APPLICATION OF LOCALIZATION TO THE MULTIVARIATE MOMENT PROBLEM

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Luminy, France, November 15, 2013

Contents

• Sections 1, 2 and 3 provide background, mainly from the period 2000-2003. Section 4 is more recent work.

• In Section 4 it is explained how the results in Sections 1, 2 and 3 lead to an interesting reformulation of the multivariate moment problem in terms of extension of positive semidefinite linear functionals on $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ to positive semidefinite linear functionals on the localization of $\mathbb{R}[\underline{x}]$ at

$$p = \prod_{i=1}^{n}(1+x_i^2) \text{ or } p' = \prod_{i=1}^{n-1}(1+x_i^2).$$

• It is also explained how this reformulation can be exploited to prove new results concerning existence and uniqueness of the measure μ and density of $\mathbb{C}[\underline{x}]$ in the Lebesque Space $\mathcal{L}^s(\mu)$ for various values of s and, at the same time, to give new proofs of old results of B. Fuglede [1983], A.E. Nussbaum [1965], L.C. Petersen [1982] and K. Schmüdgen [1991], results which were proved previously using the theory of strongly commuting self-adjoint operators on Hilbert space.

• The work in Section 4 is motivated by an interesting application of the result of A.E. Nussbaum [1965] appearing in a recent paper of J.-B. Lasserre [2013].

1 Positivstellensätze for Quadratic Modules

• Let A be a commutative ring with 1. $X(A)$ denotes the set of all (unitary) ring homomorphisms $\alpha : A \to \mathbb{R}$.

• For $a \in A$, $\hat{a} : X(A) \to \mathbb{R}$ is defined by $\hat{a}(\alpha) = \alpha(a)$. $X(A)$ is given the weakest topology such that the functions $\hat{a}$, $a \in A$ are continuous.

• The mapping $a \mapsto \hat{a}$ defines a ring homomorphism from A into $\mathrm{C}(X(A))$, the ring of all continuous functions from $X(A)$ to $\mathbb{R}$.

• $\mathbb{R}[\underline{x}]$ denotes the polynomial ring $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \dots, x_n]$. The only ring homomorphism from $\mathbb{R}$ to itself is the identity. Ring homomorphisms from $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\underline{p})$, $\underline{p} \in \mathbb{R}^n$. $X(\mathbb{R}[\underline{x}])$ is identified (as a topological space) with $\mathbb{R}^n$.

Definition. By a *quadratic module* of A we mean a subset M of A satisfying $1 \in M$, $M + M \subseteq M$ and $a^2 M \subseteq M$ for each $a \in A$. A *quadratic preordering* of A is a quadratic module of A which is also closed under multiplication.

• We denote by $\sum A^2$ the set of all finite sums $\sum a_i^2$, $a_i \in A$. $\sum A^2$ is the unique smallest quadratic module of A. $\sum A^2$ is closed under multiplication, so $\sum A^2$ is also the unique smallest quadratic preordering of A.

• For $g_1, \dots, g_s \in A$, the quadratic module of A generated by $g_1, \dots, g_s$ is $M = \sum A^2 + \sum A^2 g_1 + \dots + \sum A^2 g_s$.

• For any subset M of A, $X_M := \{\alpha \in X(A) \mid \hat{a}(\alpha) \geq 0 \ \forall a \in M\}$.

• If $M = \sum A^2$ then $X_M = X(A)$. If M is the quadratic module of A generated by $g_1, \ldots, g_s$ then $X_M := \{\alpha \in X(A) \mid \hat{g}_i(\alpha) \geq 0, i = 1, \ldots, s\}$.

• For any quadratic module M of A, we denote by $\geq_M$ the associated partial ordering on A, i.e., $a \geq_M b$ means $a - b \in M$. Observe that $a \geq_M 0 \Rightarrow \hat{a} \geq 0$ on X_M. *One is interested in knowing to what extent the converse is true.*

Definition. A quadratic module M in A is said to be *archimedean* if for each $a \in A$ there exists an integer k such that $k \geq_M a$.

• If M is a quadratic module of A which is archimedean then, for each $a \in A$, there exists an integer $k_a \geq 1$ such that $k_a \geq_M a$ and $k_a \geq_M -a$ (so $|\hat{a}| \leq k_a$ on X_M). Thus X_M is identified with a (closed) subspace of the product space $\prod_{a \in A} [-k_a, k_a]$, so X_M is compact. The converse is false in general.

For simplicity, we assume from now on that A is an $\mathbb{R}$-algebra. We record the following special case of the representation theorem of T. Jacobi.

Theorem 1.1 (Jacobi, 2001). *Suppose M is an archimedean quadratic module in A. Then, for $a \in A$, the following are equivalent:*

(1) $\hat{a} \geq 0$ on X_M.

(2) $a + \epsilon \in M$ for all real $\epsilon > 0$.

Note: The implication (2) $\Rightarrow$ (1) is trivial. The implication (1) $\Rightarrow$ (2) is non-trivial.

Note: (2) is not saying that $a \geq_M 0$, but only that $a \geq_M -\epsilon$ for all real $\epsilon > 0$.

• Earlier versions of Theorem 1.1 were proved by E. Becker and N. Schwartz [1983], D.W. Dubois [1967], J.-L. Krivine [1964], and M.H. Stone [1940]. A special case of Theorem 1.1 (the case where $A = \mathbb{R}[\underline{x}]$ and the quadratic module M is finitely generated) is due already to M. Putinar [1993]. The simplest proof is the one in the book by M. [2008]. See the paper by S. Burgdorf, C. Scheiderer and M. Schweighofer [2012] for a proof of Theorem 1.1, based on ideas of K.R. Goodearl and D. Handleman [1976].

• Theorem 1.1 is the key theoretical ingredient underlying J.B-Lasserre's algorithm for polynomial optimization using semidefinite programming [2001].

• We would like to be able to apply Theorem 1.1 to the quadratic module $\sum A^2$. Unfortunately this is usually not possible because $\sum A^2$ is usually not archimedean. To get around this problem we will use a certain self-strengthening of Theorem 1.1 – see Theorem 1.2 below. But first we need a definition and an example.

Definition. Let M be a quadratic module of A. We will say that M is *p-archimedian* if the following three conditions hold:

(i) p is a unit of A.

(ii) $p \geq_M 1$.

(iii) For all $a \in A$ there exists $b \in \mathbb{Z}[p]$ such that $b \geq_M a$.

• Note that M is archimedian iff M is 1-archimedian.

• We examine the condition $p-1 \in M$ a bit more: If $p-1 \in M$ then it follows that $p^2-p=(p-1)^2+(p-1) \in M$ and, multiplying each of $p-1, p^2-p$ by even powers of p, that $p^k-p^{k-1} \in M$ for all integers $k \geq 1$. It follows that the set $\{kp^\ell \mid k, \ell$ are integers $\geq 0\}$ is cofinal in the subring $\mathbb{Z}[p]$ of A, with respect to the partial ordering associated to M. Consequently, in the presence of the condition (ii), condition (iii) is equivalent to

(iii′) For all $a \in A$ there exist integers $k, \ell \geq 0$ such that $kp^\ell \geq_M a$.

Example: Consider the polynomial ring $A=\mathbb{R}[\underline{x}]$. Take

$$p=(1+x_1^2)\cdots(1+x_n^2) \quad \text{or} \quad p=1+x_1^2+\cdots+x_n^2$$

(roughly speaking, any polynomial that grows sufficiently rapidly on $\mathbb{R}^n$). Then p satisfies (ii) and (iii), for any quadratic module M of $\mathbb{R}[\underline{x}]$. Of course, p does not satisfy (i). To rectify this one needs to replace the ring $A=\mathbb{R}[\underline{x}]$ by its localization at p, which is

$$\mathbb{R}[\underline{x}]_p := \{\frac{f}{p^k} \mid f \in \mathbb{R}[\underline{x}], k \geq 0\},$$

with operations defined in the obvious way, and M by any quadratic module of $\mathbb{R}[\underline{x}]_p$. Note also that, like $X(\mathbb{R}[\underline{x}])$, $X(\mathbb{R}[\underline{x}]_p)$ is naturally identified with $\mathbb{R}^n$.

Theorem 1.2 (M. 2003). *Suppose M is a p-archimedian quadratic module of A. Then, for any $a \in A$, the following are equivalent:*

(1) $\hat{a} \geq 0$ on X_M.

(2) There exists an integer $k \geq 0$ such that, for all real $\epsilon > 0$, $a + \epsilon p^k \in M$.

Note: Theorem 1.2 includes Theorem 1.1 as a special case, taking $p = 1$.

The proof of Theorem 1.2 follows along the lines of the proof of a similar result for preorderings, given by by M. in 2001.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Suppose there exists $k \geq 0$ such that for all real $\epsilon > 0$, $a + \epsilon p^k \in M$. Then, for any $\alpha \in X_M$, $\alpha(a) + \epsilon\alpha(p)^k \geq 0$, so $\alpha(a) \geq 0$.

(1) $\Rightarrow$ (2). Let

$$B = \{f \in A \mid \exists \text{ a positive integer } k \text{ such that } k + f, k - f \in M\}.$$

One can show that B is a (unitary) subring of A (more precisely, an $\mathbb{R}$-subalgebra). The quadratic module M' of B defined by $M' = M \cap B$ is obviously archimedean. Also, $1 - 1/p = (p^2 - p)/p^2$ and $1 + 1/p = (p^2 + p)/p^2$ both belong to M so $1/p \in B$. If $a \in A$ then $kp^j - a \in M$ and $kp^j + a \in M$ for some integers $j \geq 0$, $k \geq 1$. Replacing j by $j + 1$ if necessary, we can assume j is even, i.e., $kp^{2\ell} - a, kp^{2\ell} + a \in M$ for some integer $\ell \geq 0$. It follows that, for each $a \in A$, $a/p^{2\ell} \in B$ for some integer $\ell \geq 0$. This implies that A is the localization of B at $1/p$. Thus a ring homomorphism $\alpha : B \to \mathbb{R}$ lifts to a ring homomorphism $\alpha : A \to \mathbb{R}$ iff $\alpha(1/p) \neq 0$ and, in this case, the extension is unique. Moreover, if $\alpha \in X_{M'}$, and $\alpha(1/p) \neq 0$, then $\alpha(1/p) > 0$ and the extension of α to A (which we also denote by α) is in X_M. Suppose now that $\alpha(a) \geq 0$ holds for all $\alpha \in X_M$. Then, for each $\alpha \in X_{M'}$, either $\alpha(1/p) = 0$, so $\alpha(a/p^{2\ell+2}) = \alpha(a/p^{2\ell})\alpha(1/p)^2 = 0$, or $\alpha(1/p) > 0$ and, extending α, $\alpha(a/p^{2\ell+2})\alpha(p)^{2\ell+2} = \alpha(a) \geq 0$, so $\alpha(a/p^{2\ell+2}) \geq 0$. Thus $\alpha(a/p^{2\ell+2}) \geq 0$ holds in all cases so, by Theorem 2.1, $a/p^{2\ell+2} + \epsilon \in M'$ holds for all real $\epsilon > 0$. Clearing fractions, this yields $a + \epsilon p^{2\ell+2} \in M$. □

• There are also refined versions of Theorem 1.1 and Theorem 1.2 (due to Schmüdgen [1991], Wörmann [1998], Jacobi and Prestel [2001] and others in the case of Theorem 1.1, and M. [2001] and M. [2003] in the case of Theorem 2.2) which involve Tarski's Transfer Principle. We do not require these results and will not refer to them further.

2 Positive Linear Functionals and Measures

• The results of the previous section have application to the Moment Problem. To see this application, we make use of the following extension of Haviland's theorem:

Theorem 2.1 (M. 2003). *Suppose A is an $\mathbb{R}$-algebra, X a Hausdorff space and $\hat{}: A \to C(X)$ is an $\mathbb{R}$-algebra homomorphism. Suppose $(*)$ there exists $p \in A$ such that $\hat{p} \geq 0$ on X and, for each integer $k \geq 1$, the set $X_k := \{\alpha \in X \mid \hat{p}(\alpha) \leq k\}$ is compact. Then, for any linear function $L : A \to \mathbb{R}$ satisfying $(**)$ $L(\{a \in A \mid \hat{a} \geq 0 \text{ on } X\}) \subseteq [0, \infty)$, there exists a positive Borel measure μ on X such that, $\forall\ a \in A$, $L(a) = \int \hat{a}\, d\mu$.*

• Theorem 2.1 applies in various cases. It applies, for example, when $A = \mathbb{R}[\underline{x}]$, X is a closed subset of $\mathbb{R}^n$ and $\hat{}: A \to C(X)$ is the restriction map, taking $p = x_1^2 + \cdots + x_n^2$. This special case of Theorem 2.1 is due to E.K. Haviland [1935/36].

• The converse of Theorem 2.1 is more or less obvious: If μ is any positive Borel measure on X such that $\int \hat{a} d\mu$ is well-defined and finite for all $a \in A$, then $a \mapsto L(a) := \int \hat{a} d\mu$ is a well-defined linear functional on A satisfying $(**)$.

• It is possible to deduce Theorem 2.1 from a theorem of Choquet [Lectures in Analysis Vol 2, Theorem 34.6, 1969]. Rather than attempt to explain how this is done, it is easier to give a direct proof.

Proof. For $\alpha \in X$, X_k is a neighborhood of α for k sufficiently large, so X is locally compact. Denote by $\mathrm{C}'(X)$ the algebra of all continuous functions $f : X \to \mathbb{R}$ which are bounded by some $\hat{a}$, $a \in A$ in the sense that there exists $a \in A$ such that $|f| \leq \hat{a}$ on X. We begin by proving the existence of a positive linear functional $\overline{L} : \mathrm{C}'(X) \to \mathbb{R}$ such that $\overline{L}(\hat{a}) = L(a)$ for all $a \in A$. Let $A_0 = \{\hat{a} \mid a \in A\}$. If $\hat{a} = 0$ on X then, by our hypothesis, $L(a) = 0$. Thus we have a well-defined linear map $\overline{L} : A_0 \to \mathbb{R}$ given by $\overline{L}(\hat{a}) = L(a)$. Use Zorn's lemma to pick a pair $(V, \overline{L})$ where V is a subspace of $\mathrm{C}'(X)$ containing A_0 and $\overline{L}$ is an extension of $\overline{L}$ to V maximal with the property that

$$\forall\ f \in V, f \geq 0 \text{ on } X \ \Rightarrow\ \overline{L}(f) \geq 0.$$

We claim that $V = \mathrm{C}'(X)$. Otherwise, we have some $g \in \mathrm{C}'(X)$, $g \notin V$. If $f_1, f_2 \in V$ are such that $f_1 \leq g$, $g \leq f_2$ on X, then $f_1 \leq f_2$ on X so $\overline{L}(f_1) \leq \overline{L}(f_2)$. Such f_1, f_2 exist, e.g., pick $f_1 = -\hat{a}$, $f_2 = \hat{a}$ where $a \in A$ is such that $\hat{a} \geq |g|$. Thus there exists a real number e such that

$$\sup\{\overline{L}(f_1) \mid f_1 \in V, f_1 \leq g\} \leq e \leq \inf\{\overline{L}(f_2) \mid f_2 \in V, f_2 \geq g\}.$$

Then $\overline{L}$ extends to $V' = V + \mathbb{R}g$ via $\overline{L}(f + dg) = \overline{L}(f) + de$, a contradiction.

$\mathrm{C}'(X)$ contains all continuous functions with compact support so, by the Riesz representation theorem, we have a unique positive Borel measure μ on X such that $\overline{L}(f) = \int f \, d\mu$ holds for all continuous f with compact support. It remains to show that this is true for any f in $\mathrm{C}'(X)$. Suppose $f \in \mathrm{C}'(X)$ is given. Decomposing f as $f = f_+ - f_-$, $f_+ = (|f| + f)/2$,

$f_- = (|f| - f)/2$, we can assume $f \geq 0$. Take $q = f + \hat{p}$ and, for each integer $k \geq 1$, set $X'_k = \{\alpha \in X \mid q(\alpha) \leq k\}$. X'_k is closed and $X'_k \subseteq X_k$ so X'_k is compact. Obviously $X'_i \subseteq X'_{i+1}$ and $\cup_{i\geq 1} X'_i = X$. Using Urysohn's lemma, we have continuous functions f_i with $0 \leq f_i \leq f$, $f_i = f$ on X'_i, $f_i = 0$ off X'_{i+1}. Since $q > i$ off X'_i we see that $q^2/i \geq f - f_i \geq 0$ on X, so $\overline{L}(q^2)/i \geq \overline{L}(f) - \overline{L}(f_i) \geq 0$. This proves $\overline{L}(f) = \lim_{i\to\infty} \overline{L}(f_i)$ which in turn implies that $\int f\, d\mu = \lim_{i\to\infty} \int f_i\, d\mu = \lim_{i\to\infty} \overline{L}(f_i) = \overline{L}(f)$. □

Combining Theorem 1.2 and Theorem 2.1 yields the following:

Theorem 2.2 (M. 2003)**.** *Suppose A is an $\mathbb{R}$-algebra and M is a p-archimedean quadratic module of A. Then, for any linear function $L : A \to \mathbb{R}$ satisfying $L(M) \subseteq [0, \infty)$, there exists a unique positive Borel measure μ on X_M such that $\forall\ a \in A$, $L(a) = \int \hat{a}\, d\mu$.*

Proof. We use the notation in the proof of Theorem 1.2. For each $k \geq 1$,

$$\{\alpha \in X_M \mid \hat{p}(\alpha) \leq k\} = \{\alpha \in X_{M'} \mid (\hat{\frac{1}{p}})(\alpha) \geq \frac{1}{k}\}$$

is a closed subset of a compact space, so $\{\alpha \in X_M \mid \hat{p}(\alpha) \leq k\}$ is compact. This proves $(*)$. Let $a \in A$, $\hat{a} \geq 0$ on X_M. By Theorem 1.2 $\exists\ k \geq 0$ such that for all real $\epsilon > 0$ $a + \epsilon p^k \in M$. Thus $L(a + \epsilon p^k) = L(a) + \epsilon L(p^k) \geq 0$. Letting $\epsilon \to 0$, we see that $L(f) \geq 0$. This proves $(**)$. By Theorem 2.1, there exists a positive Borel measure μ on X_M such that

$L(f) = \int \hat{a} d\mu$ for all $a \in A$. To prove uniqueness of μ, let $\phi : X_M \to \mathbb{R}$ be any continuous function with compact support. X_M is naturally identified with an open subspace of $X_{M'}$. Extend ϕ to $X_{M'}$ by setting $\phi = 0$ on $X_{M'} \backslash X_M$. By the Stone-Weierstrass approximation theorem $\exists$ a sequence $f_k \in B$ such that $|\hat{f}_k - \phi| \leq \frac{1}{k}$ pointwise on $X_{M'}$. This implies, in particular, that $|\int (\hat{f}_k - \phi)\, d\mu| \leq \frac{1}{k}\mu(X_M)$, so $\int \phi\, d\mu = \lim_{k\to\infty} L(f_k)$. Uniqueness of μ follows now, by the Riesz representation theorem. □

Definition (Lebesque Spaces). For a positive Borel measure μ on a locally compact Hausdorff space X and a Borel measurable function $f : X \to \mathbb{C}$, define

$$\|f\|_{s,\mu} := \Big[\int |f|^s d\mu \Big]^{1/s},$$

and define

$$\mathcal{L}^s(\mu) := \{f : X \to \mathbb{C} \mid f \text{ is Borel measurable and } \|f\|_{s,\mu} < \infty\}.$$

We conclude this section by noting the following complement to Theorem 2.2:

Theorem 2.3 (Density). *Suppose A is an $\mathbb{R}$-algebra, M is a p-archimedean quadratic module of A and μ is a positive Borel measure on X_M such that $\int \hat{a} d\mu$ is well-defined and finite for all $a \in A$. Then for any $1 \leq s < \infty$ the obvious map $A \otimes \mathbb{C} \to \mathcal{L}^s(\mu)$ has dense image, equivalently, the image of A is dense in the real part of $\mathcal{L}^s(\mu)$.*

Proof. It suffices to show that the step functions $\sum_{j=1}^{m} a_j \chi_{S_j}$, $a_j \in \mathbb{C}$, $S_j \subseteq X_M$ a Borel set, belong to the closure of the image of A. Using the triangle inequality we are reduced further to the case $m = 1$, $a_1 = 1$. Let $S \subseteq X_M$ be a Borel set. Choose K compact, U open such that $K \subseteq S \subseteq U$, $\mu(U \backslash K) < \epsilon$. We make use of the terminology introduced in the proof of Theorem 1.2. By Urysohn's lemma there exists a continuous function $\phi : X_{M'} \to \mathbb{R}$ such that $0 \leq \phi \leq 1$ on $X_{M'}$, $\phi = 1$ on K, $\phi = 0$ on $X_{M'} \backslash U$. Extend μ to a positive Borel measure μ' on $X_{M'}$ defined by $\mu'(E) := \mu(E \cap X_M)$. Then $\|\chi_S - \phi\|_{s,\mu'} \leq \epsilon^{1/s}$. Use the Stone-Weierstrass approximation theorem to get $a \in B$ such that $\|\phi - \hat{a}\|_\infty < \epsilon$, where $\| \cdot \|_\infty$ denotes the sup-norm. Then $\|\phi - \hat{a}\|_{s,\mu'} \leq \epsilon \mu(X_M)^{1/s}$. Putting these things together yields $\|\chi_S - \hat{a}\|_{s,\mu} = \|\chi_S - \hat{a}\|_{s,\mu'} \leq \|\chi_S - \phi\|_{s,\mu'} + \|\phi - \hat{a}\|_{s,\mu'} \leq \epsilon^{1/s} + \epsilon \mu(X_M)^{1/s}$. $\square$

3 Generalized Cylinder Result

• Denote by $A[y]$, polynomial ring in a single variable y with coefficients in the $\mathbb{R}$-algebra A. The cylinder $X(A) \times \mathbb{R}$ is naturally identified with $X(A[y])$. For any quadratic module M of A, the cylinder $X_M \times \mathbb{R}$ is naturally identified with $X_{\tilde{M}}$, where $\tilde{M}$ denotes the extension of M to $A[y]$.

A result of S. Kuhlmann and M. [2002], for archimedean preorderings and cylinders with compact cross-section, extends to p-archimedian quadratic modules as follows:

Theorem 3.1 (M. 2003)**.** *Suppose M is a p-archimedian quadratic module of A. Then, for any $f \in A[y]$, the following are equivalent:*

(1) $\hat{f} \geq 0$ on $X_M \times \mathbb{R}$.

(2) $\exists$ integers $k, \ell \geq 0$ such that $\forall$ real $\epsilon > 0$, $f + \epsilon p^k (1+y^2)^\ell \in \tilde{M}$, where $\tilde{M}$ denotes the extension of M to $A[y]$.

Note: We are not assuming here that $\tilde{M}$ is $p(1+y^2)$-archimedian. In fact it is not, since $1+y^2$ is not invertible in $A[y]$. The advantage of Theorem 3.1 over Theorem 1.2 is that it is not necessary to invert $1+y^2$.

Proof.

Case $p = 1$: Suppose f satisfies (1). Take k to be any integer such that $2k \geq \deg(f)$. Following exactly the proof in the preordering case given in S. Kuhlmann and M. [2002], we see that $f + \epsilon q \in \tilde{M}$ holds for any real $\epsilon > 0$ where $q = 3 + y + 3y^2 + y^3 + \cdots + 3y^{2k}$. Observing that

$$q + \frac{1}{2}(\sum_{i=0}^{\ell-1}(1-y)^2 y^{2i} + 1 + y^{2k}) = 4(1 + y^2 + \cdots + y^{2k}),$$

it is clear that $f + 4\epsilon(1+y^2)^k \in \tilde{M}$ also holds.

General Case: Define B, M' as in the proof of Theorem 1.2 and denote by $\tilde{M}'$ the quadratic module in $B[y]$ generated by M'. Suppose $f \in A[y]$, $\hat{f} \geq 0$ on $X_M \times \mathbb{R}$. Say $f = a_0 + \cdots + a_{2d}y^{2d}$, $a_i \in A$. Choose ℓ so large that $a_i/p^{2\ell} \in B$ for each i. Then $f/p^{2\ell} \in B[y]$ and one checks, as in the proof of Theorem 1.2, that $f/p^{2\ell+2} \geq 0$ on $X_{M'} \times \mathbb{R}$. By case (i), $f/p^{2\ell+2} + \epsilon(1+y^2)^d \in \tilde{M}'$ holds for all real $\epsilon > 0$. The result follows now, multiplying by $p^{2\ell+2}$. □

Theorem 2.2 extends as follows:

Theorem 3.2. *Suppose M is a p-archimedean quadratic module of A. If $L : A[y] \to \mathbb{R}$ is any linear map satisfying $L(\tilde{M}) \subseteq [0,\infty)$ then there exists a positive Borel measure μ on $X_M \times \mathbb{R}$ such that $L(f) = \int \hat{f} d\mu$ for all $f \in A[y]$.*

Proof. Argue as in the proof of Theorem 2.2 but use Theorem 3.1 now instead of Theorem 1.2. □

Remark 3.3.

(i) In the special case $(A, M) = (\mathbb{R}, \mathbb{R}_{\geq 0})$ Theorem 3.2 is due to H. Hamburger [1920].

(ii) There is no claim that the measure μ is unique or that the map $A[y] \otimes \mathbb{C} \to \mathcal{L}^s(\mu)$ has dense image. In fact, this is not true in general.

(iii) A sufficient condition for the measure μ to be unique is that there exists a sequence p_k in $A[y] \otimes \mathbb{C}$ such that $\|1-(y-i)p_k\|_{2,\mu} \to 0$ as $k \to \infty$. The proof of this will be omitted, although it is not difficult to prove.

(iv) If $(A, M) = (\mathbb{R}, \mathbb{R}_{\geq 0})$ the sufficient condition in (iii) is also necessary, by classical results of H. Hamburger, M. Riesz, and R. Nevalinna [1920-1922]. It is not clear (to me at least) if the condition in (iii) is necessary in the general case.

(v) The condition in (iii) is not so easy to check. A simpler condition to check is

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(y^{2k})}} = \infty,$$

the so-called the **Carleman condition** [T. Carleman, 1892-1949]. The Carleman condition implies the uniqueness of μ. It also implies that the map $A[y] \otimes \mathbb{C} \to \mathcal{L}^s(\mu)$ has dense image, for all $1 \leq s < \infty$.

4 Applications to the Multivariate Moment Problem

• For $n \geq 1$, we denote the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ by $\mathbb{R}[\underline{x}]$ for short.

• For a linear map $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, we consider the set of positive Borel measures μ on $\mathbb{R}^n$ such that $L(f) = \int f d\mu \; \forall \; f \in \mathbb{R}[\underline{x}]$. The *multivariate moment problem* is to understand this set of measures, for a given linear map $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$. In particular, one wants to know:

(i) When is this set non-empty?

(ii) In case it is non-empty, when is it a singleton set?

• For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by $\underline{x}^\alpha$ for short.

• The positive Borel measures μ that we are interested in have *finite moments,* i.e., $\int \underline{x}^\alpha d\mu$ is a well-defined finite real number $\forall \; \alpha \in \mathbb{N}^n$.

• Conversely, if μ is any positive Borel measure on $\mathbb{R}^n$ having finite moments then $L_\mu : \mathbb{R}[\underline{x}] \to \mathbb{R}$ defined by $L_\mu(f) = \int f d\mu \; \forall \; f \in \mathbb{R}[\underline{x}]$ is a well-defined linear map. This is clear.

• For positive Borel measures μ, ν on $\mathbb{R}^n$, each having finite moments, we write $\mu \sim \nu$ to indicate that μ and ν have the same moments, i.e., $L_\mu = L_\nu$. We say μ is *determinate* if $\mu \sim \nu \Rightarrow \mu = \nu$ and *indeterminate* if this is not the case.

• A linear map $L : A \to \mathbb{R}$, where A is an $\mathbb{R}$-algebra, is said to be PSD (positive semidefinite) if $L(f^2) \geq 0 \; \forall \; f \in A$, i.e., if $L(\sum A^2) \subseteq [0, \infty)$.

• For a linear map $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, a necessary condition for the set in (i) to be non-empty is that L is PSD.

Define

$$p := (1 + x_1^2) \cdots (1 + x_n^2).$$

We apply our earlier results in the case where $A = \mathbb{R}[\underline{x}]_p$ (localization of $\mathbb{R}[\underline{x}]$ at p) and $M = \sum A^2$, so $X_M = X(A) = \mathbb{R}^n$. We begin with applications of Theorem 2.2.

Corollary 4.1. *If $L : \mathbb{R}[\underline{x}]_p \to \mathbb{R}$ is a PSD linear map there exists a unique positive Borel measure μ on $\mathbb{R}^n$ such that $L(f) = \int f d\mu$ for all $f \in \mathbb{R}[\underline{x}]_p$.*

Proof. This is a direct application of Theorem 2.2. □

Corollary 4.2. *For any linear map $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, the set of positive Borel measures μ on $\mathbb{R}^n$ such that $L = L_\mu$ is in natural one-to-one correspondence with the set of PSD linear maps $L' : \mathbb{R}[\underline{x}]_p \to \mathbb{R}$ extending L.*

Proof. If μ is a positive Borel measure on $\mathbb{R}^n$ such that $L = L_\mu$, the corresponding extension of L to a PSD linear map $L' : \mathbb{R}[\underline{x}]_p \to \mathbb{R}$ is defined by $L'(f) = \int f d\mu$. The correspondence $\mu \mapsto L'$ has the desired properties by Corollary 4.1. □

Corollary 4.2 allows one to reformulate the multivariate moment problem in an obvious way.

Our next result is an easy application of Corollary 4.2. At the same time, it marginally extends earlier results of L.C. Petersen [1982] and B. Fuglede [1983].

Corollary 4.3. *Suppose* $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ *is linear and, for each* $j \in \{1, \dots, n\}$, *there exists a sequence* $p_{jk} \in \mathbb{C}[\underline{x}]$ *such that* $L(|1 - (x_j - i)p_{jk}|^2) \to 0$ *as* $k \to \infty$. *Then there is at most one positive Borel measure* μ *on* $\mathbb{R}^n$ *such that* $L = L_\mu$.

We turn now to the applications of Theorem 2.3.

Corollary 4.4. *Let* μ *be a positive Borel measure on* $\mathbb{R}^n$ *having finite moments. Then, for any* $1 \leq s < \infty$, $\mathbb{C}[\underline{x}]_p$ *is dense in* $\mathcal{L}^s(\mu)$, *equivalently,* $\mathbb{R}[\underline{x}]_p$ *is dense in the real part of* $\mathcal{L}^s(\mu)$.

Corollary 4.5. *Let* μ *be a positive Borel measure on* $\mathbb{R}^n$ *having finite moments. For* $1 \leq s < \infty$, *the following are equivalent:*

(1) $\mathbb{C}[\underline{x}]$ *is dense in* $\mathcal{L}^s(\mu)$.

(2) $\mathbb{C}[\underline{x}]$ *is dense in* $\mathbb{C}[\underline{x}]_p$ *in the topology induced by the norm* $\| \cdot \|_{s,\mu}$.

Remark 4.6.

(i) For $n = 1$ a necessary and sufficient condition for (2) to hold is that there exists a sequence $p_k \in \mathbb{C}[\underline{x}]$ so that $\|p_k - \frac{1}{x_1 - i}\|_{s,\mu} \to 0$ as $k \to \infty$.

(ii) For $n \geq 2$ a sufficient condition for condition for (2) to hold is that there exists $s' > s$ and a sequence $p_{jk} \in \mathbb{C}[\underline{x}]$ so that $\|p_{jk} - \frac{1}{x_j - i}\|_{s',\mu} \to 0$ as $k \to \infty$ for $j = 1, \dots, n$.

(iii) A slightly weaker version of (ii) is due already to L.C. Petersen [1982].

We turn finally to applications of Theorems 3.1 and 3.2. Define

$$p' := (1 + x_1^2) \cdots (1 + x_{n-1}^2).$$

Note that $\mathbb{R}[\underline{x}]_{p'} = \mathbb{R}[x_1, \ldots, x_n]_{p'} = \mathbb{R}[x_1, \ldots, x_{n-1}]_{p'}[x_n]$.

Corollary 4.7. *If $L : \mathbb{R}[\underline{x}]_{p'} \to \mathbb{R}$ is a PSD linear map there exists a positive Borel measure μ on $\mathbb{R}^n$ such that $L(f) = \int f d\mu$ for all $f \in \mathbb{R}[\underline{x}]_{p'}$.*

Corollary 4.8. *For a linear map $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, the following are equivalent:*

(1) There exists a positive Borel measure μ on $\mathbb{R}^n$ such that $L = L_\mu$.
(2) L extends to a PSD linear map $L : \mathbb{R}[\underline{x}]_{p'} \to \mathbb{R}$.
(3) $L \geq 0$ on $\sum \mathbb{R}[\underline{x}]_{p'}^2 \cap \mathbb{R}[\underline{x}]$.
(4) For all $m \geq 0$, $p'^m f \in \sum \mathbb{R}[\underline{x}]^2 \Rightarrow L(f) \geq 0$.

Proof. (1) $\Rightarrow$ (2). Extend L to $\mathbb{R}[\underline{x}]_{p'}$ in the obvious way, i.e., $L(f) = \int f d\mu$ for all $f \in \mathbb{R}[\underline{x}]_{p'}$. (2) $\Rightarrow$ (3). $L \geq 0$ on $\sum \mathbb{R}[\underline{x}]_{p'}^2$ so $L \geq 0$ on $\sum \mathbb{R}[\underline{x}]_{p'}^2 \cap \mathbb{R}[\underline{x}]$. (3) $\Rightarrow$ (4). Suppose that $f \in \mathbb{R}[\underline{x}]$, $p'^m f \in \sum \mathbb{R}[\underline{x}]^2$. Then $f = \frac{p'^m f}{p'^m} = (\frac{1}{p'})^{2m}(p'^m)(p'^m f) \in \sum \mathbb{R}[\underline{x}]_{p'}^2$, so $L(f) \geq 0$. (4) $\Rightarrow$ (1). Suppose $f \in \mathbb{R}[\underline{x}]$, $f \geq 0$ on $\mathbb{R}^n$. By Theorem 3.1, there exist integers $k, \ell \geq 0$ such that, for all $\epsilon > 0$, $f + \epsilon p'^k (1 + x_n^2)^\ell \in \sum \mathbb{R}[\underline{x}]_{p'}^2$, so $p'^{2m}(f + \epsilon p'^k (1 + x_n^2)^\ell) \in \sum \mathbb{R}[\underline{x}]^2$, for some $m \geq 0$. By (4) this implies $L(f + \epsilon p'^k (1 + x_n^2)^\ell) \geq 0$. Since this is valid for any $\epsilon > 0$, this implies $L(f) \geq 0$. Thus (1) follows, by Theorem 2.1 (Haviland's Theorem). $\square$

Remark 4.9.

(i) Corollary 4.8 strengthens Haviland's Theorem. Instead of having to check $f \geq 0$ on $\mathbb{R}^n \Rightarrow L(f) \geq 0$, one only has to check that $p'^m f \in \sum \mathbb{R}[\underline{x}]^2 \Rightarrow L(f) \geq 0$.

(ii) Observe that if $n = 1$ then $p' = 1$, so Corollary 4.8 coincides with Hamburger's Theorem in this case.

One can apply Corollary 4.8 to obtain a new proof of an old result of A.E. Nussbaum [1965].

Corollary 4.10 (Nussbaum, 1965)**.** *Suppose $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD and the Carleman condition*

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt[2i]{L(x_j^{2i})}} = \infty \tag{1}$$

holds for $j = 1, \ldots, n-1$. Then there exists a positive Borel measure μ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition (1) holds also for $j = n$ then the measure is determinate.

There is also another result, somewhat stronger than Nussbaum's result, due to K. Schmüdgen [1991], which can also be deduced using Corollary 4.8.

Corollary 4.11 (Schmüdgen, 1991). *Suppose $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ is linear and PSD. Fix a positive Borel measure μ_j on $\mathbb{R}$ such that $L|_{\mathbb{R}[x_j]} = L_{\mu_j}$ and suppose for $j = 1, \dots, n-1$ that $\mathbb{C}[x_j]$ is dense in $\mathcal{L}^4(\mu_j)$, i.e.,*

$$\exists\ p_{jk} \in \mathbb{C}[x_j] \text{ such that } \|p_{jk} - \frac{1}{x_j - i}\|_{4,u_j} \to 0 \text{ as } k \to \infty. \tag{2}$$

Then there exists a positive Borel measure μ on $\mathbb{R}^n$ such that $L = L_\mu$. If condition (2) holds also for $j = n$ then the measure is determinate.

References

[1] T. Jacobi, A representation theorem for certain partially ordered commutative rings. *Math. Z.* **237**, 259–273 (2001).

[2] M. Marshall, Approximating positive polynomials using sums of squares, *Canad. Math. Bull.* **46**, 400–418 (2003).

[3] M. Marshall, Application of localization to the multivariate moment problem, *Math. Scandinavica,* to appear.