From approximate factorizations to approximate lifts

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Linear Lifts

A linear lift of a polytope P of size k is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } a_0 + \sum a_i x_i + \sum b_i y_i \geq 0 \right\}$$

where a_i and b_i are in $\mathbb{R}^k$.

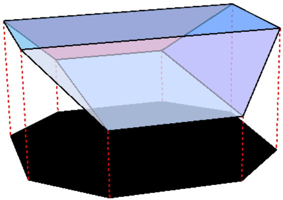
Equivalently, it is a polytope Q with k facets such that $L(Q) = P$ for some affine map L.

Example - Octagon

Consider the octagon O of vertices $\{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$

O is the set of (x, y) such that $\exists z$ for which

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} y + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} z \geq 0$$



Linear lift of size 6

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the octagon.

	$\binom{1}{2}$	$\binom{-1}{2}$	$\binom{-2}{1}$	$\binom{-2}{-1}$	$\binom{-1}{-2}$	$\binom{1}{-2}$	$\binom{2}{-1}$	$\binom{2}{1}$
$2 - x \geq 0$	1	3	4	4	3	1	0	0
$3 - x - y \geq 0$	0	2	4	6	6	4	2	0
$2 - y \geq 0$	0	0	1	3	4	4	3	1
$3 + x - y \geq 0$	2	0	0	2	4	6	6	4
$2 + x \geq 0$	3	1	0	0	1	3	4	4
$3 + x + y \geq 0$	6	4	2	0	0	2	4	6
$2 + y \geq 0$	4	4	3	1	0	0	1	3
$3 - x + y \geq 0$	4	6	6	4	2	0	0	2

Nonnegative Factorizations

Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_{+}^{n \times m}$ a k-nonnegative factorization, is a pair of matrices $A \in \mathbb{R}_{+}^{k \times n}$ and $B \in \mathbb{R}_{+}^{k \times m}$ such that

$$M = A^t \cdot B.$$

The smallest k for which M has such factorization is the nonnegative rank of M

$$\begin{bmatrix} 1 & 3 & 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 6 & 6 & 4 & 2 & 0 \\ 0 & 0 & 1 & 3 & 4 & 4 & 3 & 1 \\ 2 & 0 & 0 & 2 & 4 & 6 & 6 & 4 \\ 3 & 1 & 0 & 0 & 1 & 3 & 4 & 4 \\ 6 & 4 & 2 & 0 & 0 & 2 & 4 & 6 \\ 4 & 4 & 3 & 1 & 0 & 0 & 1 & 3 \\ 4 & 6 & 6 & 4 & 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 4 & 4 \\ 4 & 4 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 4 & 4 & 2 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 & 0 & 0 \end{bmatrix}$$

The slack matrix of a regular octagon has nonnegative rank 6

Yannakakis Theorem

Theorem (Yannakakis 1991)

A polytope P has a linear lift of size k if and only if its slack matrix has a k-nonnegative factorization.

More precisely, let $P = \{x : H^t x \leq \mathbb{1}\}$ and $S_P = A^t \cdot B$ be a k-nonnegative factorization.

$$P = \left\{ x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^k_+ \text{ s.t. } H^t x + A^t y = \mathbb{1} \right\}$$

This formulation is very overdetermined, any perturbation of A makes it unfeasible. We need a more robust version.

Robust Lifts from Factorizations

Let $P = \{x : H^t x \leq \mathbb{1}\}$ and V be the matrix whose columns are the vertices of P.

If $S_P = A^t \cdot B$ is a k-nonnegative factorization then:

$$P = \left\{ x \in \mathbb{R}^n \,:\, \exists y \in \mathbb{R}^k_+ \text{ s.t. } \mathbb{1} - H^t x - A^t y \in \mathbb{R}^f_+ \right\}$$

$$P = \left\{ Vz \,:\, z \in \mathbb{R}^v_+, \quad \mathbb{1}^t z \leq 1, \quad Bz \in \mathbb{R}^k_+ \right\}$$

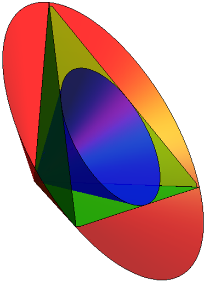
These are robust formulations, but too big.

Approximations for the nonnegative orthant

Define the cones

$$\mathcal{O}^n_{in} = \{x \in \mathbb{R}^n : \sqrt{n-1} \cdot \|x\| \leq \mathbb{1}^t x\}$$

$$\mathcal{O}^n_{out} = \{x \in \mathbb{R}^n : \|x\| \leq \mathbb{1}^t x\}.$$



Then $\mathcal{O}^n_{in} \subseteq \mathbb{R}^n_+ \subseteq \mathcal{O}^n_{out}$, and furthermore, $(\mathcal{O}^n_{in})^* = (\mathcal{O}^n_{out})$.

Effective Robust Lifts from Factorizations

Again, let $P = \{x : H^t x \leq \mathbb{1}\} \subseteq \mathbb{R}^n$ and V be the matrix whose columns are the vertices of P.

If $S_P = A^t \cdot B$ is a k-nonnegative factorization then

$$P = \mathrm{Inn}_P(A) = \left\{ x \in \mathbb{R}^n \; : \; \exists y \in \mathbb{R}^k_+ \text{ s.t. } \mathbb{1} - H^t x - A^t y \in \mathbb{R}^f_+ \; \mathcal{O}^f_{in} \right\}$$

$$P = \mathrm{Out}_P(B) = \left\{ Vz \; : \; z \in \mathbb{R}^v_+ \;\; \mathcal{O}^v_{out}, \quad \mathbb{1}^t z \leq 1, \quad Bz \in \mathbb{R}^k_+ \right\}$$

Both $\mathrm{Inn}_P(A)$ and $\mathrm{Out}_P(B)$ are actually $\mathbb{R}^k_+ \times SOC_{k+n+1}$ lifts, so we gain robustness and don't loose effectiveness.

Containment

Containment Property

For any A and B nonnegative

$$\mathrm{Inn}_P(A) \subseteq P \subseteq \mathrm{Out}_P(B).$$

So nonnegative matrices give us automatic inner and outer approximations of a polytope.

Polar Property

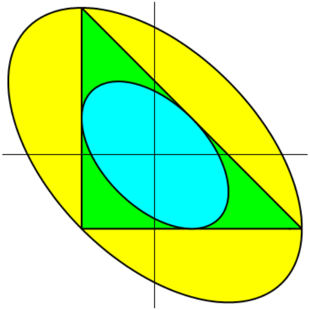
$(\mathrm{Inn}_P(A))^\circ = \mathrm{Out}_{P^\circ}(A)$ and $(\mathrm{Out}_P(B))^\circ = \mathrm{Inn}_{P^\circ}(B)$.

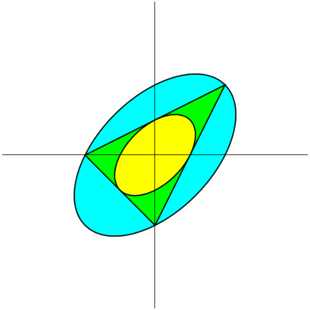
Example

Let $P = \left\{ (x, y) : \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix} \leq \mathbb{1} \right\}$

$$\mathrm{Inn}_P(0) = \left\{ (x, y) \ : \ 3(x+y)^2 + (x-y)^2 \leq 3 \right\}$$

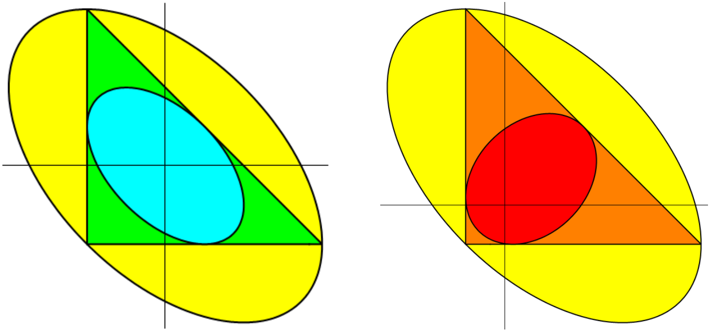
$$\mathrm{Out}_P(0) = \left\{ (x, y) \ : \ 3(x+y)^2 + (x-y)^2 \leq 12 \right\}.$$





Translation (In)variance

Note that the inner approximations depend on the choice of the center, while the outer is invariant.



Error bounds

Error bounds for the approximations

Let $\tilde{S} = A^t \cdot B$, and P a polytope such that

$$\varepsilon_1 = \|\tilde{S} - S_P\|_{\infty,2}; \quad \varepsilon_2 = \|\tilde{S} - S_P\|_{1,2}.$$

Then

$$\frac{1}{1+\varepsilon_1} P \subseteq \mathrm{Inn}_P(B) \subseteq P; \quad P \subseteq \mathrm{Out}_P(A) \subseteq (1+\varepsilon_2)P.$$

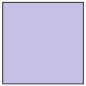
Good factorizations give good approximations.

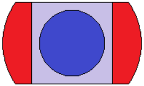
Example

Consider P the square with vertices $(\pm 1, \pm 1)$.

$$S_P = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix} \qquad \tilde{S} = A^t \cdot B = \begin{bmatrix} 4/3 & 0 \\ 4/3 & 4/3 \\ 0 & 4/3 \\ 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

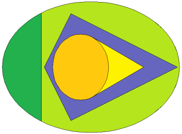
$$\varepsilon_1 = 2/3\sqrt{10}; \qquad \varepsilon_2 = 2/3\sqrt{6}$$



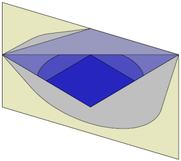


Further thoughts

- All remains true for general cones
- $\mathrm{Inn}_P(0)$ is a Dikin-like ellipsoid.
- Canonical choice for $k = 1$ is obtained by svd.



- Generalizes to “sandwiched” polytopes.



- Approximate lifts to approximate factorizations is easy.

THE END

THANK YOU