The moment-LP and moment-SOS approaches

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- Semidefinite Programming
- Why polynomial optimization?
- LP- and SDP- CERTIFICATES of POSITIVITY
- The moment-LP and moment-SOS approaches
- Two examples outside optimization:
  - Approximating sets defined with quantifiers
  - Convex polynomial underestimators

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The CONVEX optimization problem:

$$\mathbf{P} \quad \to \quad \min_{x \in \mathbb{R}^n} \{ c' x \mid \sum_{i=1}^{n} A_i x_i \succeq b \},$$

where $c \in \mathbb{R}^n$ and $b, A_i \in \mathcal{S}_m$ ($m \times m$ symmetric matrices), is called a semidefinite program.

The notation "$\cdot \succeq 0$" means the real symmetric matrix "$\cdot$" is positive semidefinite, i.e., all its (real) EIGENVALUES are nonnegative.

Example

$$\mathbf{P}: \quad \min_{x} \quad \left\{ x_1 + x_2 \; : \; \text{s.t.} \quad \begin{bmatrix} 3 + 2x_1 + x_2 & x_1 - 5 \\ x_1 - 5 & x_1 - 2x_2 \end{bmatrix} \succeq 0 \right\},$$

or, equivalently

$$\mathbf{P}: \; \min_{x} \quad \left\{ x_1 + x_2 \; : \; \text{s.t.} \quad \begin{bmatrix} 3 & -5 \\ -5 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \succeq 0 \right\}$$

P and its dual **P*** are **convex** problems that are solvable in polynomial time to arbitrary precision $\epsilon > 0$.
= generalization to the convex cone $\mathcal{S}_m^+$ ($X \succeq 0$) of Linear Programming on the convex polyhedral cone $\mathbb{R}_+^m$ ($x \geq 0$).

Indeed, with DIAGONAL matrices

Semidefinite programming = Linear Programming!

Several academic SDP software packages exist, (e.g. MATLAB "LMI toolbox", SeduMi, SDPT3, ...). However, so far, size limitation is more severe than for LP software packages.
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Why Polynomial Optimization?

After all ...

the polynomial optimization problem:

$$f^* = \min\{f(\mathbf{x}) : \quad g_j(\mathbf{x}) \geq 0, \ j = 1, \ldots, m\}$$

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True!

... if one is interested with a LOCAL optimum only!!

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When searching for a local minimum ...

Optimality conditions and descent algorithms use basic tools from REAL and CONVEX analysis and linear algebra

The focus is on how to improve f by looking at a NEIGHBORHOOD of a nominal point $\mathbf{x} \in \mathbf{K}$, i.e., LOCALLY AROUND $\mathbf{x} \in \mathbf{K}$, and in general, no GLOBAL property of $\mathbf{x} \in \mathbf{K}$ can be inferred.

The fact that f and g_j are POLYNOMIALS does not help much!

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BUT for GLOBAL Optimization

... the picture is different!

Remember that for the GLOBAL minimum f^*:

$$f^* = \sup \{ \lambda : f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K} \}.$$

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REAL ALGEBRAIC GEOMETRY helps!!!!

Indeed, POWERFUL CERTIFICATES OF POSITIVITY EXIST!

Moreover .... and importantly,

Such certificates are amenable to PRACTICAL COMPUTATION!

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SOS-based certificate

$$\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0, \quad j = 1, \ldots, m\}$$

Theorem (Putinar's Positivstellensatz)

If $\mathbf{K}$ *is compact (+ a technical Archimedean assumption) and* $f > 0$ *on* $\mathbf{K}$ *then:*

$$\dagger \quad f(\mathbf{x}) = \sigma_0(\mathbf{x}) + \sum_{j=1}^{m} \sigma_j(\mathbf{x})\, g_j(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

for some SOS polynomials $(\sigma_j) \subset \mathbb{R}[\mathbf{x}]$.

Testing whether † holds for some

SOS $(\sigma_j) \subset \mathbb{R}[\mathbf{x}]$ with a degree bound, is SOLVING an SDP!

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$$\mathbf{K} = \{\mathbf{x} : \; g_j(\mathbf{x}) \geq 0; \; (1 - g_j(\mathbf{x})) \geq 0, \quad j = 1, \ldots, m\}$$

Theorem (Krivine-Vasilescu-Handelman's Positivstellensatz)

Let $\mathbf{K}$ *be compact and the family* $\{g_j, (1 - g_j)\}$ *generate* $\mathbb{R}[\mathbf{x}]$*. If* $f > 0$ *on* $\mathbf{K}$ *then:*

$$\star \quad f(\mathbf{x}) = \sum_{\alpha,\beta} c_{\alpha\beta} \prod_{j=1}^{m} g_j(\mathbf{x})^{\alpha_j} (1 - g_j(\mathbf{x}))^{\beta_j}, , \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

for some NONNEGATIVE scalars $(c_{\alpha\beta})$.

Testing whether $\star$ holds for some NONNEGATIVE $(c_{\alpha\beta})$ with $|\alpha + \beta| \leq M$, is SOLVING an LP!

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allow to infer GLOBAL Properties of
FEASIBILITY and OPTIMALITY,

... the analogue of (well-known) previous ones

valid in the CONVEX CASE ONLY!

- Farkas Lemma $\rightarrow$ Krivine-Stengle
- KKT-Optimality conditions $\rightarrow$ Schmüdgen-Putinar

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• In addition, polynomials NONNEGATIVE ON A SET $\mathbf{K} \subset \mathbb{R}^n$ are ubiquitous. They appear in many important applications, and not only in global optimization!

For instance, one may also want:

• To approximate sets defined with QUANTIFIERS, like .e.g.,

$$R_f := \{x \in \mathbf{B} \quad : \quad f(x, y) \leq 0 \text{ for all } y \text{ such that } (x, y) \in \mathbf{K}\}$$

$$D_f := \{x \in \mathbf{B} \quad : \quad f(x, y) \leq 0 \text{ for some } y \text{ such that } (x, y) \in \mathbf{K}\}$$

where $f \in \mathbb{R}[x, y]$, $\mathbf{B}$ is a simple set (box, ellipsoid).

• To compute convex polynomial underestimators $p \leq f$ of a polynomial f on a box $\mathbf{B} \subset \mathbb{R}^n$. (Very useful in MINLP.)

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The moment-LP and moment-SOS approaches

consist of using a certain type of positivity certificate (Krivine-Stengle's or Putinar's certificate) in potentially any application where such a characterization is needed. (Global optimization is only one example.)

In may situations this amounts to

solving a HIERARCHY of :

- LINEAR PROGRAMS, or
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... of increasing size!.

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LP- and SDP-hierarchies for optimization

Replace $f^* = \sup_{\lambda,\sigma_j} \{ \lambda : f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K} \}$ with:

The SDP-hierarchy indexed by $d \in \mathbb{N}$:

$$f_d^* = \sup \{ \lambda : f - \lambda = \underbrace{\sigma_0}_{SOS} + \sum_{j=1}^{m} \underbrace{\sigma_j}_{SOS} g_j ; \quad \deg(\sigma_j g_j) \leq 2d \}$$

or, the LP-hierarchy indexed by $d \in \mathbb{N}$:

$$\theta_d = \sup \{ \lambda : f - \lambda = \sum_{\alpha,\beta} \underbrace{c_{\alpha\beta}}_{\geq 0} \prod_{j=1}^{m} g_j^{\alpha_j} (1 - g_j)^{\beta_j} ; \quad |\alpha + \beta| \leq 2d \}$$

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Theorem

Both sequence (f^*_d), and (θ_d), $d \in \mathbb{N}$, are MONOTONE NON DECREASING and when $\mathbf{K}$ is compact (and satisfies a technical Archimedean assumption) then:

$$f^* = \lim_{d\to\infty} f^*_d = \lim_{d\to\infty} \theta_d.$$

• What makes this approach exciting is that it is at the crossroads of several disciplines/applications:

- Commutative, Non-commutative, and Non-linear ALGEBRA
- Real algebraic geometry, and Functional Analysis
- Optimization, Convex Analysis
- Computational Complexity in Computer Science,

which BENEFIT from interactions!

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• As mentioned ... potential applications are ENDLESS!

• Has already been proved useful and successful in applications with modest problem size, notably in optimization, control, robust control, optimal control, estimation, computer vision, etc.

• HAS initiated and stimulated new research issues:

- in Convex Algebraic Geometry (e.g. semidefinite representation of convex sets, algebraic degree of semidefinite programming and polynomial optimization)
- in Computational algebra (e.g., for solving polynomial equations via SDP and Border bases)
- Computational Complexity where LP- and SDP-HIERARCHIES have become an important tool to analyze Hardness of Approximation for 0/1 combinatorial problems ($\rightarrow$ links with quantum computing)

A remarkable property of the SOS hierarchy: I

When solving the optimization problem

$$\mathbf{P}: \quad f^* = \min \{ f(\mathbf{x}) : \; g_j(\mathbf{x}) \geq 0, \; j = 1, \ldots, m \}$$

one does NOT distinguish between CONVEX, CONTINUOUS NON CONVEX, and 0/1 (and DISCRETE) problems! A boolean variable x_i is modelled via the equality constraint "$x_i^2 - x_i = 0$".

In Non Linear Programming (NLP),

modeling a 0/1 variable with the polynomial equality constraint
"$x_i^2 - x_i = 0$"
and applying a standard descent algorithm would be considered "stupid"!

Each class of problems has its own *ad hoc* tailored algorithms.

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Each class of problems has its own *ad hoc* tailored algorithms.

Even though the moment-SOS approach DOES NOT SPECIALIZES to each class of problems:

- It recognizes the class of (easy) SOS-convex problems as FINITE CONVERGENCE occurs at the FIRST relaxation in the hierarchy. (Finite convergence also occurs for general convex problems.)
  $\rightarrow$ (NOT true for the LP-hierarchy.)
- The SOS-hierarchy dominates other lift-and-project hierarchies (i.e. provides the best lower bounds) for hard 0/1 combinatorial optimization problems!

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A remarkable property: II

FINITE CONVERGENCE of the SOS-hierarchy is GENERIC!

... and provides a GLOBAL OPTIMALITY CERTIFICATE,

the analogue for the NON CONVEX CASE of the KKT-OPTIMALITY conditions in the CONVEX CASE!

Theorem (Marshall, Nie)

Let $\mathbf{x}^* \in \mathbf{K}$ *be a global minimizer of*

$$\mathbf{P}: \quad f^* = \min\{f(\mathbf{x}) : g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}.$$

and assume that:

(i) *The gradients* $\{\nabla g_j(\mathbf{x}^*)\}$ *are linearly independent,*

(ii) *Strict complementarity holds* ($\lambda_j^* \, g_j(\mathbf{x}^*) = 0$ *for all* j.)

(iii) *Second-order sufficiency conditions hold at* $(\mathbf{x}^*, \lambda^*) \in \mathbf{K} \times \mathbb{R}_+^m$.

Then $f(\mathbf{x}) - f^* = \sigma_0^*(\mathbf{x}) + \sum_{j=1}^m \sigma_j^*(\mathbf{x}) g_j(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$, *for some SOS polynomials* $\{\sigma_j^*\}$.

Moreover, the conditions (i)-(ii)-(iii) HOLD GENERICALLY!

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In summary:

KKT-OPTIMALITY when f and $-g_j$ are CONVEX	PUTINAR's CERTIFICATE in the non CONVEX CASE
$\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j^* \nabla g_j(\mathbf{x}^*) = 0$	$\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \sigma_j(\mathbf{x}^*) \nabla g_j(\mathbf{x}^*) = 0$
$f(\mathbf{x}) - f^* - \sum_{j=1}^m \lambda_j^* g_j(\mathbf{x})$	$f(\mathbf{x}) - f^* - \sum_{j=1}^m \sigma_j^*(\mathbf{x}) g_j(\mathbf{x})$
≥ 0 for all $\mathbf{x} \in \mathbb{R}^n$	$(= \sigma_0^*(\mathbf{x})) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
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$f(\mathbf{x}) - f^* - \sum_{j=1}^m \lambda_j^* g_j(\mathbf{x})$ ≥ 0 for all $\mathbf{x} \in \mathbb{R}^n$	$f(\mathbf{x}) - f^* - \sum_{j=1}^m \sigma_j^*(\mathbf{x}) g_j(\mathbf{x})$ $(= \sigma_0^*(\mathbf{x})) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. for some SOS $\{\sigma_j^*\}$, and $\sigma_j^*(\mathbf{x}^*) = \lambda_j^*$.

II. Approximation of sets with quantifiers

Let $f \in \mathbb{R}[x, y]$ and let $\mathbf{K} \subset \mathbb{R}^n \times \mathbb{R}^p$ be the semi-algebraic set:

$$\mathbf{K} := \{(x, y) : \quad g_j(x, y) \geq 0, \quad j = 1, \ldots, m\},$$

and let $\mathbf{B} \subset \mathbb{R}^n$ be the unit ball or the $[-1, 1]^n$.

Suppose that one wants to approximate the set:

$$R_f := \{x \in \mathbf{B} : f(x, y) \leq 0 \text{ for all } y \text{ such that } (x, y) \in \mathbf{K}\}$$

as closely as desired by a sequence of sets of the form:

$$\Theta_k := \{\mathbf{x} \in \mathbf{B} : J_k(x) \leq 0\}$$

for some polynomials J_k.

With $g_0 = 1$ and with $\mathbf{K} \subset \mathbb{R}^n \times \mathbb{R}^p$ and $k \in \mathbb{N}$, let

$$Q_k(g) := \left\{ \sum_{j=0}^{m} \sigma_j(x,y)\, g_j(x,y) : \quad \sigma_j \in \Sigma[x,y], \deg \sigma_j\, g_j \leq 2k \right\}$$

Let $x \mapsto F(x) := \max \{ f(x,y) : (x,y) \in \mathbf{K} \}$, and

for every integer k consider the optimization problem:

$$\rho_k = \min_{J \in \mathbb{R}[x]_k} \left\{ \int_{\mathbf{B}} (J - F)\, dx \; : \; J(x) - f(x,y) \in Q_k(g) \right\}$$

1. The criterion

$$\int_{\mathbf{B}} (J - F)\, dx = \underbrace{\int_{\mathbf{B}} -F\, dx}_{\text{unknown but constant}} + \sum_{\alpha} J_{\alpha} \underbrace{\int_{\mathbf{B}} \mathbf{x}^{\alpha}\, dx}_{\text{easy to compute}}$$

is LINEAR in the coefficients J_{α} of the unknown polynomial $J \in \mathbb{R}[\mathbf{x}]_k$!

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IS AN SDP! Moreover, it has an optimal solution $J_k^* \in \mathbb{R}[x]_k$!

- Alternatively, if one uses LP-based positivity certificates for $J(\mathbf{x}) - f(\mathbf{x}, y)$, one ends up with solving an LP!

From the definition of J_k^*, the sublevel sets

$$\Theta_k := \{x \in \mathbf{B} : J_k^*(x) \leq 0\} \subset R_f, \quad k \in \mathbb{N},$$

provide a nested sequence of INNNER approximations of R_f.

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Theorem (Lass)

(Strong) convergence in $L_1(\mathbf{B})$-norm takes place, that is:

$$\lim_{k\to\infty} \int_{\mathbf{B}} |\, J_k^* - F \,|\, dx \;=\; 0$$

and, if in addition the set $\{x \in \mathbf{B} : \; F(x) = 0\}$ has Lebesgue measure zero, then

$$\lim_{k\to\infty} \; \mathrm{VOL}(R_f \setminus \Theta_k) \;=\; 0$$

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Ex: Polynomial Matrix Inequalities: (with D. Henrion)

Let $x \mapsto \mathbf{A}(x) \in \mathbb{R}^{p\times p}$ where $\mathbf{A}(x)$ is the matrix-polynomial

$$x \mapsto \mathbf{A}(x) = \sum_{\alpha\in\mathbb{N}^n} \mathbf{A}_\alpha\, x^\alpha \quad \left(= \sum_{\alpha\in\mathbb{N}^n} \mathbf{A}_\alpha\, x_1^{\alpha_1}\cdots x_n^{\alpha_n} \right).$$

for finitely many real symmetric matrices $(\mathbf{A}_\alpha)$, $\alpha\in\mathbb{N}^n$.

... and suppose one wants to approximate the set

$$R_{\mathbf{A}} := \{x\in\mathbf{B} \,:\, \mathbf{A}(x)\succeq 0\} = \{x \,:\, \lambda_{\min}(\mathbf{A}(x))\geq 0\}.$$

Then:

$$R_{\mathbf{A}} = \left\{ x\in\mathbf{B} \,:\, \underbrace{y^T\mathbf{A}(x)y}_{f(x,y)} \geq 0, \quad \forall\, y \text{ s.t. } \|y\|^2 = 1 \right\}$$

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Illustrative example (continued)

Let **B** be the unit disk $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ and let:

$$R_{\mathbf{A}} := \left\{ \mathbf{x} \in \mathbf{B} \; : \; \mathbf{A}(\mathbf{x}) \left(= \begin{bmatrix} 1 - 16x_1x_2 & x_1 \\ x_1 & 1 - x_1^2 - x_2^2 \end{bmatrix} \right) \succeq 0 \right\}$$

Then by solving relatively simple semidefinite programs, one may approximate $R_{\mathbf{A}}$ with sublevel sets of the form:

$$\Theta_k := \{ x \in \mathbf{B} : J_k^*(x) \geq 0 \}$$

for some polynomial J_k^* of degree $k = 2, 4, \ldots$ and with

$$\mathrm{VOL}\,(R_A \setminus \Theta_k) \to 0 \qquad \text{as } k \to \infty.$$

Illustrative example (continued)

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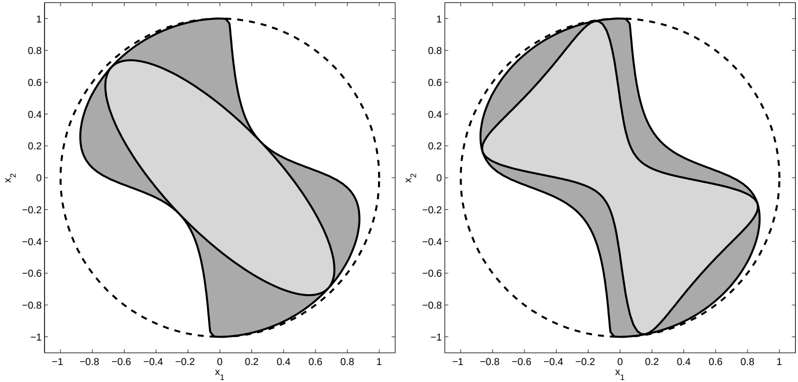
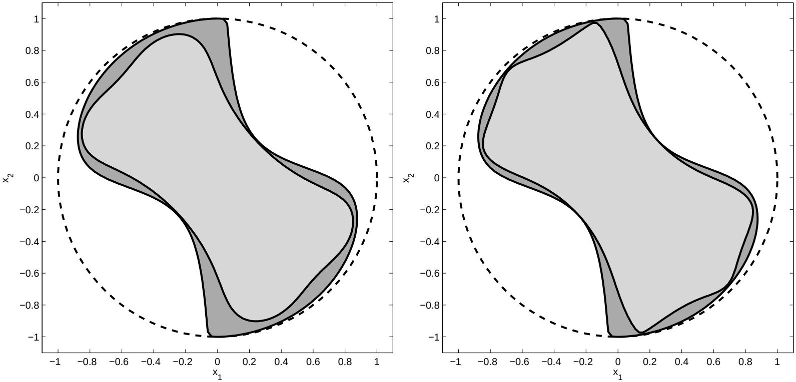


Θ_2 (left) and Θ_4 (right) inner approximations (light gray) of (dark gray) embedded in unit disk **B** (dashed).



Θ_6 (left) and Θ_8 (right) inner approximations (light gray) of (dark gray) embedded in unit disk **B** (dashed).

Similarly, suppose that one wants to approximate the set:

$$D_f := \{x \in \mathbf{B} : \; f(x,y) \leq 0 \text{ for some } y \text{ such that } (x,y) \in \mathbf{K}\}$$

as closely as desired by a sequence of sets of the form:

$$\Theta_k := \{\mathbf{x} \in \mathbf{B} : \; J_k(x) \leq 0\}$$

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Let $x \mapsto F(x) := \min\{f(x, y) : (x, y) \in \mathbf{K}\}$, and

for every integer k the optimization problem:

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IS AN SDP with an optimal solution $J_k^* \in \mathbb{R}[\mathbf{x}]_k$.

From the definition of J_k^*, the sublevel sets

$$\Theta_k := \{x \in \mathbf{B} : J_k^*(x) \leq 0\} \supset D_f, \quad k \in \mathbb{N},$$

provide a nested sequence of OUTER approximations of D_f.

Theorem (Lass)

(Strong) convergence in $L_1(\mathbf{B})$-norm takes place, that is:

$$\lim_{k\to\infty} \int_{\mathbf{B}} |\, F - J_k^* \,|\, dx \;=\; 0$$

and, if in addition the set $\{x \in \mathbf{B} : \; F(x) = 0\}$ has Lebesgue measure zero, then

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III. Convex underestimators of polynomials

In large scale Mixed Integer Nonlinear Programming (MINLP), a popular method is to use B & B where LOWER BOUNDS at each node of the search tree must be computed EFFICIENTLY!

In such a case ... one needs

CONVEX UNDERESTIMATORS

of the objective function, say on a BOX $\mathbf{B} \subset \mathbb{R}^n$!

Message:

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Solving

$$\inf_{p\in\mathbb{R}[x]_d} \left\{ \int_{\mathbf{B}} (f(x)-p(x))\,dx : \right.$$

$$\text{s.t.} \quad \left. f-p \geq 0 \text{ on } \mathbf{B} \text{ and } p \text{ convex on } \mathbf{B} \right\}$$

will provide a degree-d POLYNOMIAL CONVEX UNDERESTIMATOR p^* of f on $\mathbf{B}$ that minimizes the $L_1(\mathbf{B})$-norm $\|f-p\|_1$!

Notice that:

- $\int_{\mathbf{B}} (f(x)-p(x))\,dx$ is LINEAR in the coefficients of p!
- p convex on $\mathbf{B} \Leftrightarrow \underbrace{\mathbf{y}^T \nabla^2 p(x)\,\mathbf{y}}_{\in\mathbb{R}[x\mathbf{y}]_d} \geq 0$ on $\mathbf{B} \times \{\mathbf{y} : \|\mathbf{y}\|^2 = 1\}$!

Hence replace the positivity and convexity constraints

$f - p \geq 0$ on $\mathbf{B}$ and p convex on $\mathbf{B}$

with the positivity certificates

$$f(x) - p(x) = \sum_{k=0}^{m} \underbrace{\sigma_j(x)}_{SOS} g_j(x)$$

$$y^T \nabla^2 p(x)\, y = \sum_{k=0}^{m} \underbrace{\psi(x,y)}_{SOS} g_j(x) + \psi_{m+1}(x,y)\,(1 - \|y\|^2)$$

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and apply the moment-SOS approach

to obtain a sequence of polynomials $p_k^* \in R[x]_d$, $k \in \mathbb{N}$, of degree d which converges to the BEST convex polynomial underestimator of degree d.

Conclusion

- The moment-SOS hierarchy is a powerful general methodology.
- Works for problems of modest size (or larger size problem with sparsity and/or symmetries)

Mixed LP-SOS positivity certificate

$$f(\mathbf{x}) = \sum_{\alpha,\beta} \underbrace{c_{\alpha\beta}}_{\geq 0} \prod_j g_j(\mathbf{x})^{\alpha_j} \prod_j (1 - g_j(\mathbf{x}))^{\beta_j} + \underbrace{\sigma_0(\mathbf{x})}_{\text{sos of degree } k}$$

where k IS FIXED!

An alternative for larger size problems ?

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THANK YOU!!