GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART I.2

ROBUST STABILITY ANALYSIS: INTERVAL UNCERTAINTY

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View of the Old Town in Prague

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Independent uncertainty

So far we have studied polynomials affected by a single uncertain parameter

$$p(s,q) = (6+q) + (4+q)s + (2+q)s^2$$

However in practice several parameters can be uncertain, such as in

$$p(s,q) = (6+q_0) + (4+q_1)s + (2+q_2)s^2$$

Independent uncertainty structure: each component q_i enters into only one coefficient

Example

Crane system $0.6 + 2s + (2.6 + 0.001m_L)s^2 + 2s^3 + s^4$ with fixed rope length $l = 10$ and uncertain load mass m_L has independent uncertainty structure

Crane system $6 + 20s + (0.6l + 21)s^2 + 2ls^3 + ls^4$ with uncertain rope length l and fixed load mass $m_L = 100$ has dependent uncertainty structure

Interval uncertainty

Interval uncertainty: independent structure and uncertain parameter vector q belongs to a given box, i.e. $q_i \in [q_i^-,\ q_i^+]$

Example
Uncertain polynomial

$$(6+q_0)+(4+q_1)s+(2+q_2)s^2, \quad |q_i| \leq 1$$

has interval uncertainty, also denoted as

$$[5,7]+[3,5]s+[1,3]s^2$$

Some coefficients can be fixed, e.g.

$$6+[3,5]s+2s^2$$

Example
Some representations can be redundant, such as

$$(3+q_0)+(6+2q_1+5q_4)s+(5+q_2+2q_3)s^2+s^3, \quad |q_i| \leq 0.5$$

Defining

$$\begin{array}{rcll} \tilde{q}_0 & = & 3+q_0 & \tilde{q}_0 \in [2.5,\ 3.5] \\ \tilde{q}_1 & = & 6+2q_1+5q_4 & \tilde{q}_1 \in [2.5,\ 9.5] \\ \tilde{q}_2 & = & 5+q_2+2q_3 & \tilde{q}_2 \in [3.5,\ 6.5] \end{array}$$

equivalent to interval polynomial

$$[2.5,\ 3.5]+[2.5,\ 9.5]s+[3.5,\ 6.5]s^2+s^3$$

Kharitonov's polynomials

Associated with the interval polynomial

$$p(s,q) = \sum_{i=0}^{n} [{q^-}_i,\ {q^+}_i] s^i$$

are four Kharitonov's polynomials

$$\begin{aligned}
p^{--}(s) &= {q^-}_0 + {q^-}_1 s + {q^+}_2 s^2 + {q^+}_3 s^3 + {q^-}_4 s^4 + {q^-}_5 s^5 + \cdots \\
p^{-+}(s) &= {q^-}_0 + {q^+}_1 s + {q^+}_2 s^2 + {q^-}_3 s^3 + {q^-}_4 s^4 + {q^+}_5 s^5 + \cdots \\
p^{+-}(s) &= {q^+}_0 + {q^-}_1 s + {q^-}_2 s^2 + {q^+}_3 s^3 + {q^+}_4 s^4 + {q^-}_5 s^5 + \cdots \\
p^{++}(s) &= {q^+}_0 + {q^+}_1 s + {q^-}_2 s^2 + {q^-}_3 s^3 + {q^+}_4 s^4 + {q^+}_5 s^5 + \cdots
\end{aligned}$$

where we assume ${q^-}_n > 0$ and ${q^+}_n > 0$

Example

Interval polynomial

$$p(s,q) = [1,2] + [3,4]s + [5,6]s^2 + [7,8]s^3$$

Kharitonov's polynomials

$$\begin{aligned}
p^{--}(s) &= 1 + 3s + 6s^2 + 8s^3 \\
p^{-+}(s) &= 1 + 4s + 6s^2 + 7s^3 \\
p^{+-}(s) &= 2 + 3s + 5s^2 + 8s^3 \\
p^{++}(s) &= 2 + 4s + 5s^2 + 7s^3
\end{aligned}$$

Kharitonov's theorem

In 1978 the Russian researcher Vladimír Kharitonov proved the following fundamental result

A continuous-time interval polynomial
is robustly stable iff its
four Kharitonov polynomials are stable

Instead of checking stability of an infinite number of polynomials we just have to check stability of four polynomials, which can be done using the classical Hurwitz criterion

Simplifications for low-degree polynomials:
less Kharitonov polynomials to be tested

- degree 5: $p^{--}(s)$, $p^{-+}(s)$, $p^{+-}(s)$
- degree 4: $p^{+-}(s)$, $p^{++}(s)$ (provided $q_0^- > 0$)
- degree 3: $p^{+-}(s)$ (provided $q_0^- > 0$)

Example

Crane system with uncertain load mass $m_L \in [50, 2395]$

$$0.6 + 2s + [2.650, 4.995]s^2 + 2s^3 + s^4$$

The two Kharitonov polynomials are the same stable polynomial

$$0.6 + 2s + 2.650s^2 + 2s^3 + s^4$$

so robust stability holds for any value of the mass

Proof of Kharitonov’s theorem

Kharitonov’s original proof of his theorem is involved, but a simpler geometrical proof can be provided based on

- polynomial value sets
- zero exclusion condition
- Mikhailov’s stability criterion

Peter and Paul fortress in St Petersburg

Value set

Given continuous-time polynomial $p(s, q)$ with uncertainty vector q in uncertainty set Q, the subset of the complex plane

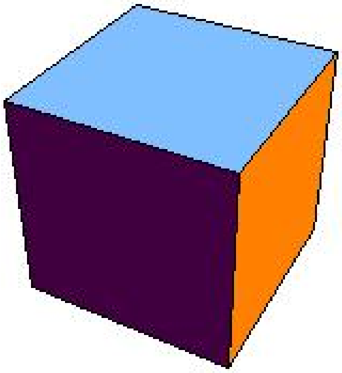
$$\{p(j\omega, q) : q \in Q\}$$

obtained at a fixed frequency ω when q ranges over Q is called the value set at ω

In the case of an interval polynomial

$$p(s, q) = \sum_{i=0}^{n} [q_i^-,\, q_i^+] s^i$$

the set Q is a hyper-rectangle $Q = \prod_{i=0}^{n} [q_i^-, q_i^+]$ with 2^n vertices and $n2^{n-1}$ edges



Set Q when $n = 3$

Kharitonov's rectangle

It turns out that the value set of an interval polynomial is also a rectangle since

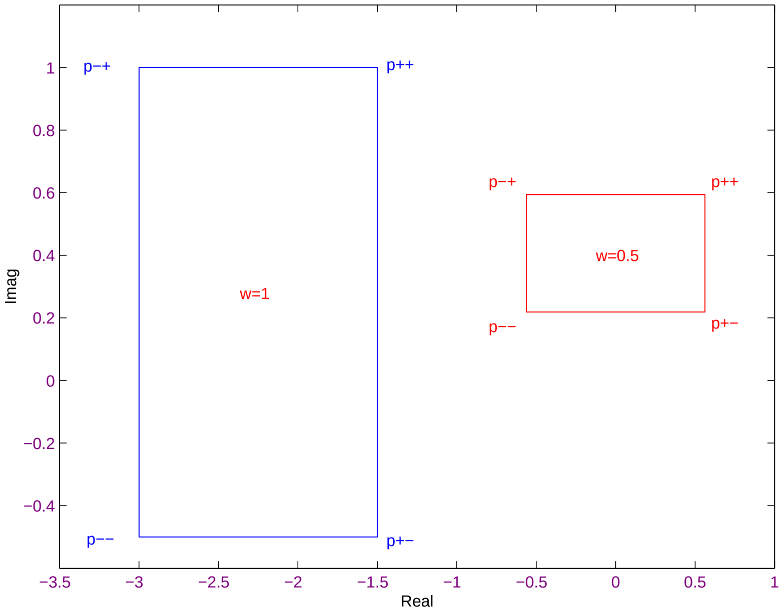
$$\begin{aligned} \text{Re}\, p(j\omega, q) &= q_0 - q_2\omega^2 + q_4\omega^4 \cdots \\ \text{Im}\, p(j\omega, q) &= q_1\omega - q_3\omega^3 + q_5\omega^5 \cdots \end{aligned}$$

and when $\omega \geq 0$

$$\begin{aligned} \min_{q\in Q} \text{Re}\, p(j\omega, q) &= q_0^- - q_2^+\omega^2 + q_4^-\omega^4 \cdots = \text{Re}\, p^{--}(j\omega) = \text{Re}\, p^{-+}(j\omega) \\ \max_{q\in Q} \text{Re}\, p(j\omega, q) &= q_0^+ - q_2^-\omega^2 + q_4^+\omega^4 \cdots = \text{Re}\, p^{++}(j\omega) = \text{Re}\, p^{+-}(j\omega) \\ \min_{q\in Q} \text{Im}\, p(j\omega, q) &= q_1^-\omega - q_3^+\omega^3 + q_5^-\omega^5 \cdots = \text{Im}\, p^{+-}(j\omega) = \text{Im}\, p^{--}(j\omega) \\ \max_{q\in Q} \text{Im}\, p(j\omega, q) &= q_1^+\omega - q_3^-\omega^3 + q_5^+\omega^5 \cdots = \text{Im}\, p^{-+}(j\omega) = \text{Im}\, p^{++}(j\omega) \end{aligned}$$

Example

$p(s,q) = [0.25,\ 1.25] + [0.75,\ 1.25]s + [2.75,\ 3.25]s^2 + [0.25, 1.25]s^3$



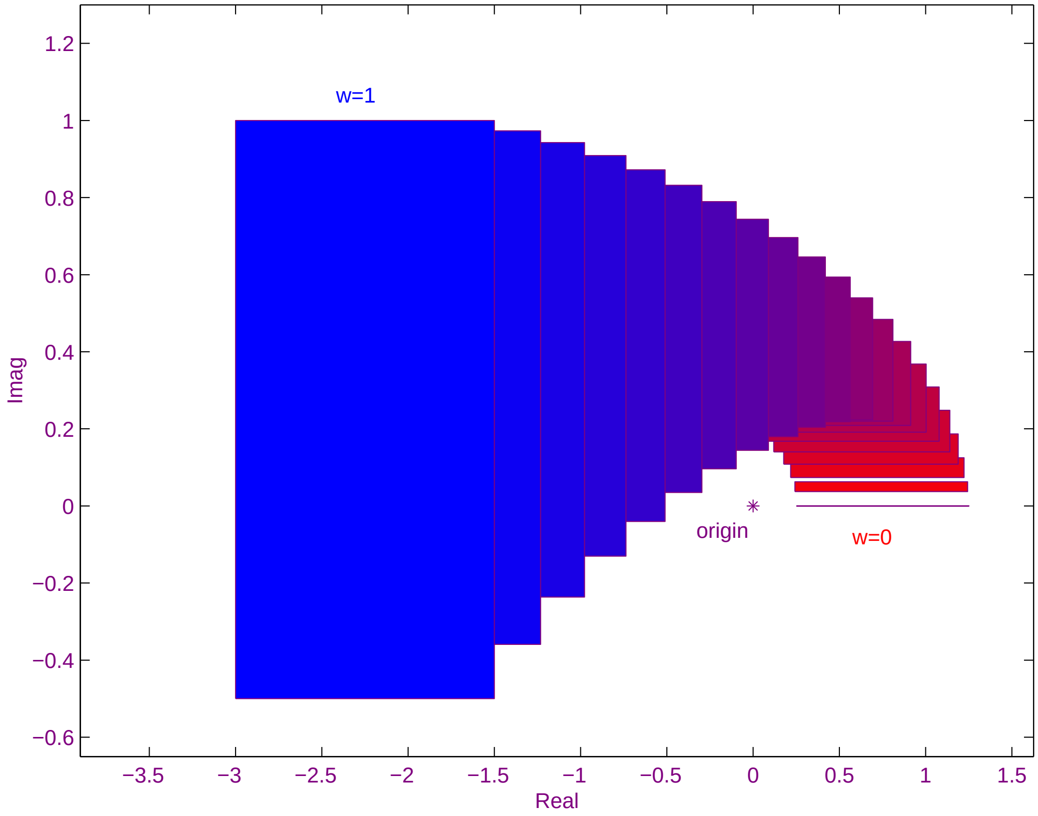
Kharitonov's rectangles at $\omega = 0.5$ and $\omega = 1$

Motion of Kharitonov's rectangle

When frequency ω sweeps the positive real axis, Kharitonov's rectangle moves around the complex plane

Example

$p(s,q) = [0.25,\ 1.25] + [0.75,\ 1.25]s + [2.75,\ 3.25]s^2 + [0.25, 1.25]s^3$



Kharitonov's rectangles for $0 \leq \omega \leq 1$

Zero exclusion condition

Consider a continuous-time interval polynomial $p(s,q)$, $q \in Q$ of invariant degree and assume that there is at least one stable member $p(s,q_0)$

$p(s,q)$ is robustly stable iff the origin is excluded from the value set, i.e. $0 \notin p(j\omega,Q)$ for all frequencies $\omega \geq 0$

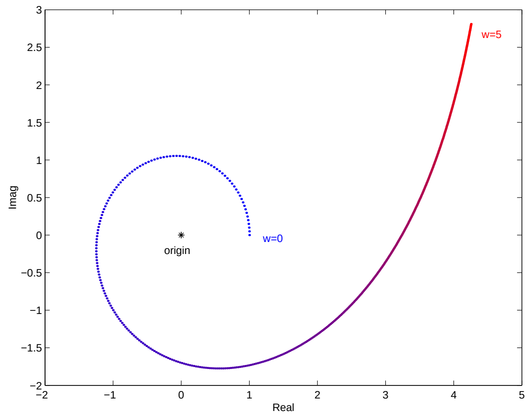
To prove the result, notice that stability is lost when crossing the stability boundary $s = j\omega$ for some $\omega = \omega^\star$ and some $q = q^\star$

- graphical condition
- not necessary to check $\omega < 0$ (symmetry)
- very general result valid also for other uncertainty models than interval uncertainty and other stability regions than the LHP

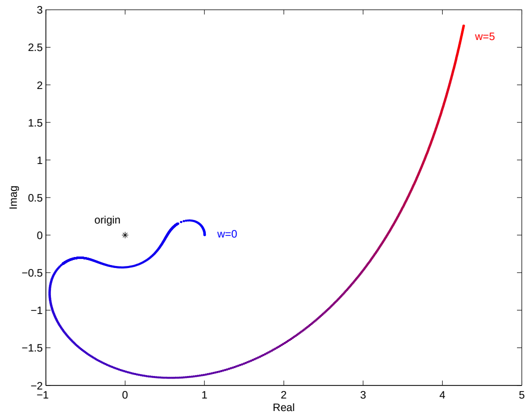
Mikhailov's criterion

In 1938 Mikhailov proposed the following stability criterion, which is the last component required to prove Kharitonov's theorem

A continuous-time polynomial
$p(s) = p_0 + p_1 s + \cdots + p_n s^n$ with $p_n > 0$
is stable iff its frequency plot $p(j\omega)$
- starts on the positive real axis
- encircles the origin in a counterclockwise direction with a phase increment of $n\pi/2$ as ω varies from 0 to $+\infty$



$1 + 5s + 10s^2 + 10s^3 + 5s^4 + s^5$
stable



$1 + s + 10s^2 + 10s^3 + 5s^4 + s^5$
unstable

Proof of Kharitonov's theorem

Necessity is trivial since stability of the interval polynomial implies stability of the 4 Kharitonov polynomials

Sufficiency can be shown as follows: assume that the 4 Kharitonov polynomials are stable

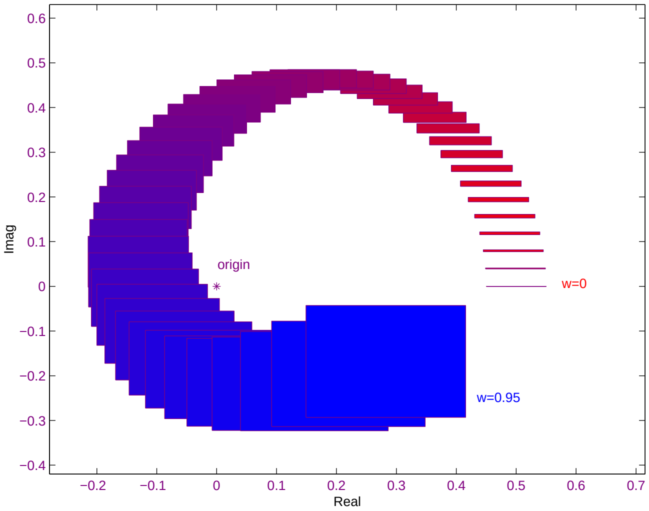
- phases of their 4 Mikhailov plots monotically increasing
- edges of Kharitonov's rectangle parallel to coordinate axis
- origin cannot enter Kharitonov's rectangle (value set)
- stability follows from zero exclusion condition

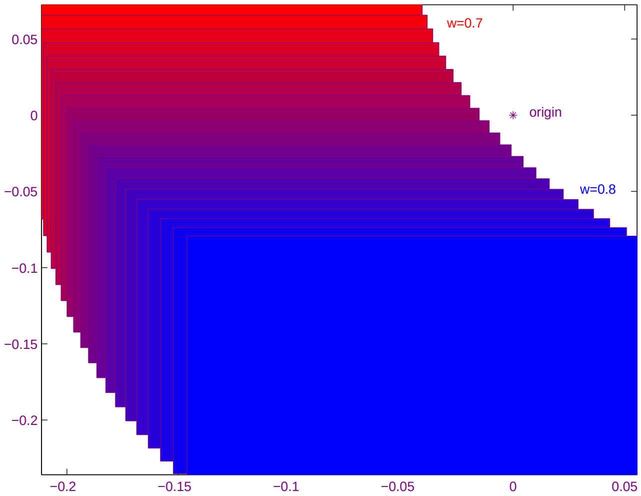
Kharitonov's theorem

Example

$p(s,q) = [0.45,\ 0.55] + [1.95,\ 2.05]s + [2.95,\ 3.05]s^2 + [5.95,\ 6.05]s^3 + [3.95,\ 4.05]s^4 + [3.95,\ 4.05]s^5 + s^6$

Midpoint is stable, so we draw Kharitonov's rectangles





Origin is excluded from Kharitonov's rectangles so interval polynomial $p(s,q)$ is robustly stable

Robustness margin

Consider the interval polynomial

$$p(s,q) = p_0(s) + r \sum_{i=0}^{n-1} [-\varepsilon_i,\ \varepsilon_i] s^i$$

parametrized in uncertainty bound $r \geq 0$ and where the $\varepsilon_i \geq 0$ are scaling factors

The maximal value r_{max} such that $p(s,q)$ is robustly stable is called the robustness margin

Defining the 4 Kharitonov polynomials

$$\begin{array}{rcl}
p_1^{--}(s) & = & -\varepsilon_0 - \varepsilon_1 s + \varepsilon_2 s^2 + \varepsilon_3 s^3 \cdots \\
p_1^{-+}(s) & = & -\varepsilon_0 + \varepsilon_1 s + \varepsilon_3 s^2 - \varepsilon_3 s^3 \cdots \\
p_1^{+-}(s) & = & \varepsilon_0 - \varepsilon_1 s - \varepsilon_2 s^2 + \varepsilon_3 s^3 \cdots \\
p_1^{++}(s) & = & \varepsilon_0 + \varepsilon_1 s - \varepsilon_2 s^2 - \varepsilon_3 s^3
\end{array}$$

and applying the eigenvalue criterion
we conclude that

$$\begin{aligned}
r_{\mathrm{max}} = \min\{ & 1/\lambda^+_{\mathrm{max}}(-H^{-1}(p_0)H(p_1^{--})), \\
& 1/\lambda^+_{\mathrm{max}}(-H^{-1}(p_0)H(p_1^{-+})), \\
& 1/\lambda^+_{\mathrm{max}}(-H^{-1}(p_0)H(p_1^{+-})), \\
& 1/\lambda^+_{\mathrm{max}}(-H^{-1}(p_0)H(p_1^{++}))\}
\end{aligned}$$

Uncertainty overbounding

Independent uncertainty structure of interval polynomials is restrictive because uncertain parameters typically enter non-linearly into more than one coefficient

Either we develop more general results (see next course) or we try to overbound the uncertainty

Example

Uncertain polynomial with $|q_i| \leq 0.25$

$$\begin{aligned} p(s,q) = \;& (0.5 - 3q_1q_2) + (6 + 6q_1 - 8q_2)s \\ & +(6 + 3q_1q_2 - 4q_2)s^2 \\ & +(5 + 0.2q_1q_2 + 0.1q_1 - 0.1q_2)s^3 + s^4 \end{aligned}$$

Compute bounds on coeffs

$$\begin{array}{ccccc} 0.3125 & \leq & 0.5 - 3q_1q_2 & \leq & 0.6875 \\ 2.5 & \leq & 6 + 6q_1 - 8q_2 & \leq & 9.5 \\ & & \cdots & & \end{array}$$

and build overbounding interval polynomial

$$\begin{aligned} \tilde{p}(s,\tilde{q}) = \;& [0.3125,\ 0.6875] + [2.5,\ 9.5]s \\ & +[4.8125,\ 7.1875]s^2 \\ & +[4.9475,\ 5.0375]s^3 + s^4 \end{aligned}$$

Applying Kharitonov's theorem, we conclude that $\tilde{p}(s,\tilde{q})$ is robustly stable, so $p(s,q)$ is also robustly stable

Failure of overbounding

Example

Consider the uncertain polynomial

$$p(s,q) = 1 + (3 - 2q_1 - 0.5q_2)s + (0.5 + q_1 + 1.5q_2)s^2 + s^3$$

with $q_i \in [0,\ 1]$

It has mutually dependent uncertainty structure, so we overbound it with the interval polynomial

$$\tilde{p}(s,\tilde{q}) = 1 + [0.5,\ 3]s + [0.5,\ 3]s^2 + s^3$$

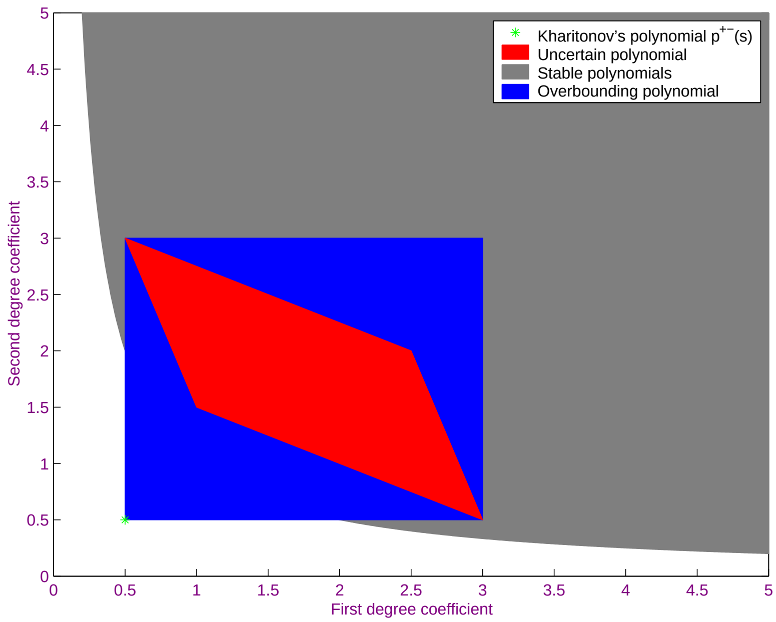
The only Kharitonov polynomial to be checked

$$p^{+-}(s) = 1 + 0.5s + 0.5s^2 + s^3$$

is unstable, so we cannot conclude:

- either $p(s,q)$ is unstable
- or overbounding $\tilde{p}(s,\tilde{q})$ is too conservative

The latter holds since one can check graphically that $p(s,q)$ is robustly stable



Open questions

Kharitonov's theorem is elegant but valid for a restricted family of uncertain polynomials

It raises more questions that it answers

- valid for polynomials with complex coeffs ?
- valid for discrete-time polynomials as well ?
- what about other stability regions ?
- valid for other uncertainty models ?
- valid for polynomial matrices ?
- etc $\cdots$

Now we will briefly answer some questions, but it should be obvious that we will need to introduce more general robust stability analysis tools in the sequel of the course

Polynomials with complex coefficients

Polynomials with complex coefficients arise
- in communication applications of signal processing (info about both amplitude and phase)
- when studying whirling shafts, vibrational systems and filters...

Diophantine equations and spectral factorization
with complex polynomials arise when designing
filters, equalizers or decouplers of mobile phones
see www.mathworks.com/products/thirdparty/poly

An extended version of Kharitonov's theorem with eight polynomials is valid to assess robust stability of a complex interval polynomial

$$p(s,q,r) = \sum_{i=0}^{n} ([q_i^-, q_i^+]s^i + j[r_i^-, r_i^+]s^i])$$

Discrete-time polynomials

Roughly speaking, Kharitonov's theorem does not hold for discrete-time polynomials

Example

Consider the discrete-time interval polynomial

$$p(z,q) = -1/3 + 3/2z^2 + [-17/8,\ 17/8]z^3 + z^4$$

One can check that both extreme polynomials $p(z,-17/8)$ and $p(z,17/8)$ are Schur stable
(roots in the unit disc)

However, the polynomial $p(z,0)$ is unstable