

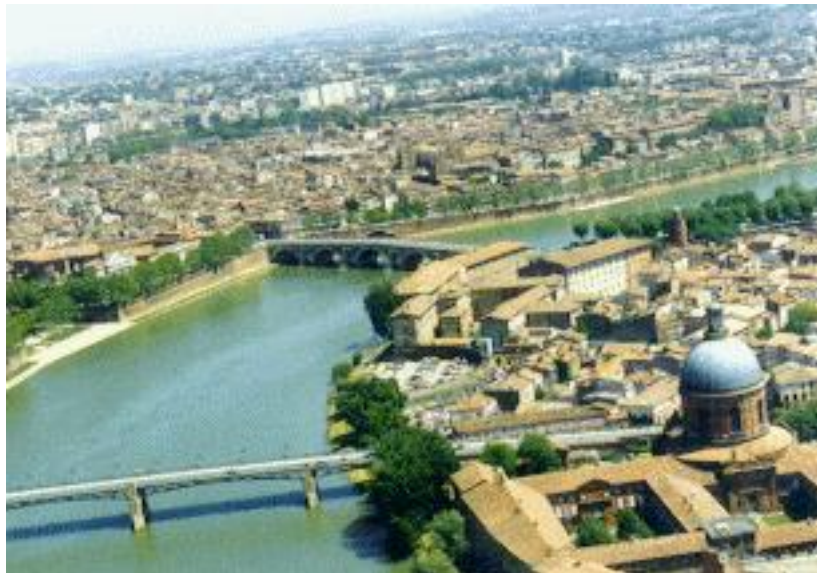
GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART III.3

**ROBUST DESIGN:
SIMULTANEOUS STABILIZATION**

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Summary

So far we have investigated two approaches for robust design

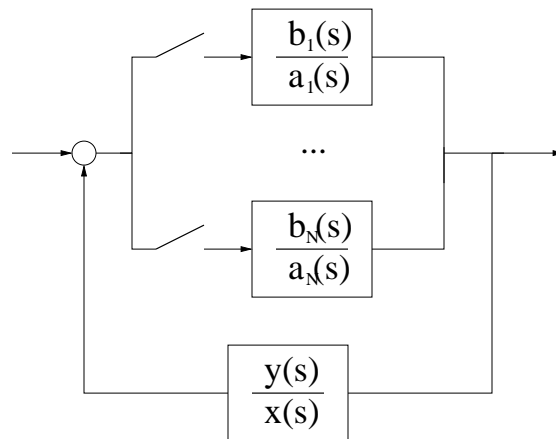
- **pole placement**, where **non-convexity** of the stability region can be overcome by a convex **approximation**, which generally results in **low-order** controllers
- **Youla-Kučera parametrization**, which is convex but where **infinite-dimensionality** must be overcome by a finite-dimensional **approximation**, which generally results in **high-order** controllers

In the last part of the course we will mix both techniques with the help of **optimization over linear matrix inequalities**

But before, we will survey briefly another **basic** but still **open** robust control problem, namely **simultaneous stabilization**

Simultaneous stabilization

Consider the system



where the same controller $y(s)/x(s)$ must stabilize **simultaneously** N plants $b_i(s)/a_i(s)$

Useful configuration if a sensor or actuator **fails**, or is **turned off** during start-up or shutdown

Despite the simplicity of the problem, it turns out that

There is no algorithmic solution to the simultaneous stabilization problem

We know that the problem is **undecidable**, i.e. there is no algorithm with rational operations that can detect whether the system is stabilizable or not

The problem can be solved in **special cases**

- two plants
- static output feedback

but remains open in its full generality, and only **heuristic** algorithms have been designed

Strong stabilization

Consider two plants $P_1(s)$, $P_2(s)$ with no common unstable real poles and define the difference plant

$$P(s) = P_1(s) - P_2(s)$$

P_1 and P_2 are simultaneously stabilizable iff P is stabilizable with a stable controller, or strongly stabilizable

More generally, the problem of knowing whether a given plant is strongly stabilizable is solved as follows

P is strongly stabilizable iff it has an even number of real poles between every pair of real zeros in the closed right half-plane

There exists a more elaborate version for three plants:

Three plants P_1 , P_2 , P_3 are simultaneously stabilizable iff some plant P built from the P_i s is stabilizable by a stable controller whose inverse is also stable

Strong stabilization

First example

$$P(s) = \frac{s - 1}{s(s - 2)}$$

Positive real zeros at $s = 1$ and $s = \infty$, with one pole at $s = 2$ in between, so the plant is **not strongly stabilizable**

Second example

$$P(s) = \frac{(s - 1)^2(s^2 - s + 1)}{(s - 2)^2(s + 1)^3}$$

Positive real zeros at $s = 1$ (two) and $s = \infty$, with two poles at $s = 2$ in between, so the plant is **strongly stabilizable**

Third example

$$P_1(s) = \frac{1}{s + 1} \quad P_2(s) = \frac{1}{(s + 1)(s - 1)}$$

Two plants simultaneously stabilizable iff

$$P(s) = P_2(s) - P_1(s) = -\frac{s - 2}{(s + 1)(s - 1)}$$

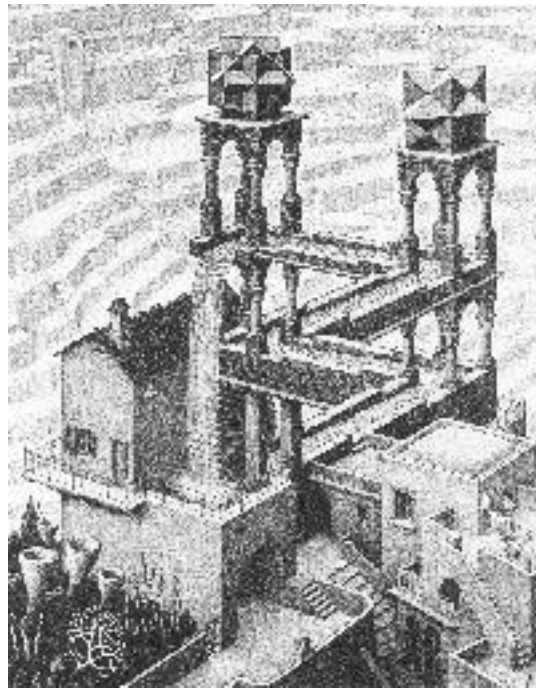
strongly stabilizable, which is the case since it has two zeros at $s = 2$ and $s = \infty$ and one unstable pole at $s = 1$

Strong stabilization

Whether a plant is strongly stabilizable or not can be checked **very easily**: either we compute the poles and zeros, or we use a modified version of the Routh table

Finding a stable controller is **more difficult**, available algorithms are based on **interpolation of units** in R and generally result in high-order controllers

It has been shown that there is actually **no upper bound** on the degree of the strongly stabilizing controller



Infinite stream
Maurits Cornelis Escher
(Leeuwarden 1898 - Baarn 1972)

Simultaneous stabilization with static output feedback

In the special case of **static output feedback** the problem can be solved with the help of **Hermite's stability criterion**



Charles Hermite
(Dieuze 1822 - Paris 1901)

The polynomial $c(s) = c_0 + c_1s + \dots + c_ns^n$ is **stable** iff the symmetric matrix

$$H(c) = \sum_{i=0}^n \sum_{j=0}^n c_i c_j H_{ij}$$

is **positive definite**, where matrices H_{ij} are given and depend on the root clustering region only

Example

discrete-time stability and $n = 3$

$$H(c) = \begin{bmatrix} c_3^2 - c_0^2 & c_2c_3 - c_0c_1 & c_1c_3 - c_0c_2 \\ c_2c_3 - c_0c_1 & c_2^2 + c_3^2 - c_0^2 - c_1^2 & c_2c_3 - c_0c_1 \\ c_1c_3 - c_0c_2 & c_2c_3 - c_0c_1 & c_3^2 - c_0^2 \end{bmatrix}$$

Simultaneous stabilization with static output feedback

We build the **Hermite matrices** corresponding to each closed-loop characteristic polynomials

$$c_i(s) = a_i(s) + kb_i(s) \quad i = 1, \dots, N$$

They are **quadratic polynomial matrices** in the static output feedback gain

$$H_i(k) = H_{i0} + H_{i1}k + H_{i2}k^2$$

and all of them must be positive definite, i.e.

$$H(k) = \text{block diag}_{i=1, \dots, N} H_i(k) > 0$$

So if we arrange all distinct real zeros of matrix $H(k)$ in increasing order

$$k_0 = -\infty \leq k_1 \leq \dots \leq k_m \leq k_{m+1} = +\infty$$

where $m \leq 2nN$ and we denote by π_i the number of positive eigenvalues of H in the interval $I_i =]k_i, k_{i+1}[$ we obtain the following result

The static output feedback simultaneous stabilization problem is solved iff
 $k \in I_i$ when $\pi_i = nN$

Computing the eigenvalues of a quadratic polynomial matrix can be done **efficiently**

Simultaneous stabilization with static output feedback

Example

We want to simultaneously stabilize $N = 4$ operating points of the longitudinal short period mode of the F4E fighter aircraft

$$\begin{aligned} b_1(s)/a_1(s) &= (-351.1 - 367.6s)/(-113.0 + 51.46s + 31.84s^2 + s^3) \\ b_2(s)/a_2(s) &= (-676.5 - 346.6s)/(-31.50 + 38.53s + 31.32s^2 + s^3) \\ b_3(s)/a_3(s) &= (-455.4 - 978.4s)/(-262.5 + 84.85s + 33.12s^2 + s^3) \\ b_4(s)/a_4(s) &= (-538.7 - 790.3s)/(576.7 + 71.46s + 31.74s^2 + s^3) \end{aligned}$$

We obtain the following eigenvalue and inertia pattern

i	0	1	2	3	4
k_i	$-\infty$	-0.5764	-0.3219	-0.0466	0.0689
π_i	12	11	10	9	7
i	5	6	7	8	9
k_i	0.0962	0.1216	0.1543	1.0705	$+\infty$
π_i	7	7	7	8	

In the interval I_0 there are $nN = 12$ positive eigenvalues so the 4 plants are simultaneously stabilizable by a static output feedback

$$k < -0.5764$$



F-4E fighter

Simultaneous stabilization

In general, when the compensator is not static and when there are three plants or more to be stabilized, the problem is **still open**

Applying Hermite's stability criterion on each closed-loop characteristic polynomial

$$c_i(s) = a_i(s)x(s) + b_i(s)y(s) \quad i = 1, \dots, N$$

and gathering controller coeffs x and y into a vector z , we obtain the algebraic condition

$$H(c_i) = \sum_j \sum_k z_j z_k H_{ijk} > 0 \quad i = 1, \dots, N$$

which is a **bilinear matrix inequality** (BMI) problem **non-convex** in unknown vector

BMI problems arise frequently in robustness, this is the **most difficult problem** we can face

LMI relaxations or **heuristics** can be devised

Open problems related to simultaneous stabilization

Vincent Blondel from Université Catholique de Louvain in Belgium

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is offering a bottle of good French **champagne** to the first author who answers the problem

For what values of δ is the system

$$\frac{s^2 - 1}{s^2 - 2\delta s + 1}$$

stabilizable by a stable controller whose inverse is also stable ?



Open problems related to simultaneous stabilization

One kilogram of famous Belgian **chocolates** are offered for the solution of the previous problem when $\delta = 0.9$



In the same vein, it is known that the 3 dt plants

$$P_1(z) = \frac{1}{z} \quad P_2(z) = \frac{1}{z} + \beta \quad P_3(z) = \frac{1}{z} - \beta$$

are simultaneously stabilizable iff $|\beta| < \frac{\Gamma^4(1/4)}{4\pi^2} = 4.377$

Two bottles of **whisky** are offered by Blondel for general stabilizability conditions for three arbitrary first order systems