then $S_A \subseteq S_B$.

Elementary proof

For $x \in S_A$, the last two conditions imply

$$B(x) = I_l + \sum_{p=1}^{n} x_p B_p \; = \; \sum_{i=1}^{k} C_{ii} + \sum_{p=1}^{n} \sum_{i,j=1}^{k} x_p \, a_{ij}^{p} \, C_{ij}$$

$$= \sum_{i,j=1}^{k} C_{ij} \, (I_k)_{ij} + \sum_{i,j=1}^{k} C_{ij} \sum_{p=1}^{n} x_p \, a_{ij}^{p} \; = \; \sum_{i,j=1}^{k} (A(x))_{ij} \, C_{ij} \, .$$

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$$= \sum_{i,j=1}^{k} C_{ij} (I_k)_{ij} + \sum_{i,j=1}^{k} C_{ij} \sum_{p=1}^{n} x_p a_{ij}^p = \sum_{i,j=1}^{k} (A(x))_{ij} C_{ij} .$$

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Since $A(x)$ and C are psd, the Kronecker product $A(x) \otimes C \in \mathbb{R}^{k^2 l \times k^2 l}$ is psd. Consider the principal submatrix given by the (i,j)-th sub-block of every (i,j)-th block,

$$\left((A(x))_{ij} C_{ij}\right)_{i,j=1}^{k} \in \mathcal{S}_{kl}[x].$$

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Setting $\mathbb{I} = [I_l, \ldots, I_l]^T$, for any $v \in \mathbb{R}^l$ we have

$$\begin{aligned} v^T B(x) v &= v^T \left(\mathbb{I}^T \left((A(x))_{ij} \, C_{ij}\right)_{i,j=1}^{k} \mathbb{I} \right) v \\ &= \left(v^T \; \ldots \; v^T\right) \left((A(x))_{ij} \, C_{ij}\right)_{i,j=1}^{k} (v \; \ldots \; v)^T \; \geq \; 0 \, . \end{aligned}$$

$\square$

Extend to spectrahedra (Variation)

Corollary

The theorem still holds when system (1) is relaxed to the system

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0, \; I_l - \sum_{i=1}^{k} C_{ii} \succeq 0, \quad \forall p = 1, \ldots, n : \; B_p = \sum_{i,j=1}^{k} a_{ij}^{p} C_{ij} \,. \tag{2}$$

Exact cases and quality

Questions:

- When is the criterion exact?
- How good/useful is the criterion?
- Hierarchy?

Given a linear pencil $A(x) \in \mathcal{S}_k[x]$, call the linear pencil

$$\hat{A}(x) := \begin{bmatrix} 1 & 0 \\ 0 & A(x) \end{bmatrix} \in \mathcal{S}_{k+1}[x]$$

the *extended linear pencil* of $S_A = S_{\hat{A}}$. The entries of $\hat{A}_p$ in $\hat{A}(x) = \hat{A}_0 + \sum_{p=1}^{n} x_p \hat{A}_p$ are denoted by $\hat{a}_{ij}^p$ for $i, j = 0, \ldots, k$.

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Cases where exactness holds

Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils. In the following cases the criteria (1) as well as (2) are necessary and sufficient for the inclusion $S_A \subseteq S_B$:

1. if $A(x)$ and $B(x)$ are normal forms of ellipsoids (both centrally symmetric, axis-aligned semi-axes),

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For semi-axis lengths $a_1, \ldots, a_n$: $A(x) = I_{n+1} + \sum_{p=1}^{n} \frac{x_p}{a_p} S_{p,n+1}$ with $S_{p,n+1} \in \mathcal{S}_{n+1}$ with 1 in the entries $(p, n+1), (n+1, p)$.

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1. if $A(x)$ and $B(x)$ are normal forms of ellipsoids (both centrally symmetric, axis-aligned semi-axes),

2. if $A(x)$ and $B(x)$ are normal forms of a ball and an $\mathcal{H}$-polyhedron, respectively,

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Main exactness results:

3. if $B(x)$ is the normal form of a polytope,
4. if $\hat{A}(x)$ is the extended form of a spectrahedron and $B(x)$ is the normal form of a polyhedron.

Cases where the criterion fails

$$\text{Let } A(x) = I_3 + x_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Both define the unit disk, i.e., $S_A = S_B$.

Proposition

The containment question $S_B \subseteq S_A$ is certified by criterion (1), while the reverse containment question $S_A \subseteq S_B$ is not certified by the criterion.

Q: Does there exist a scaling factor r for one of the spectrahedra so that the containment criterion is satisfied after this scaling?

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Scaled containment

Theorem

Let $A(x)$ and $B(x)$ be monic linear pencils such that S_A is bounded. Then there exists a constant $\nu > 0$ such that for the scaled spectrahedron νS_A the inclusion $\nu S_A \subseteq S_B$ is certified by any of the criteria (1) and (2).

- For containment of cubes in spectrahedra there exists a quantitative version (Ben-Tal and Nemirovski).

Containment via PMI hierarchies

$$S_A \subseteq S_B \quad \Leftrightarrow \quad \forall x \in S_A : B(x) \succeq 0 \quad \Leftrightarrow \quad \forall x \in S_A : z^T B(x) z \geq 0$$

Lemma

Let $A(x) \subseteq \mathcal{S}_k[x]$ and $B(x) \subseteq \mathcal{S}_l[x]$ with $S_A \neq \emptyset$, and let $r > 0$. Then $S_A \subseteq S_B$ if and only if the infimum μ of the polynomial optimization problem

$$\begin{aligned} \mu \; = \; \inf \;& z^T B(x) z \\ \text{s.t. } \;& A(x) \succeq 0 \\ & z \in \mathbb{B}_r(0) \end{aligned}$$

is zero, where $\mathbb{B}_r(0)$ denotes the ball in $\mathbb{R}^l$ with radius $r > 0$ centered at the origin.

Moment relaxation (Lasserre, Parrilo; Putinar, Henrion; Hol, Scherer ..)

- Linearize polynomials in the PMI:

$$b(x) = (1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, \ldots) \quad \rightarrow \quad \bar{y} = (1, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}, \ldots)$$

$$p(x) = \bar{p}^T b(x) \quad \mapsto \quad L_{\bar{y}}(p) = \bar{p}^T \bar{y}$$

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$$p(x) = \bar{p}^T b(x) \quad \mapsto \quad L_{\bar{y}}(p) = \bar{p}^T \bar{y}$$

- Model interplay of variables using moment matrix:

$$M(\bar{y}) = L_{\bar{y}}(b(x)b(x)^T) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_1^2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x_1 & x_2 & x_1^2 \end{array} \\ \begin{pmatrix} 1 & y_{10} & y_{01} & y_{20} & \cdots \\ y_{10} & y_{20} & y_{11} & y_{30} & \\ y_{01} & y_{11} & y_{02} & y_{21} & \\ y_{20} & y_{30} & y_{21} & y_{40} & \\ \vdots & & & & \ddots \end{pmatrix} \end{array}$$
$$\succeq 0$$

- Model constraints using localization matrices:

$$g(x) \geq 0 \quad \rightarrow \quad M(g\bar{y}) = L_{\bar{y}}(g(x)b(x)b(x)^T) \succeq 0$$
$$A(x) \succeq 0 \quad \rightarrow \quad M(A\bar{y}) = L_{\bar{y}}(b(x)b(x)^T \otimes A(x)) \succeq 0$$

Moment relaxation (Lasserre, Parrilo; Putinar, Henrion; Hol, Scherer ..)

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$$p(x) = \bar{p}^T b(x) \quad \mapsto \quad L_{\bar{y}}(p) = \bar{p}^T \bar{y}$$

- Truncate to matrices coming from monomials of degree at most $2t$:

$$\text{Truncated moment matrix: } M_t(\bar{y})$$
$$\text{Truncated localization matrices: } M_t(g\bar{y})$$
$$M_t(A\bar{y})$$

By restricting to moments up to a certain degree, we get a semidefinite program.

Moment relaxation (Lasserre, Parrilo; Putinar, Henrion; Hol, Scherer ..)

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$$p(x) = \bar{p}^T b(x) \quad \mapsto \quad L_{\bar{y}}(p) = \bar{p}^T \bar{y}$$

- Relaxation for the containment PMI:

$$\begin{array}{rll} \mu & = \inf & z^T B(x) z \\ & \text{s.t.} & A(x) \succeq 0 \\ & & 1 - z^T z \geq 0 \end{array} \quad \to \quad \begin{array}{rll} \mu_{\text{mom}}(t) & = \inf & L_{\bar{y}}(z^T B(x) z) \\ & \text{s.t.} & M_t(\bar{y}) \succeq 0 \\ & & M_{t-1}(A(x)\bar{y}) \succeq 0 \\ & & M_{t-1}(1 - z^T z \bar{y}) \succeq 0 \end{array}$$

Lemma

The sequence $\mu_{\text{mom}}(t)$ for $t \geq 2$ is monotone increasing. If for some t^* the condition $\mu_{\text{mom}}(t^*) \geq 0$ is satisfied then $S_A \subseteq S_B$.

Moment relaxation (Lasserre, Parrilo; Putinar, Henrion; Hol, Scherer ..)

- Linearize polynomials in the PMI:

$$b(x) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2, \ldots) \quad \rightarrow \quad \bar{y} = (1, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}, \ldots)$$
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Theorem

Let $A(x) \in \mathcal{S}_k[x]$ be a linear pencil such that the spectrahedron S_A is bounded. Then $\mu_{\text{mom}}(t) \uparrow \mu$ as $t \uparrow \infty$.

Relations between relaxations

Theorem

Assume S_A is bounded. For the properties

(1) The SDFP (1) has a solution $C \succeq 0$,

(2) The optimal value $\mu_{\text{mom}}(2) \geq 0$ (and thus $\mu_{\text{mom}}(t) \geq 0$ for all $t \geq 2$),

(3) The optimal value of the PMI is zero and thus $S_A \subseteq S_B$,

we have the implications

$$(1) \Longrightarrow (2) \Longrightarrow (3)\,.$$

- Exactness cases carry over

Relations between relaxations II

Theorem (Longer version)

Let S_A is bounded and $A_0, \ldots, A_n$ lin. independent. For the properties

(1) $\hat{\Phi}_{AB}$ is completely positive,

(1') The SDFP (1) has a solution $C \succeq 0$,

(2) The optimal value $\mu_{\text{mom}}(2) \geq 0$ (and thus $\mu_{\text{mom}}(t) \geq 0 \ \forall t \geq 2$),

(3) The optimal value of the PMI is zero and thus $S_A \subseteq S_B$,

(3') $\hat{\Phi}_{AB}$ is positive.

we have the implications

$$(1') \iff (1) \implies (2) \implies (3) \iff (3').$$

- $\hat{\Phi} : \mathcal{A} \to \mathcal{B}$ natural linear map, on extended linear pencil.
- (2) $\implies$ (1) is open (initial relaxation step $t = 2$ exact $\iff$ $\hat{\Phi}_{AB}$ is completely positive?)
- exactness of t-th relaxation step and $(k + 2 - t)$-positivity?

Conclusion and question

- Containment problems and their complexity
- Sufficient criteria and PMI hierarchy for containment of spectrahedra
- Exact in certain cases (spectrahedron in polytope, for extended form: spectrahedron in polyhedron)

Q:

- Further/more general quantitative versions (for certain classes)?
- Projections, rotations . . .