GRADUATE COURSE ON
POLYNOMIAL METHODS FOR
ROBUST CONTROL

Didier HENRION
www.laas.fr/~henrion
henrion@laas.fr

Laboratoire d’Analyse et d’Architecture des Systèmes
Centre National de la Recherche Scientifique
Toulouse

Ústav Teorie Informace a Automatizace
Akademie Věd České Republiky
Praha

Universidad de los Andes
Mérida, Venezuela
October-November 2001
Course outline

I Robust stability analysis (Part I)
I.1 General introduction - linear systems, polynomial methods and robust control
I.2 Single parameter uncertainty - eigenvalue criteria
I.3 Interval uncertainty - Kharitonov’s theorem

II Robust stability analysis (Part II)
II.1 Polytopic uncertainty - edge theorem
II.2 Multilinear and polynomic uncertainty - mapping theorem

III Robust design and convex optimization
III.1 Robust pole placement - approximation of stability region
III.2 Rank-one robust stabilization - Youla-Kučera and Rantzer-Megretski parametrizations
III.3 Simultaneous stabilization - strong stabilization, Hermite criterion, open problems

IV New results on robust analysis and design
IV.1 Robust stability analysis - Linear matrix inequalities and positive polynomial matrices
IV.2 Robust stability design - Numerical examples
Course Outline

I Robust stability analysis (Part I)
II Robust stability analysis (Part II)
III Robust design and convex optimization
IV New results on robust analysis and design

Scope of the course

<table>
<thead>
<tr>
<th>Systems</th>
<th>Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>frequential</td>
</tr>
<tr>
<td>non-linear</td>
<td>state-space</td>
</tr>
<tr>
<td></td>
<td>polynomial</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Control scheme</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>adaptive</td>
<td>non-parametric</td>
</tr>
<tr>
<td>stochastic</td>
<td>parametric</td>
</tr>
<tr>
<td>robust</td>
<td></td>
</tr>
</tbody>
</table>
Linear systems

Theory much better developed than for non-linear systems

Provides background for nonlinearists

Availability of powerful CACSD tools to solve numerical linear algebra problems

(Old) reference books

- Kailath. Linear systems. Prentice Hall, 1980
- Chen. Linear system theory and design. HRW, 1984

Computer tools used for the course

Matlab 6.0

Polynomial Toolbox 2.5

[Matlab](http://www.mathworks.com)  [Polynomial Toolbox](http://www.polyx.cz)
Polynomial methods

Based on the algebra of polynomials and polynomial matrices, typically involve
• linear Diophantine equations
• quadratic spectral factorization

Pioneered in central Europe during the 70s mainly by Vladimír Kučera from the former Czechoslovak Academy of Sciences

Network funded by the European commission EURPOLY

www.utia.cas.cz/europoly

Polynomial matrices also occur in Jan Willems' behavioral approach to systems theory

Alternative to state-space methods developed during the 60s most notably by Rudolf Kalman in the USA, rather based on
• linear Lyapunov equations
• quadratic Riccati equations
A scalar transfer function can be viewed as the ratio of two polynomials

**Example**
Consider the mechanical system

\[ G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + k_1 s + k_2} \]

Neglecting static and Coloumb frictions, we obtain the linear transfer function
Ratio of polynomial matrices

Similarly, a MIMO transfer function can be viewed as the ratio of polynomial matrices

\[ G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s) \]

the so-called matrix fraction description (MFD)

Lightly damped structures such as oil derricks, regional power models, earthquakes models, mechanical multi-body systems, damped gyroscopic systems are most naturally represented by second order polynomial MFDs

\[(D_0 + D_1 s + D_2 s^2)y(s) = N_0 u(s)\]

Example
The (simplified) oscillations of a wing in an air stream is captured by properties of the quadratic polynomial matrix [Lancaster 1966]

\[
D(s) = \begin{bmatrix}
121 & 18.9 & 15.9 \\
0 & 2.7 & 0.145 \\
11.9 & 3.64 & 15.5
\end{bmatrix}
+ \begin{bmatrix}
7.66 & 2.45 & 2.1 \\
0.23 & 1.04 & 0.223 \\
0.6 & 0.756 & 0.658
\end{bmatrix}s
+ \begin{bmatrix}
17.6 & 1.28 & 2.89 \\
1.28 & 0.824 & 0.413 \\
2.89 & 0.413 & 0.725
\end{bmatrix}s^2
\]
**First-order polynomial MFD**

**Example**

**RCL network**

- $y_1$ voltage through inductor
- $y_2$ current through inductor
- $u$ voltage

Applying Kirchoff’s laws and Laplace transform we get

$$
egin{bmatrix}
1 & -L_s \\
C_s & 1 + RC_s
\end{bmatrix}
\begin{bmatrix}
y_1(s) \\
y_2(s)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
C_s
\end{bmatrix}
u(s)
$$

and thus the first-order left system MFD

$$
G(s) = \begin{bmatrix}
1 & -L_s \\
C_s & 1 + RC_s
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
C_s
\end{bmatrix}.
$$
Second-order polynomial MFD

Example
mass-spring system

Vibration of system governed by 2nd-order differential equation \( M\ddot{x} + C\dot{x} + Kx = 0 \) where e.g. \( n = 250, m_i = 1, \kappa_i = 5, \tau_i = 10 \) except \( \kappa_1 = \kappa_n = 10 \) and \( \tau_1 = \tau_n = 20 \)

Quadratic matrix polynomial
\[
D(s) = Ms^2 + Cs + K
\]
with
\[
M = I \\
C = \text{tridiag}(-10, 30, -10) \\
K = \text{tridiag}(-5, 15, -5).
\]
Another second-order polynomial MFD

Example
Inverted pendulum on a cart

Linearization around the upper vertical position yields the left polynomial MFD

\[
\begin{bmatrix}
(M + m)s^2 + bs \\
J + l^2m+s^2 + ks - lmg
\end{bmatrix}
\begin{bmatrix}
x(s) \\
\phi(s)
\end{bmatrix}
= \begin{bmatrix}
1 \\
0
\end{bmatrix} f(s)
\]

With \( J = mL^2/12, \ l = L/2 \) and \( g = 9.8, \ M = 2, \ m = 0.35, \ l = 0.7, \ b = 4, \ k = 1 \), we obtain the denominator polynomial matrix

\[
D(s) = \begin{bmatrix}
5s + 3s^2 & 0.35s^2 \\
0.35s^2 & -3.4 + s + 0.16s^2
\end{bmatrix}
\]
More examples of polynomial MFDs

Higher degree polynomial matrices can also be found in aero-acoustics (3rd degree) or in the study of the spatial stability of the Orr-Sommerfeld equation for plane Poiseuille flow in fluid mechanics (4rd degree)

For more info see Nick Higham’s homepage at

www.ma.man.ac.uk/~higham
Uncertainty

When modeling systems we face several sources of uncertainty, including

- non-parametric (unstructured) uncertainty
  - unmodeled dynamics
  - truncated high frequency modes
  - non-linearities
  - effects of linearization, time-variation..

- parametric (structured) uncertainty
  - physical parameters vary within given bounds
  - interval uncertainty ($l_\infty$)
  - ellipsoidal uncertainty ($l_2$)
  - $l_1$ uncertainty

How can we overcome uncertainty?

- model predictive control
- adaptive control
- robust control
Robustness

We seek a control law valid over the whole range of admissible uncertainty

- off-line
- simple
- cheap
- secure

Basically we will study two classes of problems

Robust stability analysis
Parts I, II & IV of this course

Robust controller design
Parts III & IV of this course
Course material

Information on the course can be found at [www.laas.fr/~henrion/courses/polyrobust.html](http://www.laas.fr/~henrion/courses/polyrobust.html)

Most of the material of the first two parts of the course (robust stability analysis) is taken from the textbooks

- J. Ackermann. Robust control: systems with uncertain physical parameters. Springer, 1993

The third part (robust design) describes recent results published from 1992 to 1999 in

- IEEE Transactions on Automatic Control
- IFAC Automatica
- System and Control Letters
- International Journal of Control
- SIAM Journal on Control and Optimization

The last part of the course contains mostly new, previously unpublished material