Occupation measures and semi-definite relaxations for optimal control

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Table of contents

Optimal control

$$J = \inf_u \int_0^T h(t, x(t), u(t))\, dt$$

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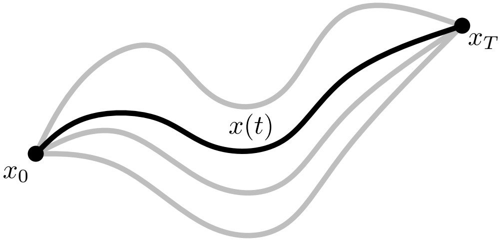
$$x(0) = x_0, \quad x(T) = x_T$$

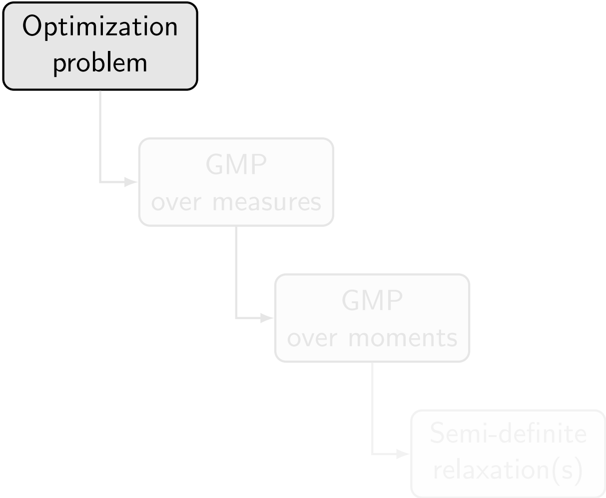


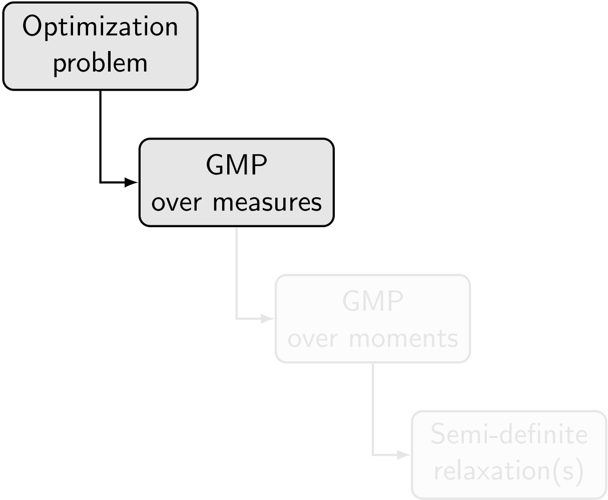
Some difficulties

- ∞-dim decision variable
- Local optimality
- Non smooth behaviors
- State constraints
- Practical implementation

The moment approach (1/2)

The moment approach (2/2)




Optimization problem
GMP over measures
GMP over moments
Semi-definite relaxation(s)

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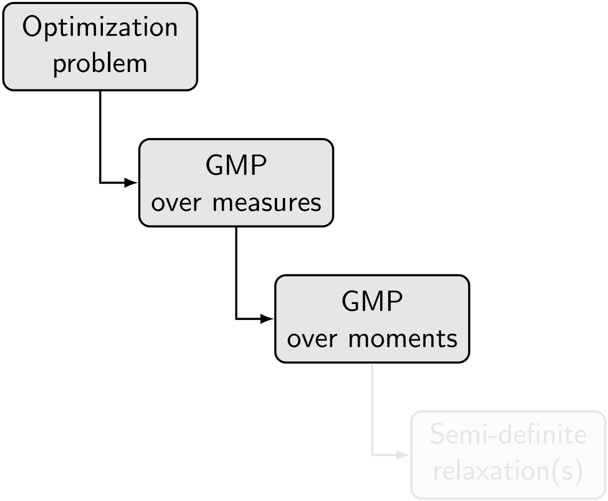

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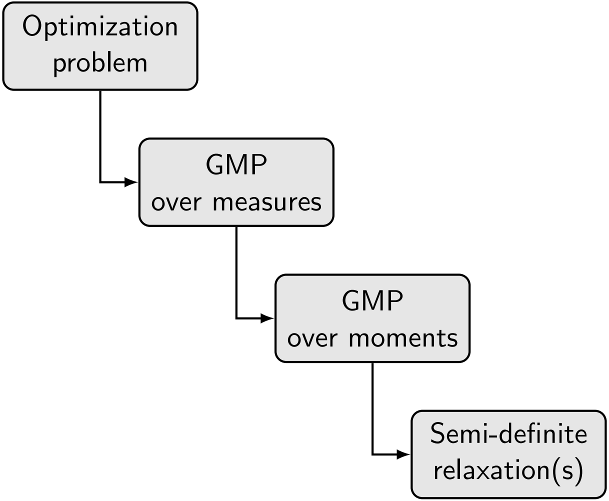

Table of contents

The control problem

$$J = \inf_u \int_0^T |u(t)|\, dt$$

$$\text{s.t. } \dot{x}(t) = A(t)x(t) + B(t)u(t)$$

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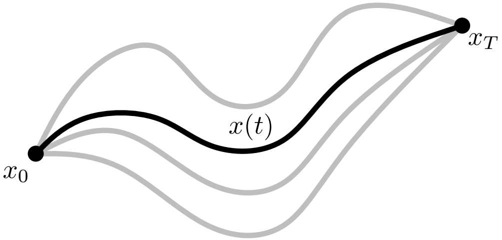
$$u(t) \in L^1([0,T];\mathbb{R}^m)$$



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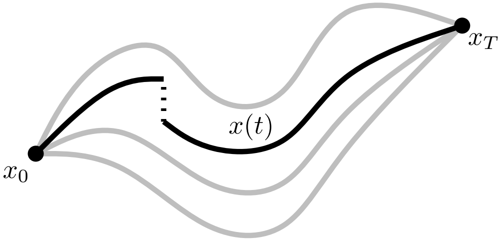

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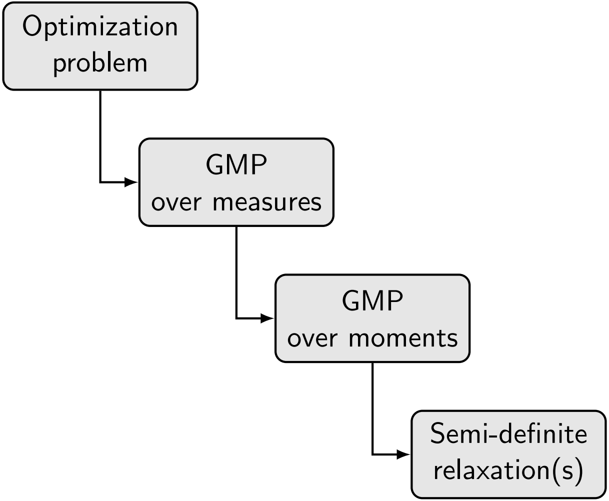

Towards a generalized moment problem

Approach [Neustadt, Luenberger, . . .] :

1. ODE integration:

$$\underbrace{\Phi^{-1}(T)\,x(T) - \Phi^{-1}(0)\,x(0)}_{c} = \int_0^T \underbrace{\Phi^{-1}(s)\,B(s)}_{F(s)}\,u(s)\,ds$$

2. Yields:

$$J = \inf_{u(t)} \|u\|$$

$$\text{t.q.} \quad \int_0^T F(t)\,u(t)\,dt = c$$

3. Appropriate $u(t) \in E$

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Measures in $\mathbb{R}^n$

Finite, Borel measures on $\mathbf{X} \subset \mathbb{R}^n$:

$$\mathcal{M}(\mathbf{X})$$

Theorem (Riesz)

$[C(\mathbf{X})]^*$ *isomorphic to* $\mathcal{M}(\mathbf{X})$

Measures: examples

- $\mu \ll \lambda_{[a,b]}$:

$$\mu([a,t]) := \int_a^t u(s)\,ds, \quad a \le t \le b$$

$$\langle v, \lambda \rangle = \int_a^b v(s) u(s)\,ds$$

- Dirac measure δ :

$$\delta_y(\mathbf{B}) = \begin{cases} 1 & \text{if } y \in \mathbf{B} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle v, \delta_y \rangle = v(y)$$

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The generalized moment problem

$$\int_0^t u(s)\, ds \quad \longrightarrow \quad \mu([0,t])$$

$$\inf_u \|u\| \qquad\qquad \min_\mu \|\mu\|$$

$$\text{s.t. } \int_0^T F(t)\, u(t)\, dt = c \quad \longrightarrow \quad \text{s.t. } \langle F, \mu \rangle = c$$

$$u \in L^1([0,T]; R^m) \qquad\qquad \mu \in \mathcal{M}([0,T]; R^m)$$

Theorem (Neustadt)

No relaxation gap.

Theorem

$\exists$ admissible $\mu \implies \exists$ n-atomic optimal solution.

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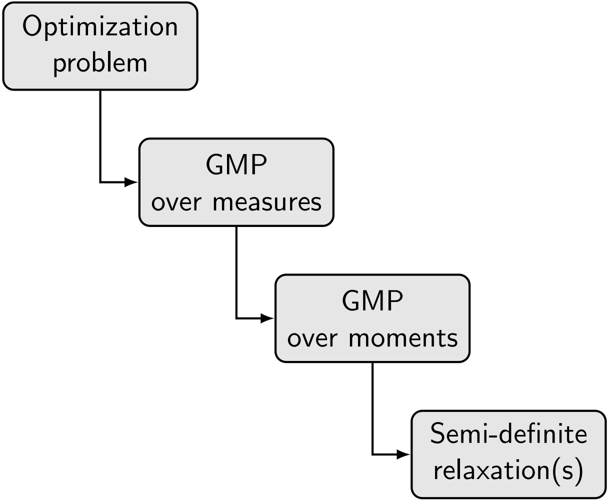

Moments

- Moments: $y_\alpha = \langle x^\alpha, \mu \rangle$

- Moment matrix: $M(y) = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots \\ y_1 & y_2 & y_3 & \\ y_2 & y_3 & y_4 & \\ \vdots & & & \ddots \end{bmatrix}$

- Let $\mathbf{X} := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \quad i = 1, ..., m\}$

Theorem (Putinar)

$\mu \in \mathcal{M}^+(\mathbf{X})$ *iff:*

$$M(y) \succeq 0, \qquad M(g_i * y) \succeq 0 \quad \forall i$$

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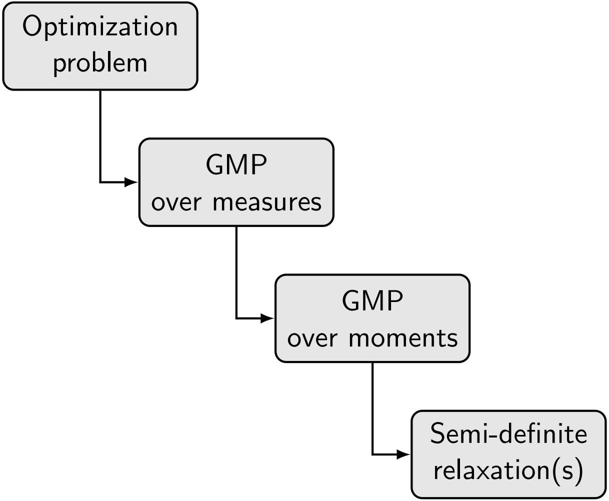

Semi-definite relaxations

Use only $(y_\alpha)_{|\alpha|\leq 2r}$.

Theorem (Lasserre)

$$J^r_{mom} \uparrow J_{meas}$$

Theorem

If $\operatorname{rank}(M_{j-1}) = \operatorname{rank}(M_j) = k$, $\exists$ *k-atomic optimal measure.*

Particular case: if $n = 1$, first relaxation is necessary and sufficient.

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Polynomial approximations

- $\|F - \tilde{F}\| = \epsilon$

$$\min_{\mu} \|\mu\| \qquad\qquad \min_{\mu} \|\mu\|$$

$$\text{s.t. } \langle F, \mu \rangle = c \quad \longrightarrow \quad \text{s.t. } |\langle \tilde{F}, \mu \rangle - c| \leq \epsilon \|\mu\|$$

- Application to orbital RDV:
  - Polynomials of degree 100
  - Computation time: 1.1 seconde
  - Direct LP method: 0.4 seconde
- [C., Arzelier, Henrion, Lasserre: CDC'13]

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Table of contents

Non-linear impulsive problems

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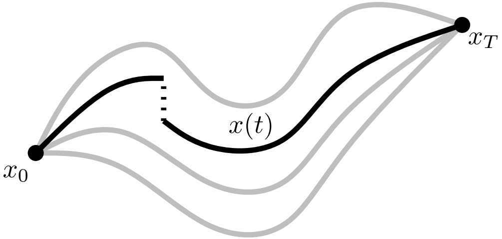

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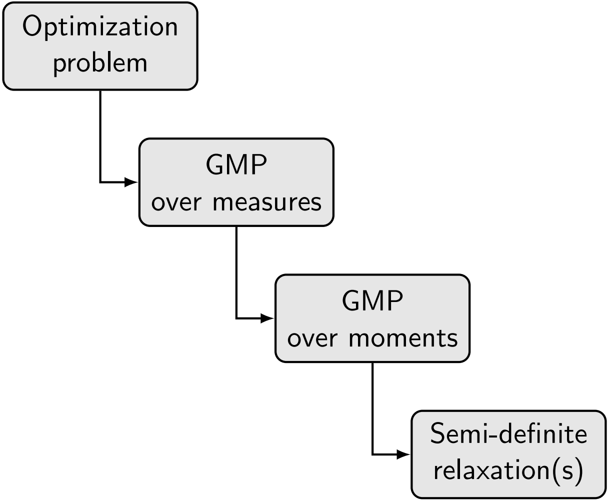

Procedure

1. Extended concept of $u(t)$

   $\rightarrow$ "Strong" problem, compact.

2. Weak integration of ODE

   $\rightarrow$ "Weak" problem, a GMP

3. Solve GMP!

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Measure driven ODE

- Generalization of

$$dx(t) = f(t, x(t))\,dt + G(t, x(t))\,u(t)\,dt$$

  into

$$dx(t) = f(t, x(t))\,dt + G(t, x(t))\,\nu(dt)$$

- [Schmaedeke]: $G(t)$.

- [Bressan et Rampazzo]: $G(t, x)$.

"Strong" form

- Decompose $\nu = \nu^C + \nu^D$

- For each t_j, associate $z(\theta)$:

- Concept of solution:

$$x(t^+) = x(0^-) + \int_0^t f(s, x(s))\, ds + \int_0^t G(s, x^C(s))\, \nu^C(ds) + \sum_{t_i \in \mathbf{S}, t_i \leq t} \left(x(t_i^+) - x(t_i^-) \right)$$

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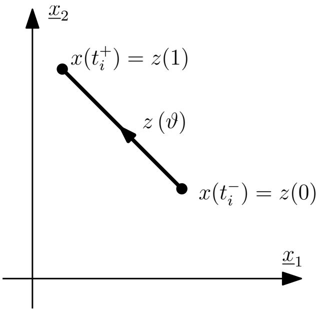

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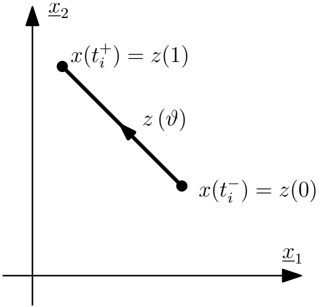

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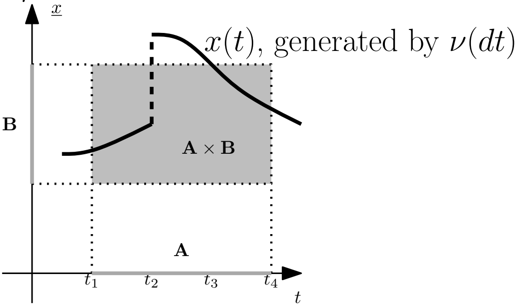
“Weak” form

- Occupation measure μ et ω:



Proposition

μ, ω satisfy $[v(\cdot, x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f, \mu \rangle + \langle \frac{\partial v}{\partial x} G, \omega \rangle$

- [C, Arzelier, Henrion, Lasserre: ACC'12]

"Weak" form

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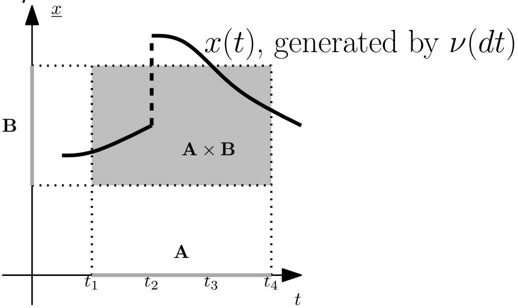

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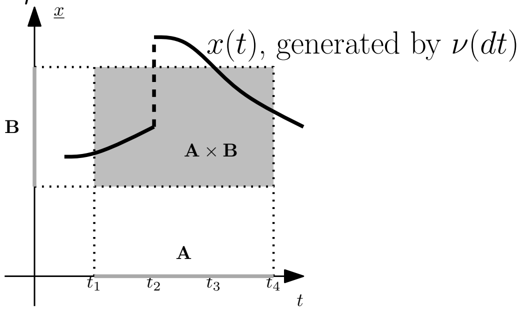

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Table of contents

Switched systems

- Control-affine problems $\rightarrow$ control measures ?

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$$\dot{x} = \sum_{j=1}^{m} f_j\,(t, x(t))\; u_j(t)$$

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1. Extended concept of $u(t)$

   Young measure: $g(u(t)) \to \langle g(s), \nu(ds) \rangle, \quad \nu \in \mathcal{P}(U)$.

2. Weak integration of ODE

   $\to$ [Rubio, Lewis, Vinter]: $[v(\cdot, x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f, \mu \rangle$

3. Solve GMP: [Lasserre, Henrion, Prieur, Trélat]

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3. Solve GMP: [Lasserre, Henrion, Prieur, Trélat]

Procedure

1. Extended concept of $u(t)$

   Young measure: $g(u(t)) \to \langle g(s), \nu(ds) \rangle, \quad \nu \in \mathcal{P}(U)$.

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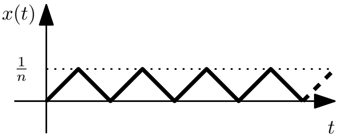

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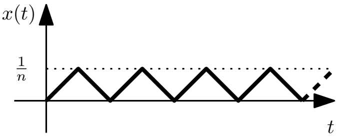

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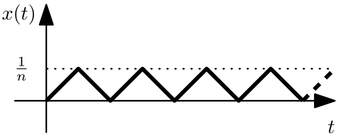

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Modal occupation measures

Proposition

$$[v(\cdot, x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \sum_{j=1}^{m} \frac{\partial v}{\partial x} f_j \, u_j, \ \mu(dt, dx, du) \rangle$$

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- $\Rightarrow$: $\mu_j(\mathbf{A} \times \mathbf{B}) := \int_{\mathbf{A}\times\mathbf{B}\times\mathbf{U}} u_j \, d\mu$
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Example: contrast problem (1/2)

- [Bonnard, C., Cots, Martinon: CDC'13]

$$
\begin{aligned}
\inf\ & -x_3^2(T) - x_4^2(T) \\
\text{s.t. } & \dot{x}_1 = -\Gamma_1 x_1 - x_2\, u \\
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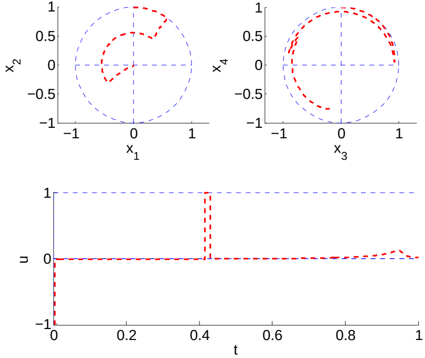

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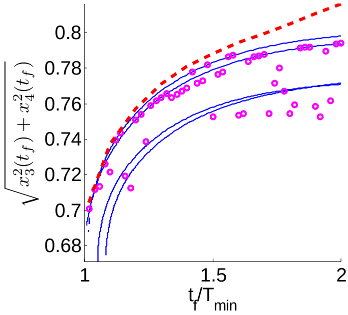

	Measured control		Control measure	
r	$\sqrt{-J_M^r}$	t_r	$\sqrt{-J_M^r}$	t_r
1	1.000	1	0.9827	0.6
2	0.8984	2	0.8756	1.0
3	0.8707	9	0.8599	6.6
4	0.8256	265	0.7973	113
5	0.7881	5147	0.7891	1298
6	0.7867	50027	0.7871	10831

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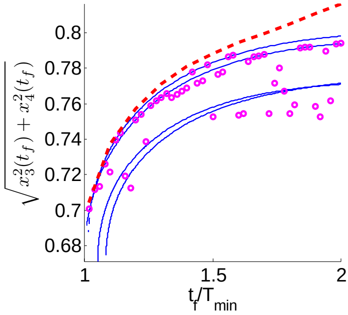

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Example: electric motorbike (1/2)

- [C., Sager, Messine]

$$\inf_{u(t)} \int_0^{10} (V_{alim}\, x_1 u + R_{bat}\, x_1^2)\, dt$$

$$\text{s.t. } \dot{x}_1 = -\frac{R_m}{L_m} x_1 - \frac{K_m}{L_m} x_2 + \frac{V_{alim}}{L_m} u,$$

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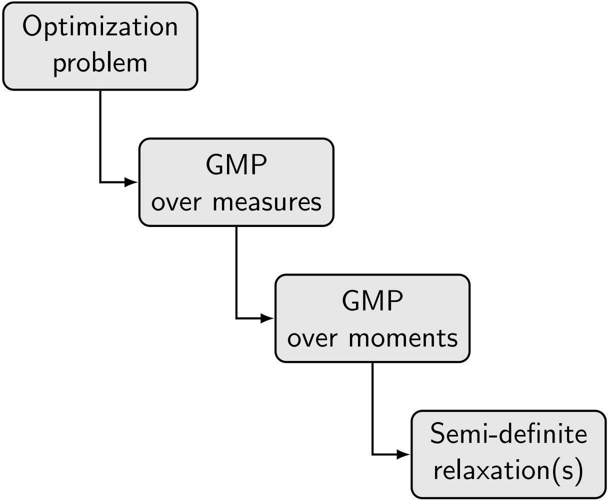

Some difficulties

- ∞-dim decision variable
- Local optimality
- Non smooth behaviors
- State constraints
- Practical implementation

Perspectives

- Relaxation gap?
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- Inverse problem?
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Thanks!

And happy birthday Jean-Bernard!