Second Order Game Dynamics

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Continuous Time Games Dynamics

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To that end:

- numerous classes of dynamical systems have been proposed from both a learning and a population perspective
  (Hofbauer and Sigmund 1988, Weibull 1995, Sandholm 2010).

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- rest points might properly contain the game's Nash set or coincide with it;
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- rest points might properly contain the game's Nash set or coincide with it;
- strictly dominated strategies might extinct or survive.

Many negative results:

- there is no class of uncoupled game dynamics that always converges to a Nash equilibrium (Hart and Mas-Collel 2003).
- weakly dominated strategies may survive (Samuelson 1993).

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- Can second order dynamics be introduced **naturally** in games?
- Do second order systems allow to have **better** convergence properties?

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- We derive a wide class of higher order imitative dynamics;
- We show that strictly dominated strategies become extinct faster than the corresponding first order dynamics;
- For Nash equilibria, an analogue of the folk theorem of evolutionary game theory holds;
- In contrast to first order, weakly dominated strategies become extinct.

Model and Notation

Basic ingredient: a multi-player game in normal form $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$:

- $\mathcal{N} = \{1, \dots, N\}$: players of the game
- $\mathcal{A}_k = \{\alpha_0, \alpha_1, \dots\}$: actions of player k
- $X_k \equiv \Delta(\mathcal{A}_k)$: mixed strategies (or population distributions) of player k
- $u_k \colon X \equiv \prod_k X_k \to \mathbb{R}$: the players' (multilinear) payoff functions

$$u_k(x) = \sum\nolimits_{\alpha}^{k} x_{k\alpha} u_{k\alpha}(x) \quad \text{where} \quad u_{k\alpha}(x) = u_k(\alpha; x_{-k})$$

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Objective: write a well-behaved dynamical system of the form $\ddot{x} = F(x, \dot{x})$.

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The problem:

These dynamics are *not* well-defined: solutions may escape the strategy space.

Towards Higher Order II: Morgan and Flåm (2004)

A solution: project this velocity on the corresponding tangent cone

$$\dot{x} = v$$
$$\dot{v} = \mathrm{proj}_{T_x X} V(x)$$

where $T_x X$ is the tangent cone to X at x:

$$T_x X = \{z \in \prod_k \mathbb{R}^{\mathcal{A}_k} : \sum_\alpha^k z_{k\alpha} = 0 \text{ and } z_{k\alpha} \geq 0 \text{ whenever } x_{k\alpha} = 0\}$$

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Drawbacks: no justification, discontinuous dynamics, problem of existence...

Towards Higher Order III - Infinite Potential Walls

Another solution: Erect infinite walls at the boundary of X:

$$\ddot{x}_{k\alpha} = x_{k\alpha}\left(u_{k\alpha}(x) - \sum_{\beta}^{k} x_{k\beta}u_{k\beta}(x)\right) + W_{k\alpha}(x_k, \dot{x}_k)$$

where:

1. $\sum_\alpha W_{k\alpha} = 0$
2. $W_{k\alpha} \to \infty$ as $x_{k\alpha} \to 0$
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Drawback: How to fix W? Which learning or population justification?

Towards Higher Order IV: Reinforcement Learning loop

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Pick strategies

$x(t) \in X$

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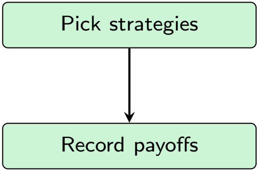

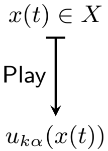

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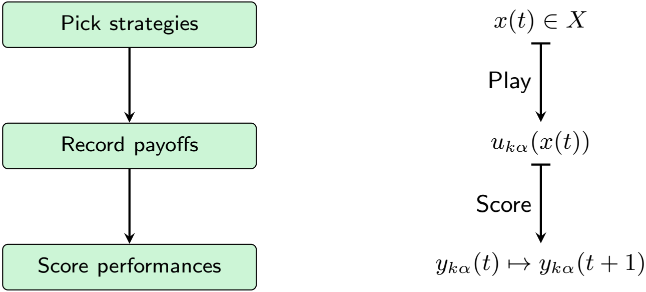

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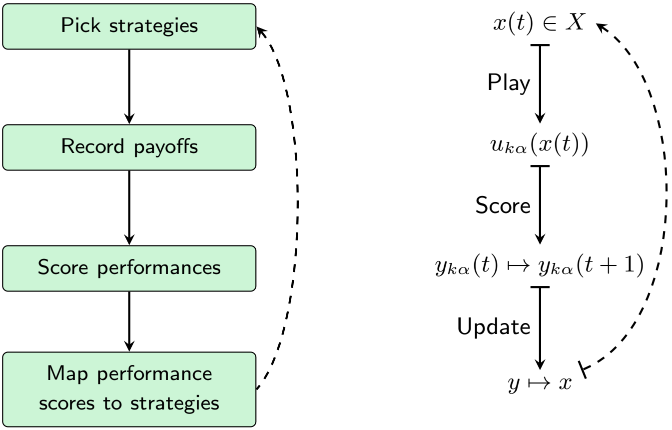

Which mapping from scores to strategies?

Key Step 1: mapping *scores* $y \in \mathbb{R}^{\mathcal{A}}$ to *strategies* $x \in \Delta(\mathcal{A})$.

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4. Invariance: $G(y_0, \ldots, y_n) = G(y_0 + c, \ldots, y_n + c)$ for any $c \in \mathbb{R}$.

   (Relative score differences is all that matters, just like adding a constant to the game's payoffs does not change the game.)

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Proposition

If $G\colon \mathbb{R}^{\mathcal{S}} \to \Delta(\mathcal{S})$ satisfies the above properties, then G is a Gibbs map:

$$G_\alpha(y) = \frac{\exp(\lambda y_\alpha)}{\sum_\beta \exp(\lambda y_\beta)} \quad \textit{for some } \lambda > 0.$$

Reinforcement Learning: Updating Scores

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A simple updating rule: score's rate of change is the instantaneous payoff:

$$\dot{y}_{k\alpha}(t) = u_{k\alpha}(x(t))$$

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A simple updating rule: score's rate of change is the instantaneous payoff:

$$\dot{y}_{k\alpha}(t) = u_{k\alpha}(x(t))$$

Coupled with the Gibbs map $x_{k\alpha} = \exp(y_{k\alpha}) \Big/ \sum_{\beta}^{k} \exp(y_{k\beta})$, this updating rule yields the (first-order) replicator dynamics:

$$\dot{x}_{k\alpha} = x_{k\alpha} \left(u_{k\alpha}(x) - \sum_{\beta}^{k} x_{k\beta} u_{k\beta}(x) \right)$$

(Hofbauer et al. 09; Mertikopoulos-Moustakas 10; Rustichini 99; Sorin, 09)

Reinforcement Learning: Second Order Effects

What if the rate of change corresponds to the cumulative payoff ?

$$\dot{y}_{k\alpha} = U_{k\alpha}$$

where $U_{k\alpha}(t) = \int_0^t u_{k\alpha}(x(s))ds$ is the cumulative payoff of strategy α.

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Coupled with the Gibbs map, we obtain the *second order replicator dynamics*:

$$\ddot{x}_{k\alpha} = x_{k\alpha}\left(u_{k\alpha}(x) - u_k(x)\right) + x_{k\alpha}\left(\dot{x}_{ka}^2/x_{k\alpha}^2 - \sum_\beta^k \dot{x}_{k\beta}^2/x_{k\beta}\right)$$

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Important observations: the initial velocity $\dot{y}(0)$ is 0.

Population justification

$\dot{x}_{k\alpha} = x_{k\alpha}\left(U_{k\alpha}(t) - U_k(t)\right)$, where $U_{k\alpha}(t) = \int_0^t u_{k\alpha}(x(s))ds$

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$\dot{x}_{k\alpha} = x_{k\alpha}\,(U_{k\alpha}(t) - U_k(t))$, where $U_{k\alpha}(t) = \int_0^t u_{k\alpha}(x(s))ds$

- By differentiating with respect to time we obtain:

$$\ddot{x}_{k\alpha} = \dot{x}_{k\alpha}\left(U_{k\alpha} - \sum_{\beta}^{k} x_{k\beta}U_{k\beta}\right) + x_{k\alpha}\left(u_{k\alpha} - \sum_{\beta}^{k} x_{k\beta}u_{k\beta}\right) - x_{k\alpha}\sum_{\beta}^{k} \dot{x}_{k\beta}U_{k\beta}$$

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- Some easy algebra yields $\sum_{\beta}^{k}\dot{x}_{k\beta}U_{k\beta} = \sum_{\beta}\dot{x}_{k\beta}^2/x_{k\beta}$
- Consequently:

$$\ddot{x}_{k\alpha} = x_{k\alpha}\,(u_{k\alpha}(x) - u_k(x)) + x_{k\alpha}\left(\dot{x}_{k\alpha}^2/x_{k\alpha}^2 - \sum\nolimits_{\beta}^{k}\dot{x}_{k\beta}^2/x_{k\beta}\right)$$

This is the second order replicator equation !

Higher Order Imitative Dynamics

More generally, players may use other "payoff observables" $w_{k\alpha}\colon X \to \mathbb{R}$ to update their scores:

$$y^{(n)}_{k\alpha} = w_{k\alpha}(x)$$

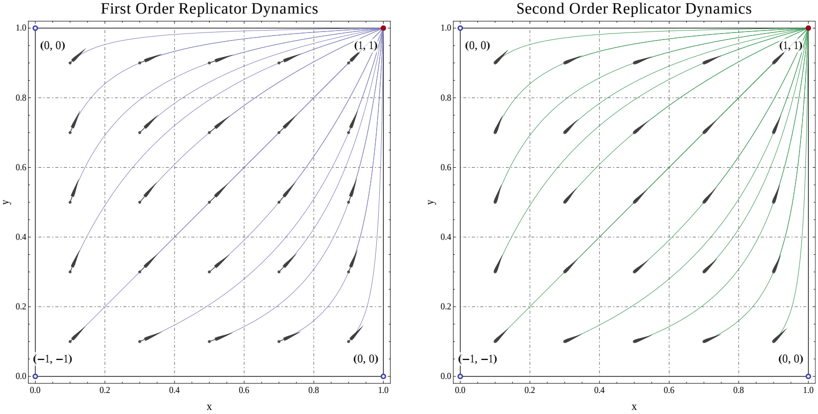
leading to the higher order imitative dynamics:

$$x^{(n)}_{k\alpha} = x_{k\alpha}\left(w_{k\alpha}(x) - \sum\nolimits_{\beta}^{k} x_{k\beta} w_{k\beta}(x)\right) + x_{k\alpha}\left(R^{n-1}_{k\alpha} - \sum\nolimits_{\beta}^{k} x_{k\beta} R^{n-1}_{k\beta}\right)$$

where $R^{n-1}_{k\alpha}$ is the higher order adjustment term of the replicator dynamics.

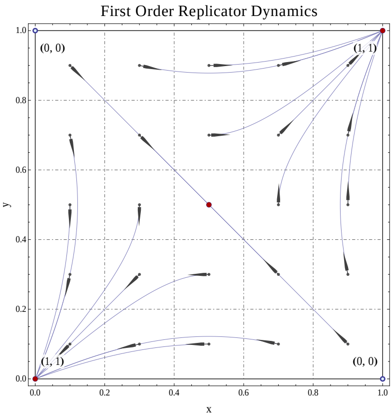
Examples I

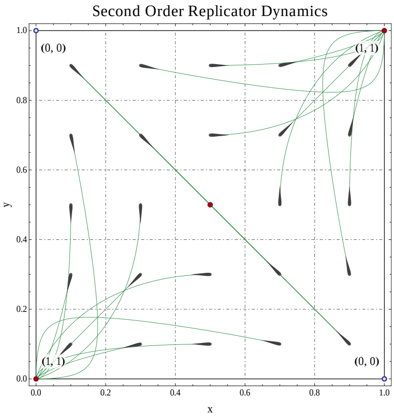
First and second order replicator dynamics in a dominance solvable game.



Examples II

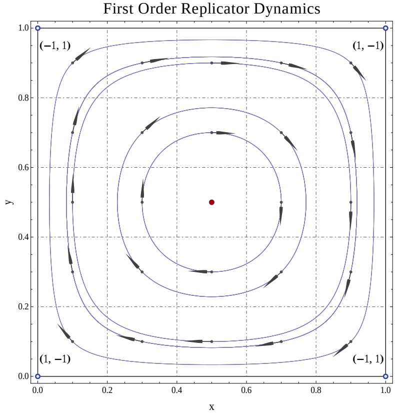
First and second order replicator dynamics in a coordination game.

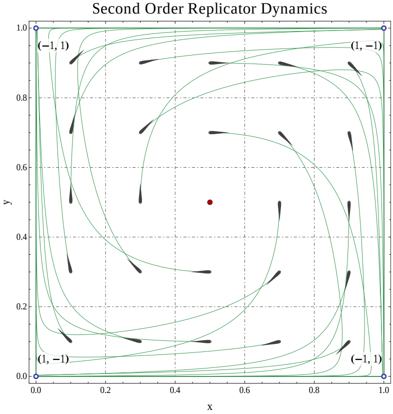




Examples III

First and second order replicator dynamics in Matching Pennies.





Extinction of Dominated Strategies I

Theorem

Let $x(t)$ be an interior solution path of the n-th order replicator dynamics. If $q_k \in X_k$ is iteratively strictly dominated, then:

$$D_{\mathrm{KL}}(q_k \,\|\, x_k(t)) \geq \lambda_k c t^n / n! + \mathcal{O}(t^{n-1}),$$

where $c > 0$ and D_{KL} is the K-L divergence $D_{\mathrm{KL}}(q_k \,\|\, x_k) = \sum_{\alpha}^{k} q_{k\alpha} \log\left(q_{k\alpha}/x_{k\alpha}\right)$.

In particular, for pure strategies $\alpha \prec \beta$, we have:

$$x_{k\alpha}(t)/x_{k\beta}(t) \leq \exp\left(-\lambda_k \Delta u_{\beta\alpha} t^n / n! + \mathcal{O}(t^{n-1})\right),$$

where $\Delta u_{\beta\alpha} = \min_{x \in X}\{u_{k\beta}(x) - u_{k\alpha}(x)\} > 0$.

In other words, iteratively strictly dominated strategies become extinct in the n-th order replicator dynamics n orders as fast as in the first order replicator dynamics.

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- If $q \prec q'$, then $u_{k\alpha}(q'_k; x_{-k}) - u_{k\alpha}(q_k; x_{-k}) \geq \delta > 0$ for all $x_{-k} \in X_{-k}$.
- The entropic difference $V(x) = D_{\mathrm{KL}}(q_k \,\|\, x_k) - D_{\mathrm{KL}}(q'_k \,\|\, x_k)$ then gives:

$$\frac{d^n}{dt^n} V(x(t)) \geq \delta > 0,$$

and the theorem follows by an n-fold application of the mean value theorem and induction on the rounds of elimination of dominated strategies. □

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Let $x(t)$ be an interior solution orbit of the n-th order ($n \geq 2$) replicator dynamics that starts at rest: $\dot{x}(0) = \ldots = x^{(n-1)}(0) = 0$.

If $q_k \in X_k$ is weakly dominated, then it becomes extinct along $x(t)$ at a rate

$$D_{\mathrm{KL}}(q_k \,\|\, x_k(t)) \geq \lambda_k c t^{n-1} / (n-1)!$$

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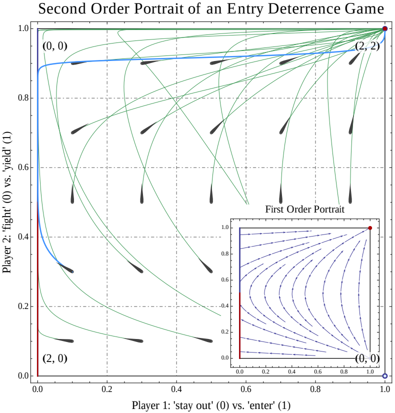
- In higher order, weakly dominated strategies extinct if players start unbiased.
- No extension for iteratively weakly dominated strategies.

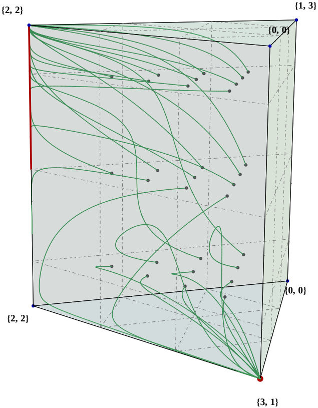
Weakly Dominated Strategies

Weakly dominated strategies in the second order replicator dynamics:

Entry Deterrence



Outside Option



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Theorem

Higher order dynamics perform one round of elimination of weakly dominated strategies followed by repeated elimination of strictly dominated strategies.

Nash Play and the Folk Theorem

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(Hofbauer and Sigmund 1988, Weibull 1995, Sandholm 2010)

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- Convergence rates to strict equilibria are n orders as fast as in first order.

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It is not well defined!... (Laraki Mertikopoulos 2013)

Thank you Amir, Hofbauer, Sorin.

Thanks to the audience.