General background on dynamical systems

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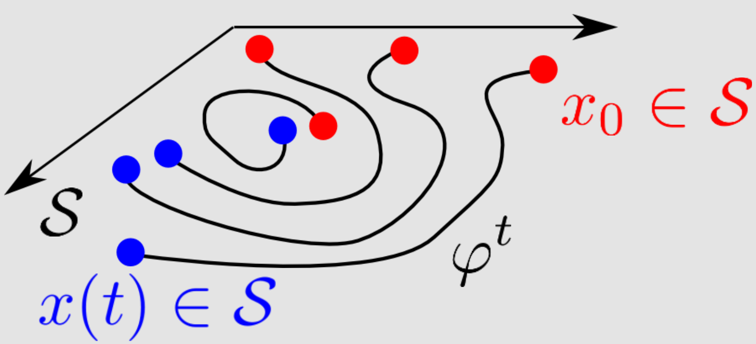
Two descriptions of dynamical systems

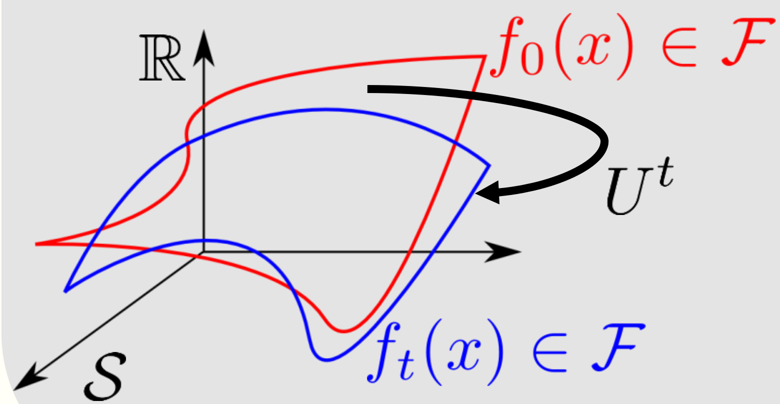
Trajectory-oriented approach

SYSTEM ≡ flow $\varphi^t : \mathcal{S} \to \mathcal{S}$
acting on the state space $\mathcal{S}$



Operator-theoretic approach (Koopman operator)

SYSTEM ≡ operator $U^t : \mathcal{F} \to \mathcal{F}$
acting on a functional space $\mathcal{F}$



➔ pointwise description

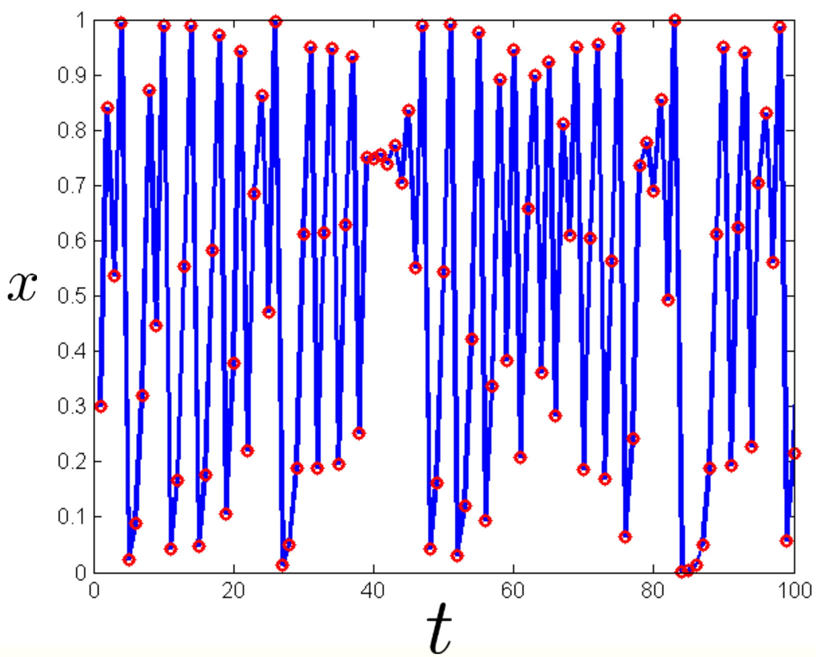
lifting

➔ global description

A first comparison...

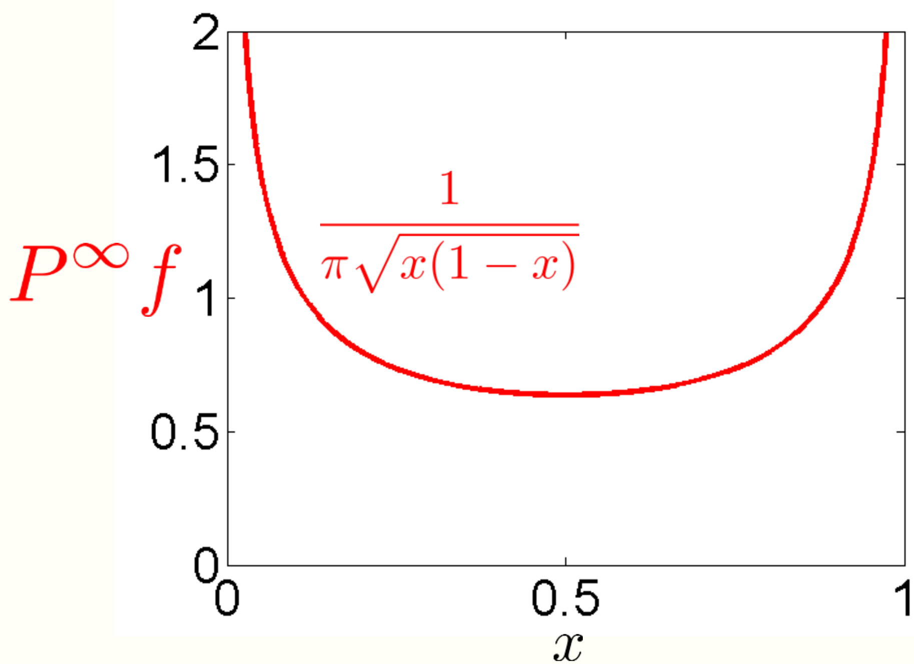
A trajectory...

$$x_{t+1} = 4x_t(1 - x_t)$$



and a density

$$P^1 f(x) = \frac{f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})}{4\sqrt{1-x}}$$



[Lasota and Mackey]

Outline

An operator-theoretic approach to dynamical systems

Spectral properties of the Koopman operator: basic results

Interplay between spectral and geometric properties

Numerical computation

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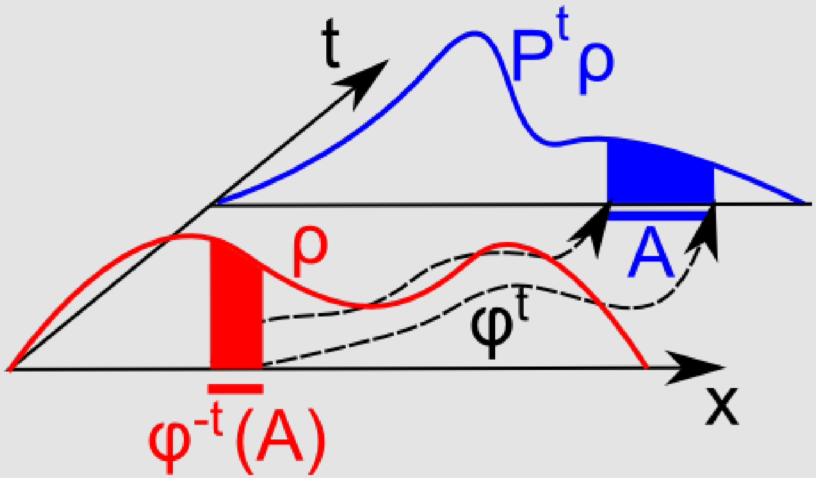
A dynamical system is described by two dual operators

Perron-Frobenius operator P^t

[Liouville 1838, Poincaré, Ulam 1960]

$\rho \in \mathcal{F}^\dagger$ is a density

$$\int_A P^t \rho(x)\, dx = \int_{\varphi^{-t}(A)} \rho(x)\, dx$$

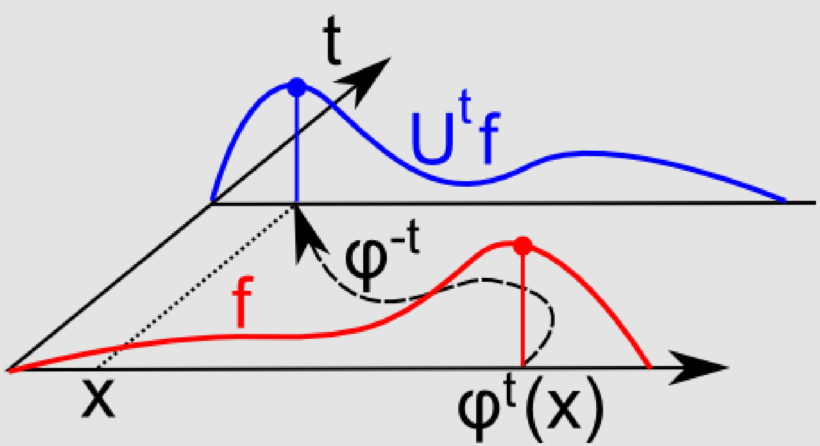


Koopman operator U^t

[Koopman 1930]

$f \in \mathcal{F}$ is an observable

$$U^t f(x) = f \circ \varphi^t(x)$$



Duality: $\int_{\mathcal{S}} (U^t f(x))\, \rho(x)\, dx = \int_{\mathcal{S}} f(x)\, (P^t \rho(x))\, dx$ $\quad \langle U^t f, \rho \rangle = \langle f, P^t \rho \rangle$

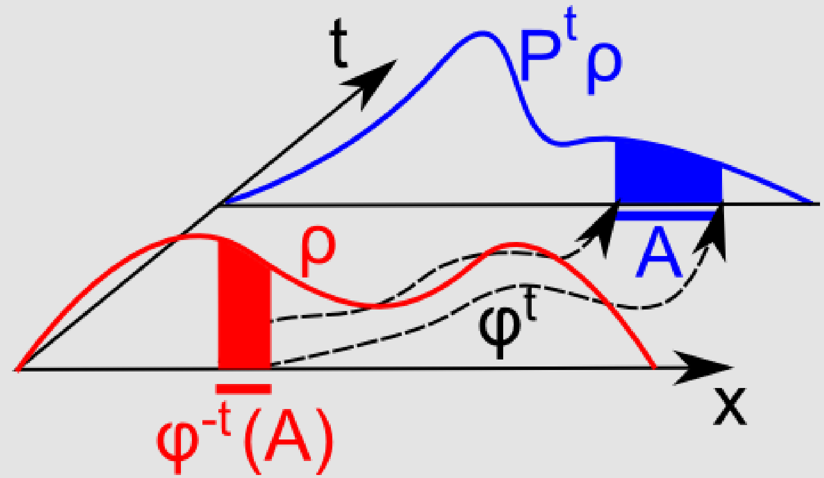
A dynamical system is described by two dual operators

Perron-Frobenius operator P^t

[Liouville 1838, Poincaré, Ulam 1960]

$\mu \in \mathcal{M}$ is a measure

$$\int_A P^t \mu(dx) = \int_{\varphi^{-t}(A)} \mu(dx)$$

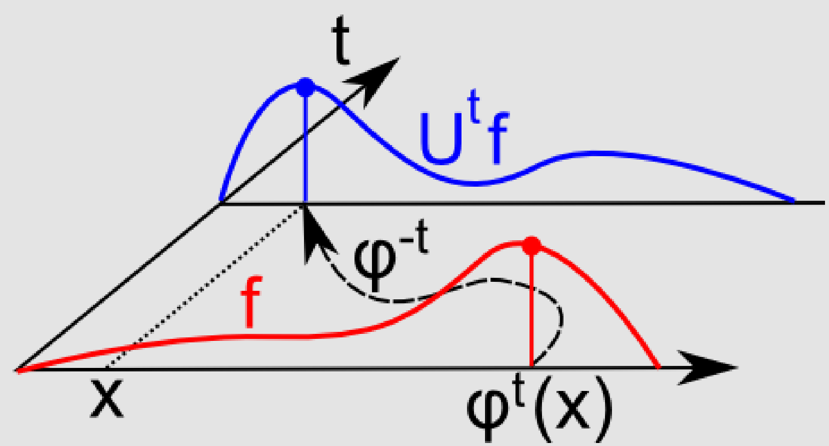


Koopman operator U^t

[Koopman 1930]

$f \in \mathcal{F}$ is an observable

$$U^t f(x) = f \circ \varphi^t(x)$$



Duality: $\int_{\mathcal{S}} (U^t f(x))\, \mu(dx) = \int_{\mathcal{S}} f(x)\, P^t \mu(dx)$ $\quad \langle U^t f, \mu \rangle = \langle f, P^t \mu \rangle$

We focus on the Koopman operator

Finite state space $\mathcal{S} = \{1, \ldots, N\}$ ➔ observable = vector $f : \{1, \ldots, N\} \to \mathbb{C}$

P is the transition matrix of the Markov chain and $U = P^T$ $\quad U : \mathbb{C}^N \to \mathbb{C}^N$

Continuous state space $\mathcal{S} = \mathbb{R}^N$ ➔ observable = function $f : \mathbb{R}^N \to \mathbb{C}$

Discrete-time system $\varphi : \mathbb{R}^N \to \mathbb{R}^N$

Koopman operator = composition operator $Uf = f \circ \varphi$ $\quad U : \mathcal{F} \to \mathcal{F}$

Continuous-time system $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ $\quad \varphi(t, \cdot) \equiv \varphi^t(\cdot)$

semi-group of Koopman operators $U^t f = f \circ \varphi^t$ $\quad t \in \mathbb{R}^+$

$\dot{x} = F(x)$ $\quad U^t = e^{Lt}$

$f \in C^1(\mathbb{R}^N)$ $\quad Lf = F \cdot \nabla f$ infinitesimal generator

Infinite-dimensional state space $\mathcal{S}$ ➔ observable = functional $f : \mathcal{S} \to \mathbb{C}$

dynamical system: PDE $\varphi : \mathbb{R}^+ \times \mathcal{S} \to \mathcal{S}$ $\quad U : \mathcal{U} \to \mathcal{U}$

The Koopman operator is linear

$$
\begin{aligned}
U^t(a_1 f_1(x) + a_2 f_2(x)) &= a_1 f_1(\varphi^t(x)) + a_2 f_2(\varphi^t(x)) \\
&= a_1 U^t f_1(x) + a_2 U^t f_2(x)
\end{aligned}
$$

There is a price to pay: the operator is infinite-dimensional

lifting

	Finite-dimensional	**Infinite-dimensional**
Dynamical system	nonlinear	linear
Optimization problem	nonconvex	convex

Outline

An operator-theoretic approach to dynamical systems

Spectral properties of the Koopman operator: basic results

Interplay between spectral and geometric properties

Numerical computation

For dissipative systems, we consider the spectral properties of the Koopman operator

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$ $\quad U^t \phi_\lambda = \phi_\lambda \circ \varphi^t = e^{\lambda t} \phi_\lambda$

Eigenvalue $\lambda \in \sigma(U) \subset \mathbb{C}$ $\quad L\phi_\lambda = F \cdot \nabla \phi_\lambda = \lambda \phi_\lambda$

Conservative systems $\mathcal{F} = L_2(\mathcal{S})$ $\quad \mathcal{F} = \mathcal{F}^\dagger$

$|e^\lambda| = 1$

$\lambda \in i\mathbb{R}$

$\forall \lambda \in \sigma(U)$ ➔ U^t is unitary

Dissipative systems (there is an attractor) $\mathcal{F} = C^1(\mathcal{S})$ or $\mathcal{F}$: analytic functions

U^t is not unitary

eigenfunctions ρ_λ of the Perron-Frobenius operator are Dirac functions

➔ consider Koopman eigenfunctions ϕ_λ

Basic properties of the Koopman eigenfunctions

The constant observable is a Koopman eigenfunction

$$\phi_0(x) = 1 \qquad\qquad \lambda = 0 \in \sigma(U)$$

Multiplication of eigenfunctions

$$\phi_{\lambda_1}\phi_{\lambda_2} = \phi_{\lambda_1+\lambda_2} \qquad\qquad \lambda_1, \lambda_2 \in \sigma(U) \Leftrightarrow \lambda_1 + \lambda_2 \in \sigma(U)$$

Power of an eigenfunction

$$(\phi_\lambda)^k = \phi_{k\lambda} \qquad\qquad \lambda \in \sigma(U) \Leftrightarrow k\lambda \in \sigma(U)$$

Example: $\dot{x} = \mu x \qquad \varphi^t(x) = e^{\mu t}x$

$$\mu \in \sigma(U) \qquad\qquad k\mu \in \sigma(U) \qquad\qquad k \in \mathbb{N}$$

$$\phi_\mu(x) = x \qquad\qquad \phi_{k\mu(x)} = x^k \in C^1$$

We can consider the spectral expansion of the Koopman operator

Regular part (related to the continuous spectrum)

$$U^t f(x) = \sum_{k=1}^{\infty} \underbrace{\overbrace{v_k}^{\text{Koopman mode}} \overbrace{\phi_{\lambda_k}(x)}^{\text{eigenfunction}}}_{f^*_\lambda = \langle f, \rho_\lambda \rangle \, \phi_\lambda \text{ : projection of } f \text{ on } \phi_{\lambda_j}} e^{\lambda_k t} + U_r^t f(x)$$

Example: $\dot{x} = \mu x \qquad \varphi^t(x) = e^{\mu t} x$

if f is analytic ➔ Taylor expansion

$$U^t f = f \circ \varphi^t(x) = \underset{\swarrow \; v_0 \phi_0 \, e^{0t}}{f_0} + \sum_{k=1}^{\infty} \underbrace{\frac{d^k f}{dx^k}(0) \, x^k}_{f^*_\lambda = \langle f, \delta^{(k)} \rangle \, x^k} e^{k\mu t}$$

We can consider the spectral expansion of the Koopman operator

Consider the identity observable $f(x) = x$

$$\varphi^t(x) = \sum_{j=1}^{\infty} \overbrace{v_j}^{\text{Koopman mode}} \overbrace{\phi_{\lambda_j}(x)}^{\text{eigenfunction}} e^{\lambda_j t} + \underbrace{U_r^t x}_{\text{Regular part (related to the continuous spectrum)}}$$

Example: linear system $\dot{x} = Ax$ $\quad Av_k = \lambda_j v_k \quad w_k A = \lambda_j w_k$

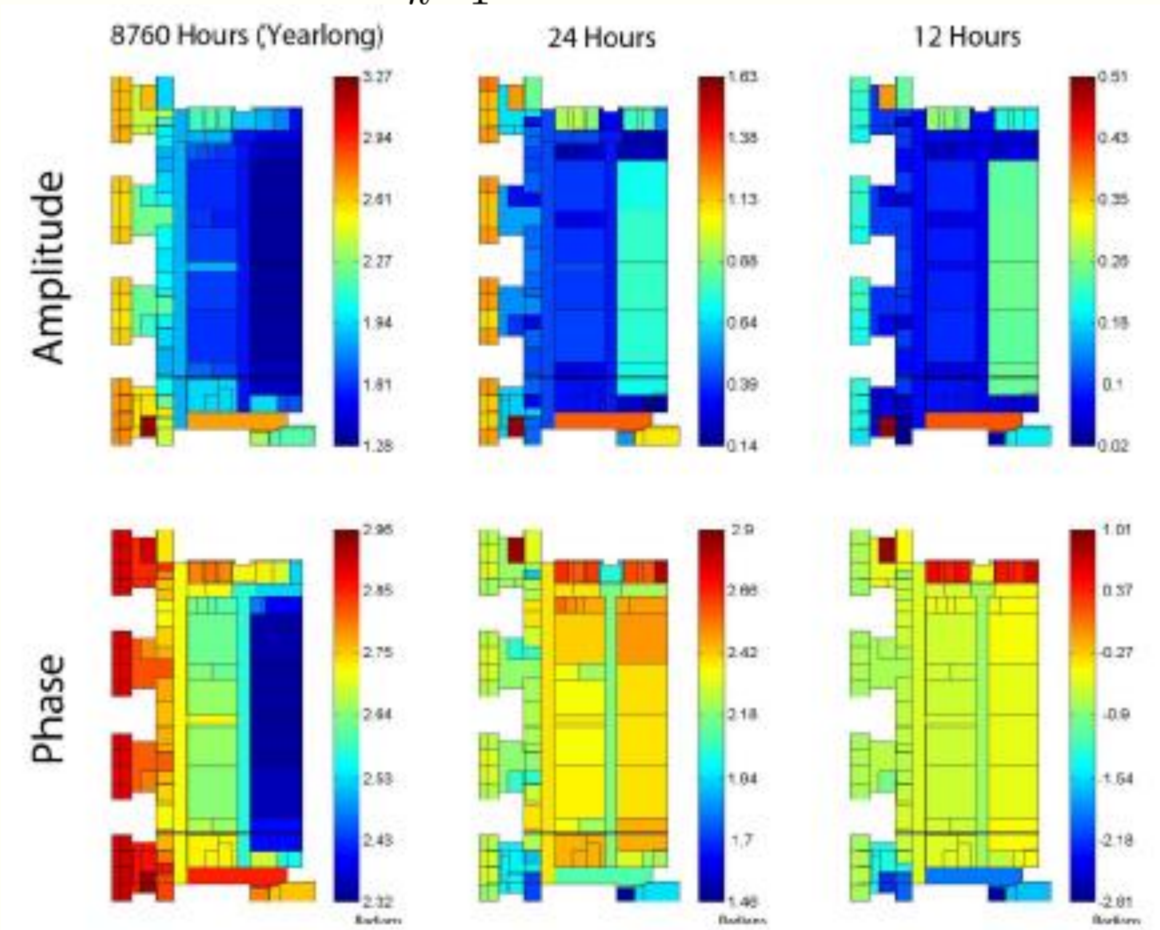
Koopman eigenvalues $\lambda_k \in \sigma(U)$

Koopman eigenfunctions $\phi_k(x) = w_k^T x$

$$\varphi^t(x) = \sum_{k=1}^{N} v_k \left(w_k^T x\right) e^{\lambda_k t}$$

Application of the Koopman mode analysis: building energy efficiency

$$U^t \mathbf{f}(x) = \sum_{k=1}^{\infty} \mathbf{v_k}\, \phi_{\lambda_k}(x)\, e^{\lambda_k t}$$



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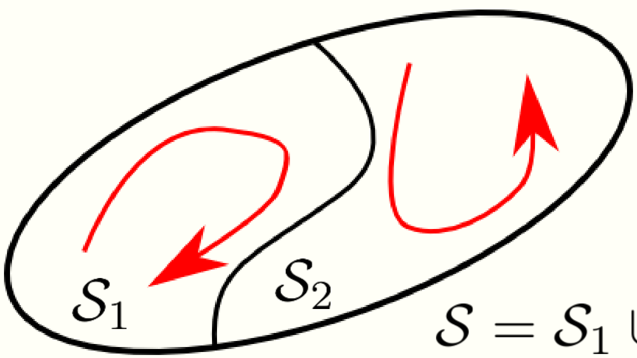
Numerical computation

Constant eigenfunctions define invariant sets (ergodic partition)

$\phi_0(x) = 1$ is always an eigenfunction

Two invariant sets $\mathcal{S}_1 \supseteq \varphi^t(\mathcal{S}_1)$ and $\mathcal{S}_2 \supseteq \varphi^t(\mathcal{S}_2)$



$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$

$$\phi_0'(x) = \begin{cases} 1 & x \in \mathcal{S}_1 \\ 0 & x \in \mathcal{S}_2 \end{cases} \qquad \phi_0''(x) = \begin{cases} 0 & x \in \mathcal{S}_1 \\ 1 & x \in \mathcal{S}_2 \end{cases}$$

$$U^t \phi_0'(x) = \phi_0' \circ \varphi^t(x) = \phi_0'(x)$$

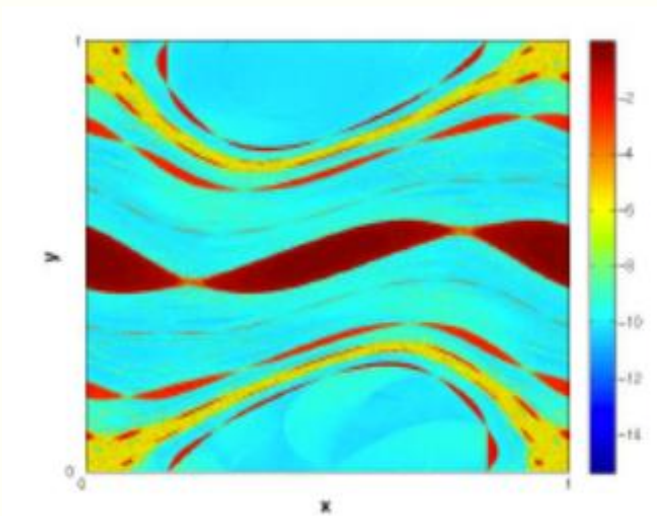
If ϕ_0 is unique ➔ the system is ergodic (every state can be reached)

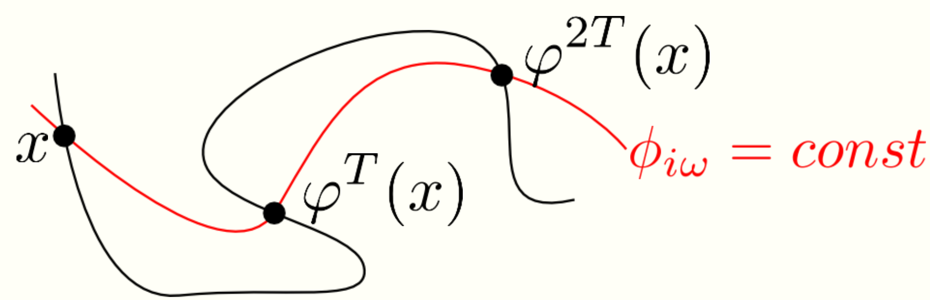
Level sets of $\phi_{\lambda_j}^{\lambda_k} \phi_{\lambda_k}^{-\lambda_j}$ are co-dimensional 1 invariant sets (trajectories in 2D)

Eigenfunctions with purely imaginary eigenvalues define periodic partitions

The level sets of $\phi_{i\omega}$ with $\omega \in \mathbb{R}^+$ define a periodic partition of period $T = 2\pi/\omega$

$$\phi_{i\omega} \circ \varphi^T(x) = e^{i\omega T}\phi_{i\omega}(x) = \phi_{i\omega}(x)$$





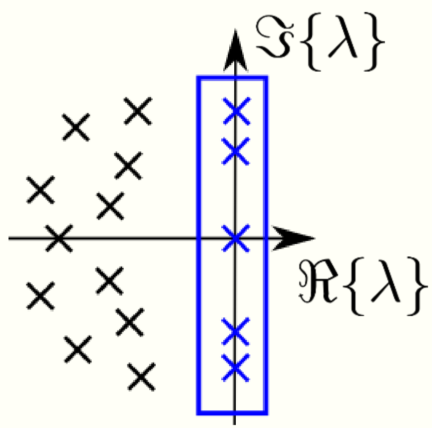
[Mezic and Banaszuk 2004, Mezic 2005]

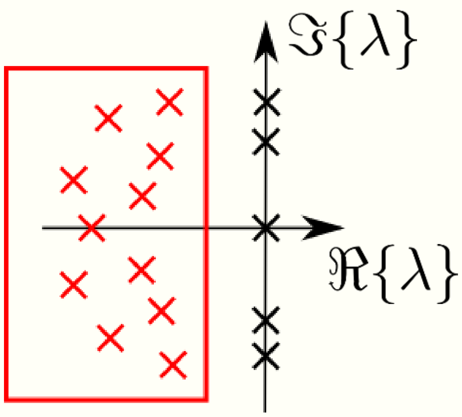
Limit cycle: the level sets of the Koopman eigenfunction $\phi_{i\omega}$ are the isochrons

[Mauroy and Mezic, 2012]

Dissipative systems: the spectrum of the Koopman operator can be decomposed in two parts

Assume that the system admits an attractor Γ





Spectrum of the Koopman operator acting on $\mathcal{F}|_\Gamma = \{f|_\Gamma : \Gamma \to \mathbb{C} | f \in \mathcal{F}\}$

- ergodic motion on Γ (frequencies)
- related to isochrons

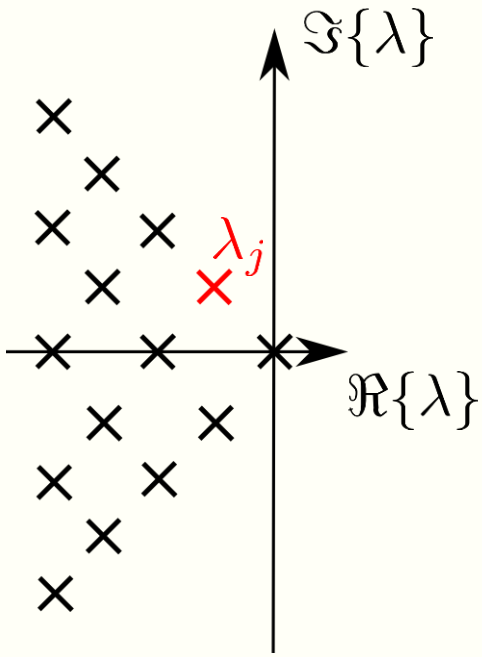
Spectrum of U^t restricted to $\mathcal{F}_{X\setminus\Gamma} = \{f \in \mathcal{F} : f(x) = 0 \, \forall x \in \Gamma\}$

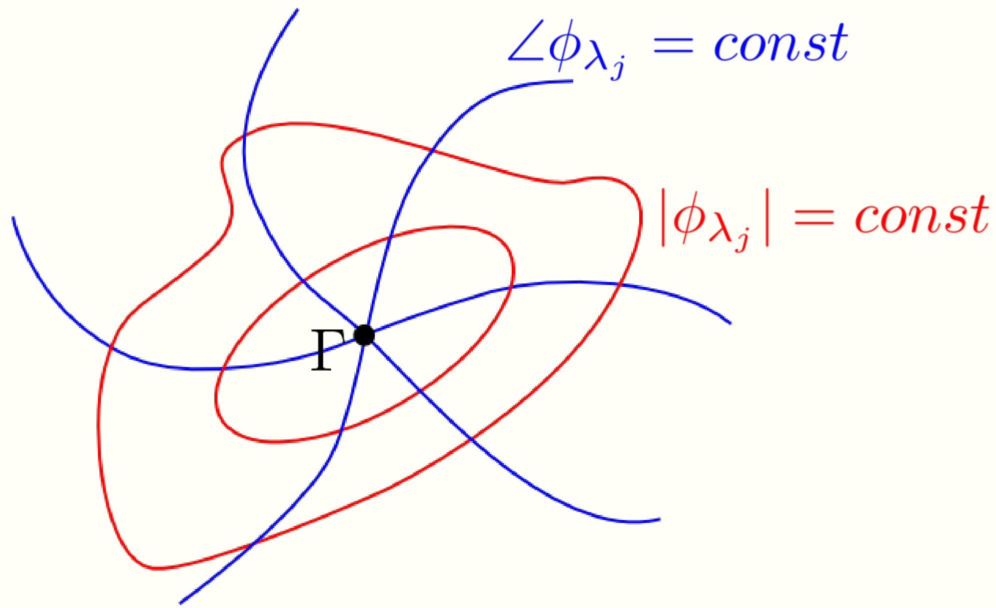
- asymptotic convergence to Γ (Floquet/Lyapunov exponents)
- related to isostables

The spectrum of the Koopman operator is related to known notions of stability

$\mathcal{F}$: space of analytic functions in a neighborhood of Γ

Γ is a fixed point





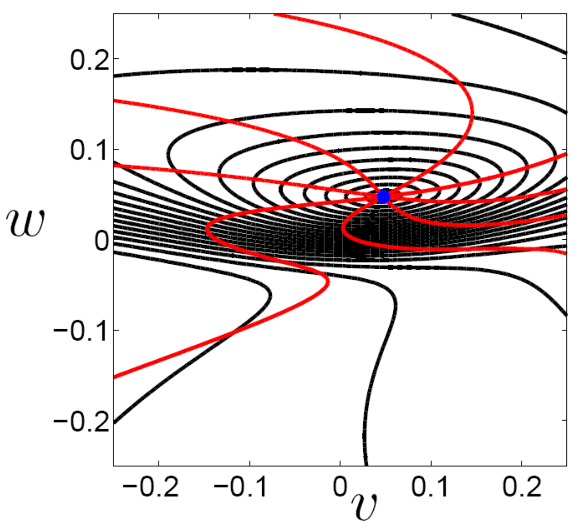
λ_j : eigenvalues of the Jacobian matrix at Γ

Action-angle coordinates and global linearization

$$\frac{d}{dt}\phi_{\lambda_k}(\varphi^t(x)) = \lambda_k \phi_{\lambda_k}(x)$$

$$\phi_{\lambda_k}(x) = r_k e^{i\theta_k}$$

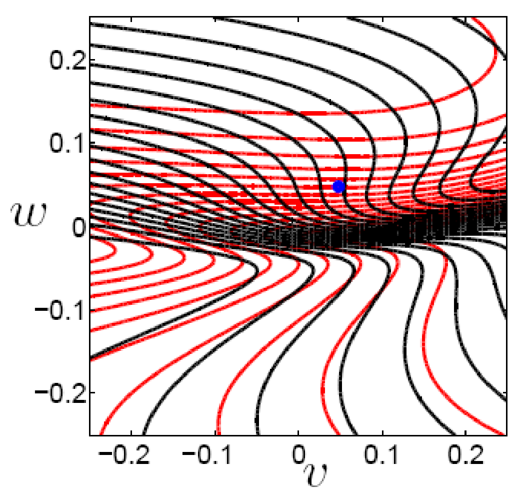
$$\dot{r}_k = \Re\{\lambda_k\} r_k$$
$$\dot{\theta}_k = \Im\{\lambda_k\}$$



$$\phi_{\lambda_k}(x) = y_k$$

$$\mathbf{z} = \mathbf{T}\mathbf{y}$$

$$\dot{\mathbf{z}} = \mathbf{J}(x^*)\mathbf{z}$$

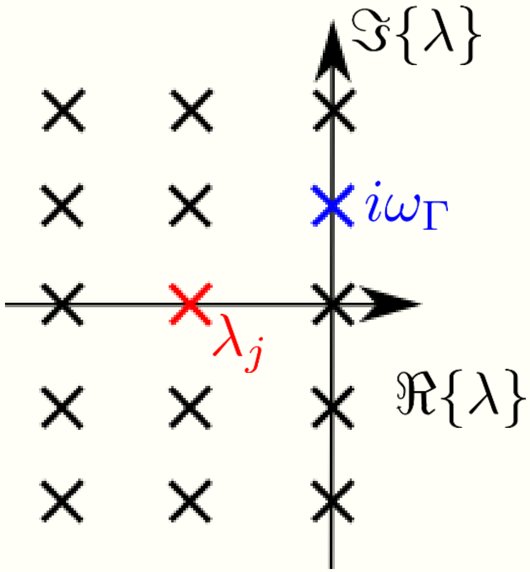


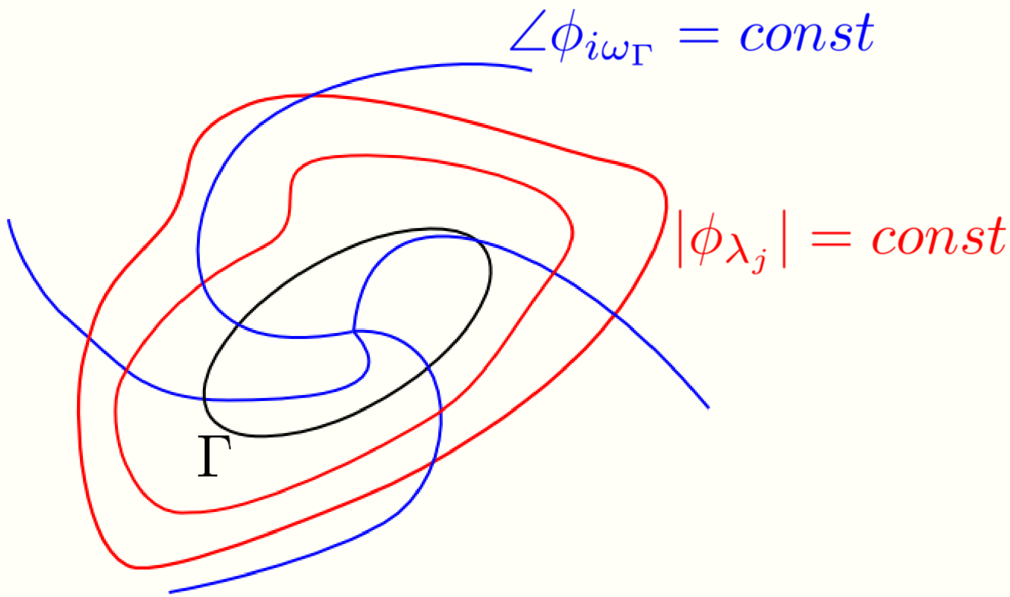
Rigorous extension of Hartman-Grobman theorem *[Lan and Mezic, 2012]*

The spectrum of the Koopman operator is related to known notions of stability

$\mathcal{F}$: space of analytic functions in a neighborhood of Γ

Γ is a limit cycle





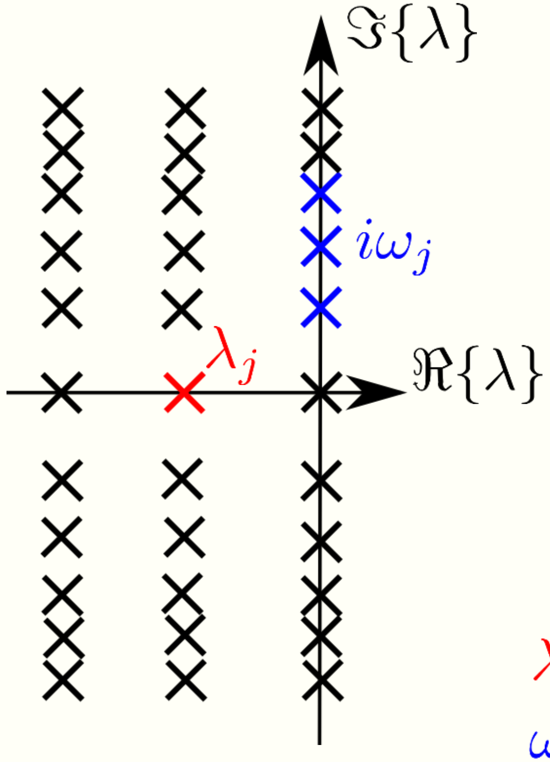
λ_j : Floquet exponents of Γ

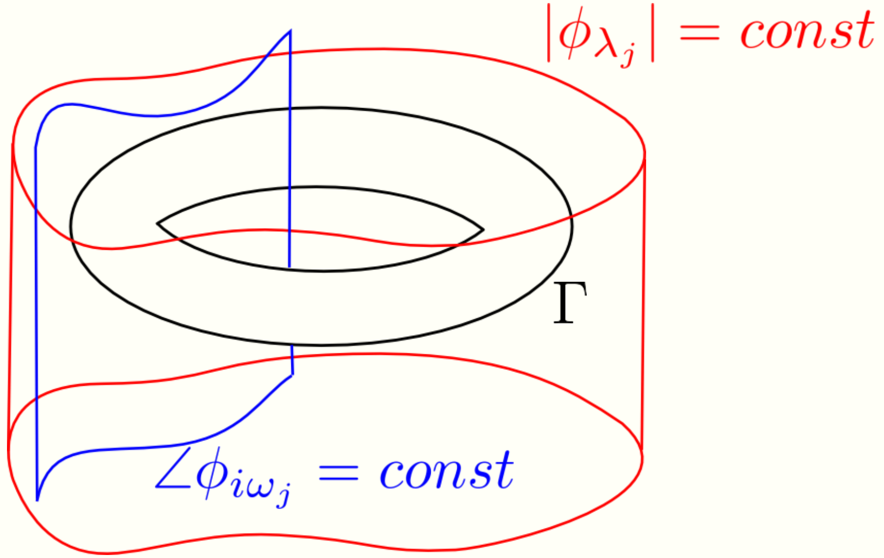
ω_Γ: frequency of Γ

The spectrum of the Koopman operator is related to known notions of stability

$\mathcal{F}$: space of analytic functions in a neighborhood of Γ

Γ is a quasiperiodic torus





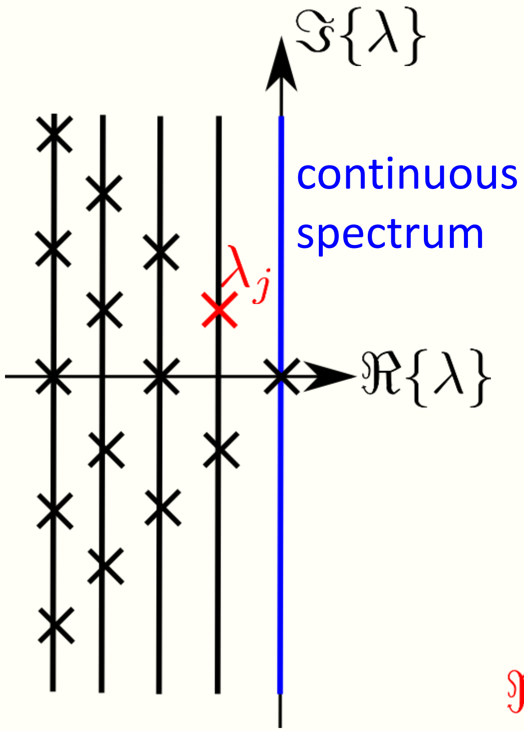
λ_j : generalized Floquet exponents of Γ

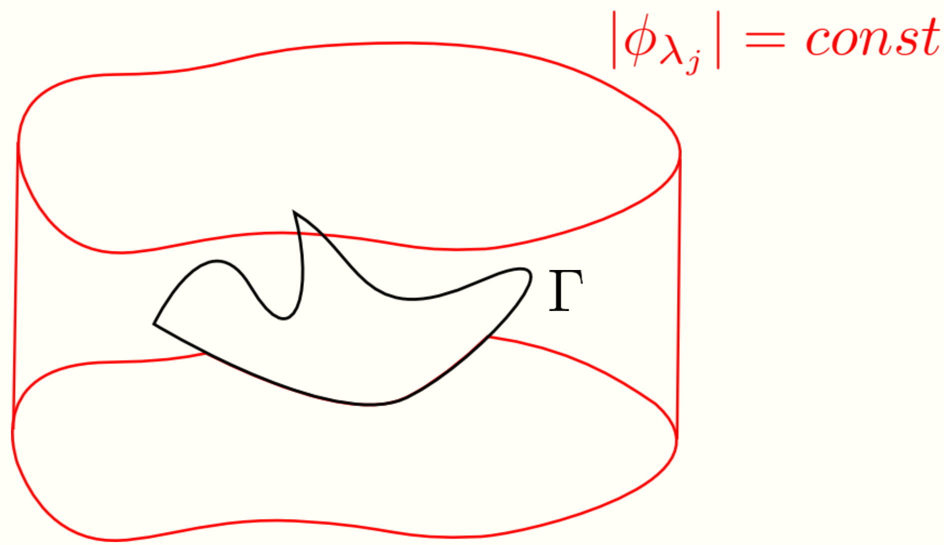
ω_Γ: basic frequencies of Γ

The spectrum of the Koopman operator is related to known notions of stability

$\mathcal{F}$: space of analytic functions in a neighborhood of Γ

Γ is a strange attractor





$\Re\{\lambda_j\}$: Lyapunov exponents of Γ

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There exist several methods for computing the Koopman modes and eigenfunctions

We know the trajectories: Fourier and Laplace averages

[Mezic 2012, Mohr and Mezic 2014]

We have data: (Extended) Dynamic Mode Decomposition

[Schmid 2010, Rowley et al. 2009, Williams et al. 2015]

We know the vector field: expansion on a finite (polynomial) basis

related to the problem of moments

[Mauroy and Mezic 2013]

The Koopman eigenfunctions can be computed with averages along the trajectories

The Fourier average

$$f^*_{i\omega}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ \varphi^t(x)\, e^{-i\omega t}\, dt$$

is the projection of f on $\phi_{i\omega}$ ➔ $f^*_{i\omega} \equiv \phi_{i\omega}$ if it is nonzero

The generalized Laplace average

$$f^*_{\lambda_j}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(f \circ \varphi^t(x) - \sum_{\substack{\lambda_k \in \sigma(U) \\ \Re\{\lambda_k\} > \Re\{\lambda_j\}}} f^*_{\lambda_k}(x) e^{\lambda_k t} \right) e^{-\lambda_j t}\, dt$$

is a projection of f on ϕ_{λ_j} ➔ $f^*_{\lambda_j} \equiv \phi_{\lambda_j}$ if it is nonzero

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The Fourier average

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is a projection of f on ϕ_{λ_j} ➔ $f^*_{\lambda_j} \equiv \phi_{\lambda_j}$ if it is nonzero

and if $f^*_{\lambda_k} = 0 \quad \forall \Re\{\lambda_k\} > \Re\{\lambda_j\}$

For $f^*_{\lambda_1}$ with $0 \neq \Re\{\lambda_1\} > \Re\{\lambda_k\}$: choose f with $f^*_0 = 0$, i.e. $f(x) = 0\, \forall x \in \Gamma$

The Koopman operator theoretic framework is conducive to data analysis

$$X = [x_1, x_2, \cdots, x_{m-1}]$$

$$Y = [x_2, x_3, \cdots, x_m]$$

$$K \triangleq Y X^{\dagger}$$

methods based on SVD

eigenvalues of K → Koopman eigenvalues

right eigenvectors of K → Koopman modes

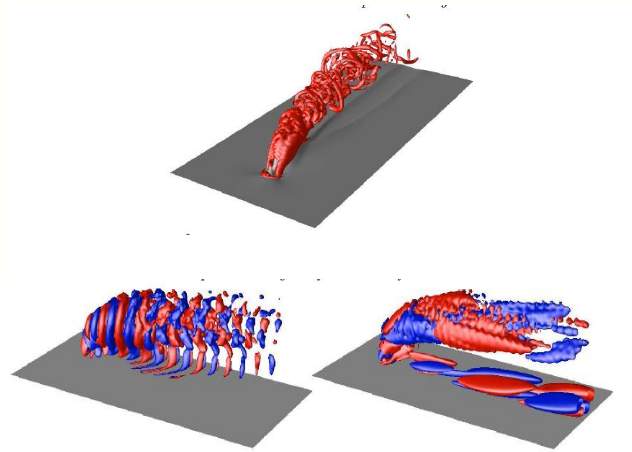
left eigenvectorss of K → Koopman eigenfunctions (linear approximation)

[Schmid 2010, Tu et al. 20014]

Recent extension(s) *[Williams et al. 2015]*

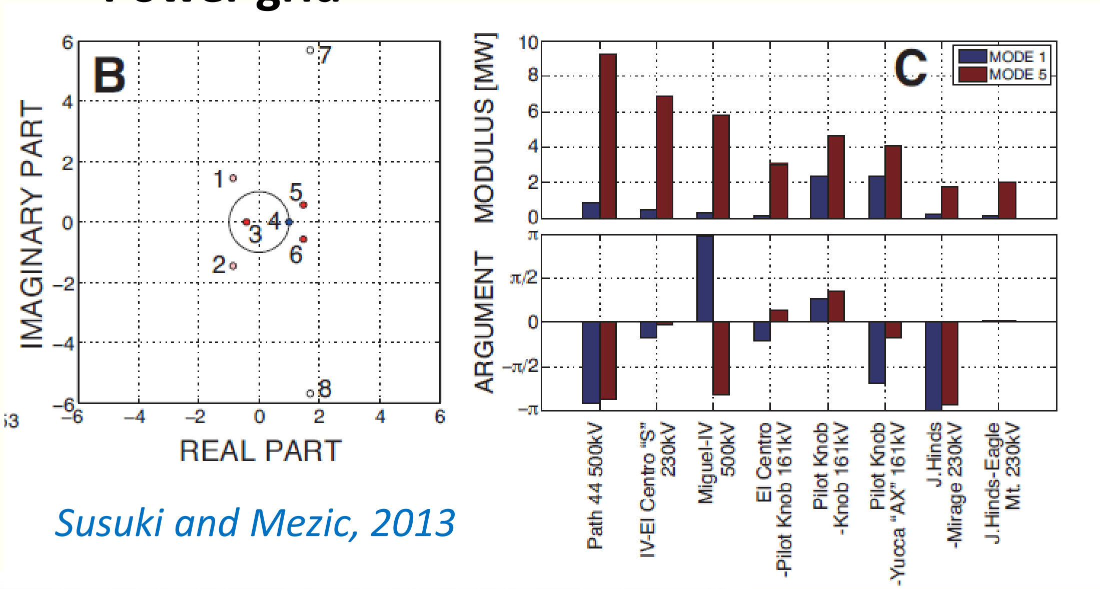
Applications of dynamic mode decomposition

Fluid dynamics



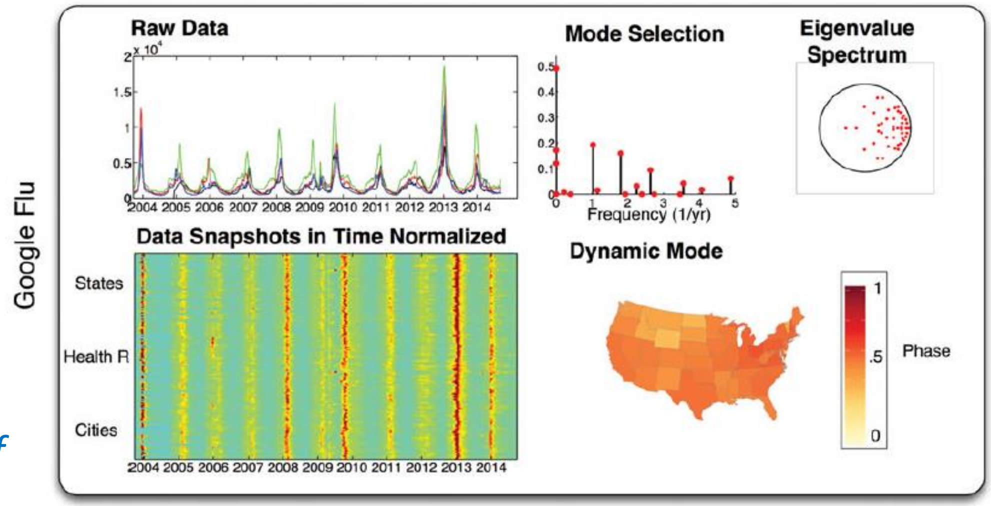
Rowley et al., 2009

Power grid



Susuki and Mezic, 2013

Epidemics



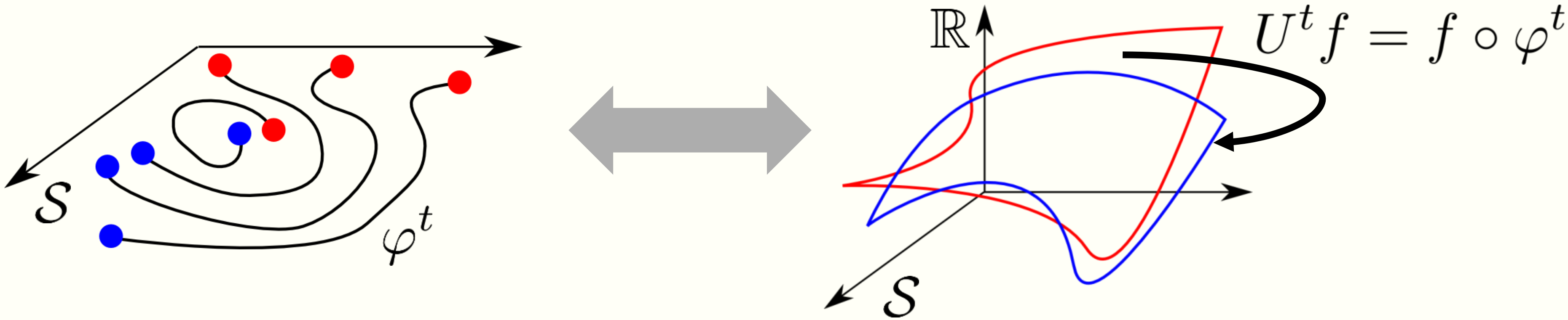
Proctor and Eckhoff

And many other applications…

Applications of Koopman analysis (to name a few):

- fluid dynamics (Rowley, Mezic, Henningson)
- power grid (Susuki and Mezic)
- energy efficiency (Georgescu and Mezic)
- system estimation (Surana and Banaszuk)
- image processing (Kutz)
- epidemics (Proctor and Eckhoff)
- neural dynamics, cardiac defibrillation (Wilson and Moehlis)
- high dimensional systems (Williams, Kevrekidis, Rowley)
- nonlinear vibration (Cirillo et al.)
- …

In summary



The Koopman operator is **linear**

The **spectral properties** of the Koopman operator are related to (geometric) properties of the dynamical system

The framework yields systematic **spectral methods** (many applications)

Recent theory in the case of dissipative systems

Perspectives

Development of efficient numerical methods
to compute the spectral properties of the Koopman operator

relationship to polynomial optimization (and SOS methods)?

Vice-versa: some (new) insight into global optimization?

An invitation to the community:

application of the Koopman framework to control theory

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Spectral properties (ergodic case)

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Dynamic mode decomposition and Koopman operator

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