

GRADUATE COURSE ON  
POLYNOMIAL METHODS FOR  
ROBUST CONTROL  
PART II.2

**ROBUST STABILITY ANALYSIS:  
MULTILINEAR UNCERTAINTY**

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## Uncertainty structure

In typical applications, uncertainty structure is **more complicated** than interval or affine

Usually, uncertainty enters highly **non-linearly** in the closed-loop characteristic polynomial

We distinguish between

- **multilinear** uncertainty, when each uncertain parameter  $q_i$  is linear when other parameters  $q_j, i \neq j$  are fixed
- **polynomial** uncertainty, when coefficients are multivariable polynomials in parameters  $q_i$

We can define the following **hierarchy** on the uncertainty structures

interval  $\subset$  affine  $\subset$  multilinear  $\subset$  polynomial

## Examples of uncertainty structures

### Examples

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2$$

has **affine** uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2$$

has **multilinear** uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2$$

has **polynomial** (here quadratic) uncertainty structure

The uncertain polynomial

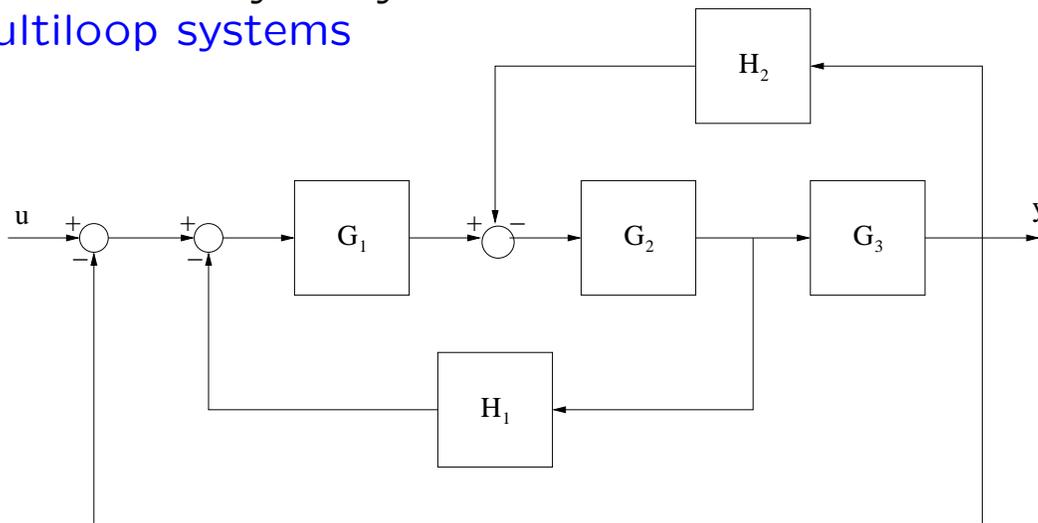
$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2$$

has **polynomial** uncertainty structure

## Multilinear uncertainty

We will focus on **multilinear uncertainty** because it arises in a wide variety of system models such as:

- multiloop systems



Closed-loop transfer function

$$\frac{y}{u} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

- state-space models with **rank-one uncertainty**

$$\dot{x} = A(q)x, \quad A(q) = \sum_{i=1}^n q_i A_i, \quad \text{rank } A_i = 1$$

and characteristic polynomial

$$p(s, q) = \det(sI - A(q))$$

- polynomial MFDs with **MIMO interval** uncertainty

$$G(s) = A^{-1}(s, q)B(s, q), \quad C(s) = Y(s)X^{-1}(s)$$

and closed-loop characteristic polynomial

$$p(s, q) = \det(A(s, q)X(s) + B(s, q)Y(s))$$

## From polynomial to multilinear

Moreover, if a polynomial  $p(s, q)$  has **polynomial** uncertainty structure defined over a **polytope**  $Q$ , then there exists an **equivalent** polynomial  $\tilde{p}(s, \tilde{q})$  with **multilinear** uncertainty defined over another **polytope**  $\tilde{Q}$

### Example

Consider the uncertain polynomial

$$p(s, q) = (6q_1q_2 + 17) + (4q_1^2 + q_2^2 + 15)s + (3q_1^3 + q_1^2q_2 + q_1q_2 + 3q_1 + 10)s^2 + s^3$$

with **polynomial** uncertainty over the polytope  $|q_1| \leq 1$  and  $|q_2| \leq 2$

We expand the uncertainty space by defining new variables  $\tilde{q}_i$  such that  $\tilde{q}_1\tilde{q}_2\tilde{q}_3 = q_1^3$  and  $\tilde{q}_4\tilde{q}_5 = q_2^2$ , so that the **equivalent** uncertain polynomial

$$\tilde{p}(s, \tilde{q}) = (6\tilde{q}_1\tilde{q}_4 + 17) + (4\tilde{q}_1\tilde{q}_2 + \tilde{q}_4\tilde{q}_5 + 15)s + (3\tilde{q}_1\tilde{q}_2\tilde{q}_3 + \tilde{q}_1\tilde{q}_2\tilde{q}_4 + \tilde{q}_1\tilde{q}_4 + 3\tilde{q}_1 + 10)s^2 + s^3$$

has **multilinear** uncertainty over the polytope  $|\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3| \leq 1$  and  $|\tilde{q}_4 = \tilde{q}_5| \leq 2$

## Lack of edge results

We have seen that there is a **lack of vertex results** for **affine** uncertainty, so that we had to develop edge results

Unfortunately, there is a **lack of edge results** for **multilinear** uncertainty

### Example

Consider the uncertain polynomial

$$\begin{aligned} p(s, q) = & (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) \\ & + (1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s \\ & + (q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 \\ & + (q_1 + q_2 + 2.56)s^3 + s^4 \end{aligned}$$

with **multilinear** uncertainty over the polytope  $q_1 \in [0, 1]$ ,  $q_2 \in [0, 3]$ , corresponding to the state-space interval matrix

$$p(s, q) = \det\left(sI - \begin{bmatrix} [-1.5, -0.5] & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & [-4, 1] \end{bmatrix}\right)$$

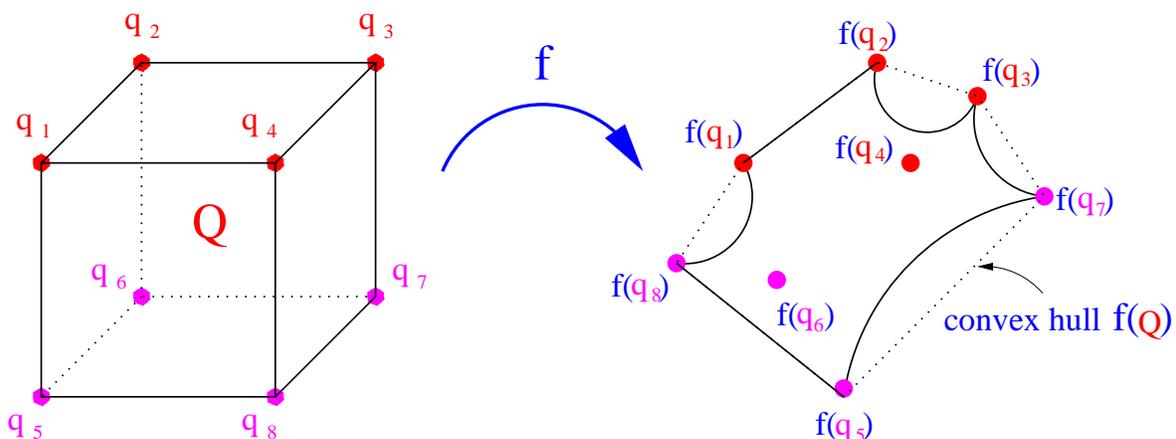
The four edges of the uncertainty bounding set are **stable**, however for  $q_1 = 0.5$  and  $q_2 = 1$  polynomial  $p(s, q)$  is **unstable**..

## The mapping theorem

To handle multilinear uncertainty we will use a special overbounding family of polynomials, as captured by the following **mapping theorem**

Let  $Q$  be a **box** with vertices  $q_i$   
Let  $f$  be a **multilinear** function with range  
 $f(Q) = \{f(q) : q \in Q\}$   
Then it follows that  
**convex hull  $f(Q) = \text{convex hull } \{f(q_i)\}$**

Geometric interpretation for a 3-D box  $Q$  and a 2-D multilinear function  $f$



Any  $f(Q)$  which curves outward is not realizable

Can be proved by **induction** or with the help of **support functions** in linear programming

## Applications of mapping theorem

If  $p(s, q)$  is an uncertain polynomial with **coefficient** vector  $a(q)$  depending multilinearly on a parameter  $q$  living in a box  $Q$  with vertices  $q_i$  then

$$\text{convex hull } a(Q) = \text{convex hull } \{a(q_i)\}$$

The same property holds for the **value set**

$$\text{convex hull } p(z, Q) = \text{convex hull } \{p(z, q_i)\}$$

hence the zero exclusion condition

$$0 \notin \text{convex hull } \{p(z, q_i)\}$$

is a **sufficient condition** for robust stability



When testing robust stability of polynomials with multilinear uncertainty if this condition is not satisfied (i.e. zero belongs to the convex hull of the value set) then we **cannot conclude without intensive gridding**..

## Value set with multilinear uncertainty

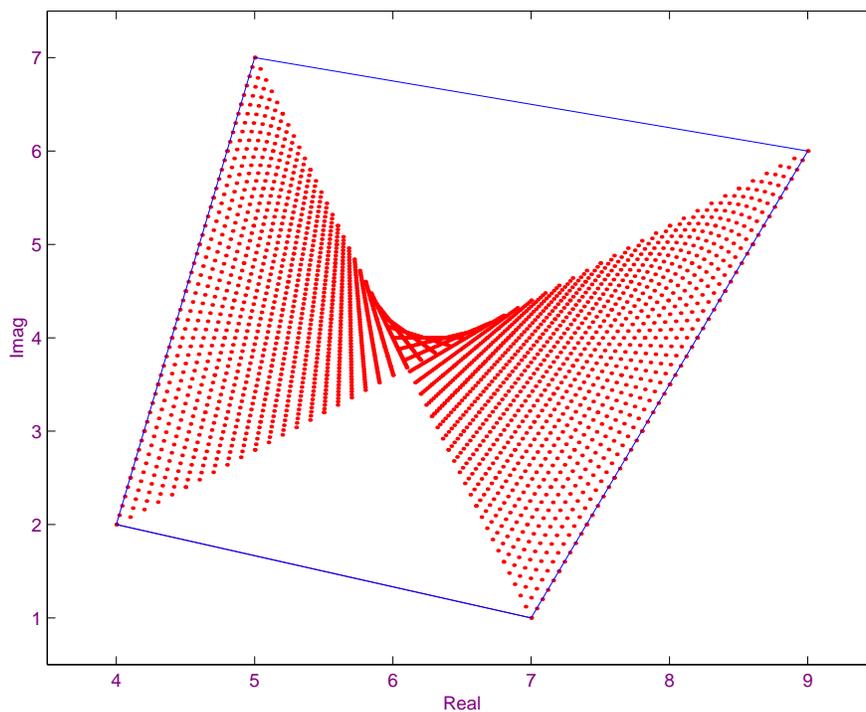
### Example

Consider the multilinear polynomial

$$(5 + 2q_1 - q_2 + 3q_1q_2) + (7 - 6q_1 - 5q_2 + 10q_1q_2)s$$

where  $q_1, q_2 \in [0, 1]$

Its **actual value set** at  $s = j$  (obtained by gridding) and its **convex hull** are shown below



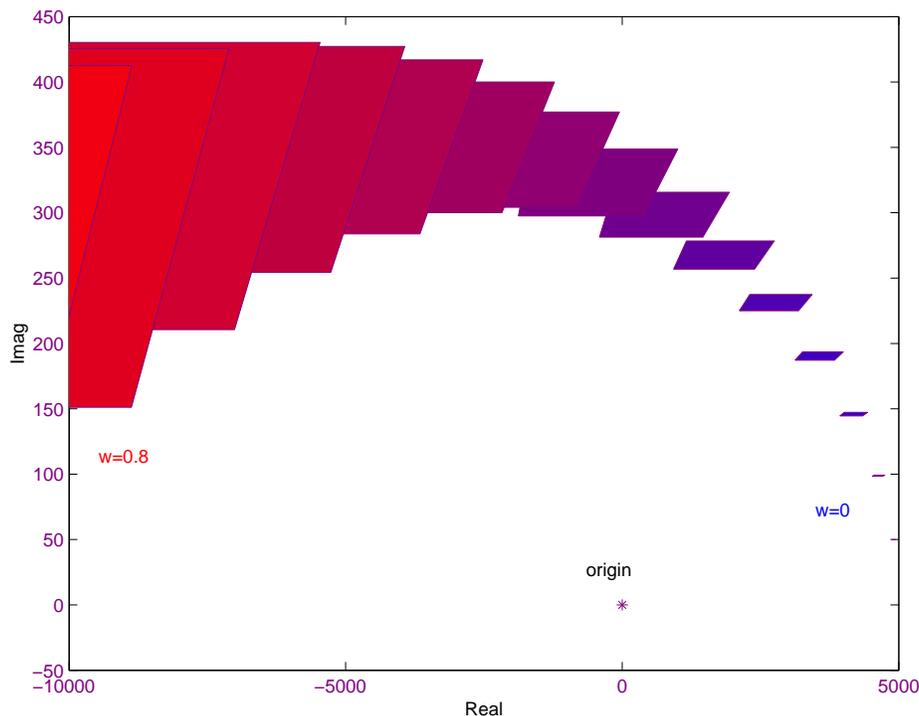
## Robust stability analysis with multilinear uncertainty

### Example

Consider the crane system with uncertain rope length  $l \in [8, 16]$ , crab mass  $m_C \in [100, 2000]$  and multilinear closed-loop characteristic polynomial

$$5000 + 1000s + (20100 + 5000l + 10m_C)s^2 + 100ls^3 + lm_Cs^4$$

Nominal stability at  $l = 8$  and  $m_C = 1000$  is easily checked, so we can build the convex hull of the value set  
= convex hull of value set of the 4 vertex polynomials  
= polygon with 4 vertices at each frequency



The convex hull of the value set excludes zero hence the polynomial is **robustly stable**

## Nasty value set

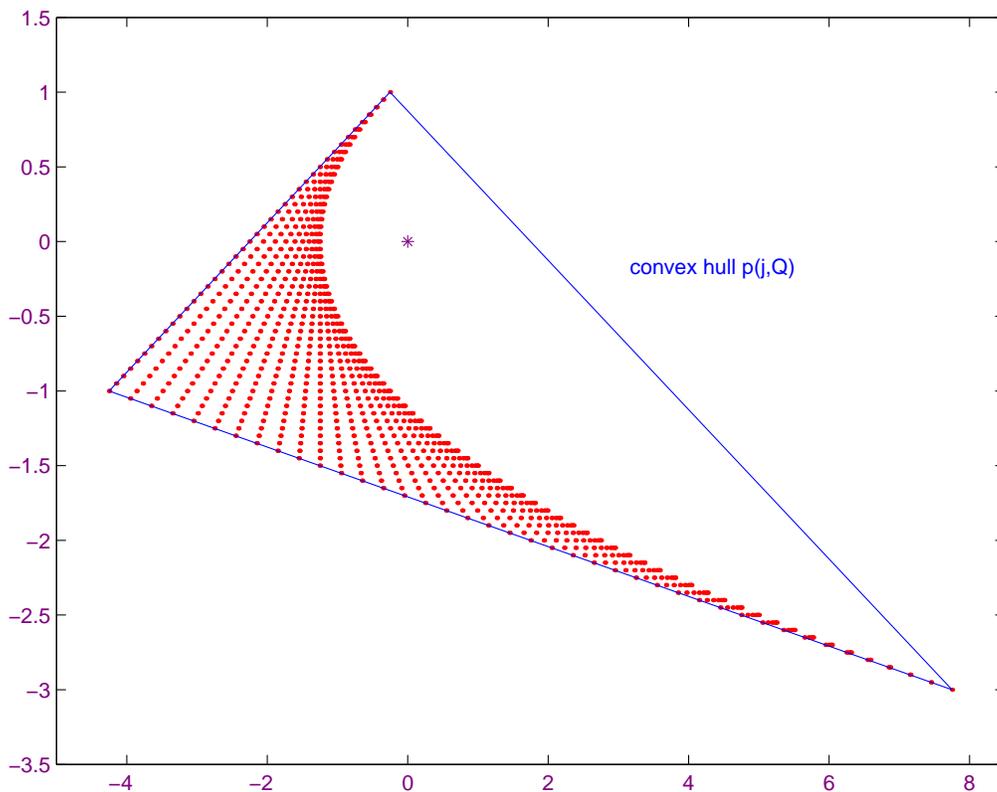
### Example

Consider now the multilinear polynomial

$$p(s, q) = 3 + 2s + (0.25 + 2q_1 + 2q_2)s^2 + 0.5(q_1 + q_2)s^3 + q_1q_2s^4$$

with uncertainty box  $Q = [1, 5] \times [1, 5]$

We build the convex hull  $p(j, Q)$  of the value set at  $s = j$

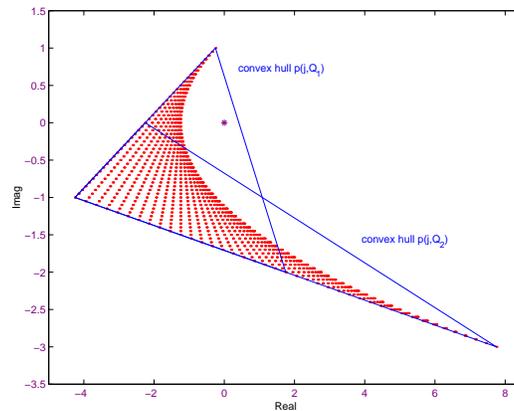


The convex hull contains the origin so we cannot conclude about robust stability

From the actual value set obtained by gridding it turns out however that the polynomial is robustly stable

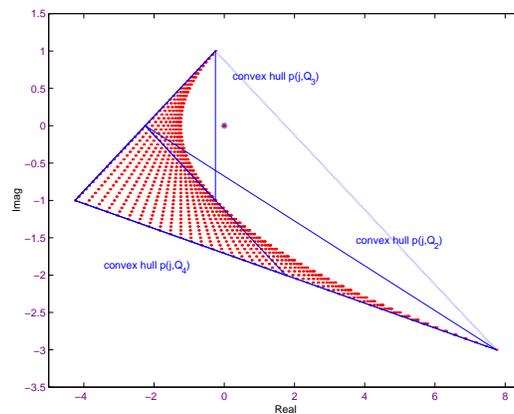
## Splitting uncertainty set

Now splitting the uncertainty interval  $Q$  into  $Q_1 = [1, 3] \times [1, 5]$  and  $Q_2 = [3, 5] \times [1, 5]$  we build **two** convex hulls  $p(j, Q_1)$  and  $p(j, Q_2)$



The convex hull of  $p(j, Q_1)$  contains the origin so once more we **cannot conclude** about robust stability

Now splitting  $Q_1$  into  $Q_3 = [1, 3] \times [1, 3]$  and  $Q_4 = [1, 3] \times [3, 5]$  we construct **three** convex hulls  $p(j, Q_3)$ ,  $p(j, Q_4)$  and  $p(j, Q_2)$



The origin is now excluded, so the polynomial is **robustly stable**

We can build a systematic algorithm to split the uncertainty intervals but the computational complexity is **exponential** in the number of parameters