Course on LMI optimization with applications in control

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Polynomial optimization (revisited)

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Let us illustrate with polynomial optimization (POP) the main steps of the moment-SOS aka Lasserre hierarchy.

Given a **nonlinear nonconvex** problem:

1. Reformulate it as a **linear** problem (at the price of enlarging or changing the space of solutions);

2. Solve approximately the linear problem with a hierarchy of tractable **convex** relaxations (of increasing size);

3. Ensure **convergence**: either the original problem is solved at a finite relaxation size, or its solution is approximated with increasing quality.

At each step, **conic duality** is an essential ingredient.

POP (Polynomial Optimization Problem)

Given polynomials $p, p_1, \ldots p_m \in \mathbb{R}[x]$ of the indeterminate $x \in \mathbb{R}^n$, consider the **nonlinear nonconvex** global optimization problem

$$\begin{array}{rcll} v^* & = & \min_x & p(x) \\ & & \text{s.t.} & x \in X \end{array}$$

defined on the bounded basic semialgebraic set

$$X := \{x \in \mathbb{R}^n : p_1(x) \geq 0, \ \ldots \ p_m(x) \geq 0\}$$

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Step 1 - Linear reformulation

Primal linear reformulation

Instead of the POP

$$v^* \;=\; \begin{array}{ll} \min_x & p(x) \\ \text{s.t.} & x \in X \end{array}$$

consider the linear problem (LP)

$$p^* \;=\; \begin{array}{ll} \inf_\mu & \langle p, \mu \rangle \\ \text{s.t.} & \langle 1, \mu \rangle = 1 \\ & \mu \in C(X)'_+ \end{array}$$

Exercise 1.1: Prove that $v^* = p^*$ and that the LP has an optimal solution equal to the Dirac measure at any optimal solution of the POP.

Dual linear reformulation

Dual to the primal LP

$$
\begin{array}{rcll}
p^* & = & \text{inf}_\mu & \langle p, \mu \rangle \\
 & & \text{s.t.} & \langle 1, \mu \rangle = 1 \\
 & & & \mu \in C(X)'_+
\end{array}
$$

is the LP

$$
\begin{array}{rcll}
d^* & = & \text{sup}_{v \in \mathbb{R}} & v \\
 & & \text{s.t.} & p - v \in C(X)'
\end{array}
$$

Exercise 1.2: Derive the dual LP from the primal LP using convex duality. Prove that strong duality holds i.e. $p^* = d^*$. Give a graphical interpretation to the dual LP.

Step 2 - Convex hierarchy

Moments and positive polynomials

POP is replaced with a primal and a dual LP

$$\begin{array}{rcll} p^* & = & \text{inf}_\mu & \langle p, \mu \rangle \\ & & \text{s.t.} & \langle 1, \mu \rangle = 1 \\ & & & \mu \in C(X)'_+ \end{array} \qquad \begin{array}{rcll} d^* & = & \text{sup}_{v \in \mathbb{R}} & v \\ & & \text{s.t.} & p - v \in C(X)_+ \end{array}$$

or equivalently

$$\begin{array}{rcll} p^* & = & \text{inf}_y & \langle p, y \rangle \\ & & \text{s.t.} & \langle 1, y \rangle = 1 \\ & & & y \in P(X)'_d \end{array} \qquad \begin{array}{rcll} d^* & = & \text{sup}_{v \in \mathbb{R}} & v \\ & & \text{s.t.} & p - v \in P(X)_d \end{array}$$

since p is a degree d polynomial.

Approximating positive polynomials

The cone of positive polynomials

$$P(X)_d := \{p \in \mathbb{R}[x]_d : p(x) \geq 0, \forall x \in X\}$$

on the bounded basic semialgebraic set

$$X := \{x \in \mathbb{R}^n : p_k(x) \geq 0, k = 1, \ldots, m\}$$

is generally **intractable**, so we will approximate it.

Denoting $p_0(x) := 1$ and enforcing (without loss of generality) $p_1(x) := R^2 - \sum_{i=1}^n x_i^2$ for R large enough, consider for $r \geq d$

$$Q(X)_r := \{p \in \mathbb{R}[x]_d : p = \sum_{k=0}^{m} s_k p_k,\ \ s_k \in \Sigma[x],\ \ s_k p_k \in \mathbb{R}[x]_r\}$$

where $\Sigma[x]$ denotes polynomial sums of squares (SOS), and observe that it is an **inner approximation**: $Q(X)_r \subset P(X)_d$

Polynomial SOS

Observe that

$$Q(X)_r := \{p \in \mathbb{R}[x]_d : p = \sum_{k=0}^{m} s_k p_k,\ \ s_k \in \Sigma[x],\ \ s_k p_k \in \mathbb{R}[x]_r\}$$

is a projection of the SOS cone

Observe also that by construction

$$Q(X)_r \subset Q(X)_{r+1}$$

Exercise 1.3: Prove that deciding whether a polynomial is SOS can be reduced to semidefinite programming.

Moment relaxations

Hence we have a hierarchy of tractable **inner approximations** for the cone of positive polynomials

$$Q(X)_r \subset Q(X)_{r+1} \subset P(X)_d$$

Using convex duality, we also have a hierarchy of tractable **outer approximations** for the cone of moments

$$Q(X)'_r \supset Q(X)'_{r+1} \supset P(X)'_d$$

Elements of $Q(X)'_r$ are sometimes called pseudo-moments

We also say that $Q(X)'_r$ is a **relaxation** of $P(X)'_d$

Moment-SOS hierarchy

Replace the intractable problems

$$\begin{array}{rcll} p^* & = & \inf_y & \langle p, y\rangle \\ & & \text{s.t.} & \langle 1, y\rangle = 1 \\ & & & y \in P(X)'_d \end{array} \qquad \begin{array}{rcll} d^* & = & \sup_{v\in\mathbb{R}} & v \\ & & \text{s.t.} & p - v \in P(X)_d \end{array}$$

with the hierarchy of tractable problems for $r = d, d+1, \ldots$

$$\begin{array}{rcll} p^*_r & = & \inf_y & \langle p, y\rangle \\ & & \text{s.t.} & \langle 1, y\rangle = 1 \\ & & & y \in Q(X)'_r \end{array} \qquad \begin{array}{rcll} d^*_r & = & \sup_{v\in\mathbb{R}} & v \\ & & \text{s.t.} & p - v \in Q(X)_r \end{array}$$

Exercise 1.4: Prove that strong duality holds: $v^*_r := p^*_r = d^*_r$

Step 3 - Convergence

Convergence

Integer r is called the **relaxation order**

Since $Q(X)_r \subset Q(X)_{r+1} \subset P(X)_d$, we have a monotone non-decreasing sequence of lower bounds on the POP value:

$$v_r^* \leq v_{r+1}^* \leq v^*$$

Theorem (Putinar 1993): $\overline{Q(X)_\infty} = P(X)_d$

Theorem (Lasserre 2001): $v_\infty^* = v^*$

The moment-SOS hierarchy is known as the **Lasserre hierarchy**

Finite convergence

Theorem (Nie 2014): Generically $\exists r < \infty$ such that $v_r^* = v^*$

In other words, a vanishing small random perturbation of the input data of a given POP ensures finite convergence of the Lasserre hierarchy

We also have sufficient linear algebra conditions to ensure finite convergence, certify global optimality and extract minimizers

The moment-SOS hierarchy is implemented in the GloptiPoly 3 package for Matlab

`homepages.laas.fr/henrion/software/gloptipoly/`