Polynomial optimal control: references, exercises, questions and answers

Didier Henrion[1,2]

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1 References

The moment-SOS hierarchy was first applied to polynomial optimal control (POC) in [20]. It fits the framework of the generalized problem of moments [19, 18]. The lecture follows closely the presentation of [14]. See also [12] for sketchy lecture notes.

The moment-SOS hierarchy can be seen as an alternative to standard numerical methods for POC. It is a global method generating lower bounds on the value function, while local methods based e.g. on the Pontryagin Maximum Principle (necessary conditions of optimality) or discretization (local optimization algorithms) generate upper bounds. If the lower and upper bounds coincide, then on the one hand, it is not necessary to go deeper in the hierarchy, and on the other hand, it is not necessary to try other initial conditions or discretize further in the local methods. This strategy was followed in [3] for solving a POC problem in data science.

The moment-SOS hierarchy is a global method bearing similarities with the Hamilton-Jacobi-Bellmann (HJB) approach to POC which consists of solving a non-linear PDE, see e.g. [9, Section 10.3] or [27, 7]. It can be interpreted as a convex relaxation of the HJB PDE.

The Brockett integrator of nonlinear systems control is used as a numerical example for the application of the moment-SOS hierarchy to POC in [20]. It is also known (up to a change of coordinates) as the unicycle or Dubins system, one of the simplest instance of a non-holonomic system in robotics, see e.g. [8] for the connection. It was studied thoroughly in [26], see also [25].

Linear formulations of optimal control problems (on ODEs and PDEs) are classical, and be tracked back to the work by L. C. Young, Filippov, Warga or Gamkrelidze. For more details see e.g. [10, Part III]. The main idea is that in calculus of variations or optimal control, the optima are not attained, i.e. the problems typically do not have solutions when formulated in smooth functional spaces (e.g. continuous functions of time, or measurable functions of

[1]CNRS; LAAS; Université de Toulouse, 7 avenue du colonel Roche, F-31400 Toulouse, France.
[2]Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, CZ-16626 Prague, Czechia.

time). Once formulated in a dual space, e.g. a space of measures, optima are generally attained [21].

Relaxed controls, also called Young measures in the calculus of variations literature, are designed to capture oscillations at the limit, when the frequency tends to infinity. Some POC problems feature a different limit behavior, namely a concentration of the control signal. This is the case for impulsive POC problems arising in space engineering, for which the moment-SOS hierarchy has been adapted [4, 5]. DiPerna-Majda measures, an extension of the Young measures, can be used to deal with the simultaneous presence of oscillations and concentrations, also with the moment-SOS hierarchy [6, 15].

More recently, efforts were dedicated to applying the moment-SOS hierarchy for analyzing and controlling nonlinear PDEs [17, 24].

2 Exercises

2.1 Exercise 3.1

2.1.1 Statement

Relaxed controls capture limit behavior such as e.g. oscillations

$$\lim_{r\to\infty}\int_0^1 v(u_{rt})dt = \int_0^1\int_U v(u)\omega_t(du)dt, \quad \forall v \in C(U)$$

What is the limit $\omega_t(du)$ for $u_{rt} = \cos(2\pi rt)$, $r = 1, 2, \ldots$?

2.1.2 Solution

For all $r \in \mathbb{N}$ function $\cos(2\pi rt)$ is periodic on $[0,1]$ with values in $U := [-1,1]$. The relaxed control or Young measure ω_t should be such that

$$\lim_{r\to\infty}\int_0^1 v(\cos(2\pi rt))dt = \int_0^1\int_{-1}^1 v(u)\omega_t(du)dt$$

for all test functions $v \in C(U)$. Denoting $s = rt$, the first integral writes

$$\int_0^1 v(\cos(2\pi rt))dt = \frac{1}{r}\int_0^r v(\cos(2\pi s))ds = \frac{1}{r}\sum_{k=1}^r\int_{k-1}^k v(\cos(2\pi s))ds = \int_0^1 v(\cos(2\pi s))ds$$

where the last relation follows from the periodicity of $f : s \mapsto \cos(2\pi s)$ on $[0,1]$. The identity

$$\int_0^1 v(f(t))dt = \int_0^1\int_{-1}^1 v(u)\omega_t(du)dt.$$

then follows by letting $\omega_t(du)$ be the image measure of the uniform measure $\lambda_{[0,1]}(dt)$ through the map f. Note that $\omega_t(du)$ does not depend on t in this case.

Let us now calculate the density $\phi(u)$ of $\omega_t(du)$ – also called Radon-Nikodým derivative – with respect to the Lebesgue measure du. Letting $\omega_t(du) = \phi(u)du$, it holds

$$\int_{-1}^{u} \phi(y)dy = \int_{f^{-1}([-1,u])} dt$$

where

$$f^{-1}([-1,u]) := \{t \in [0,1] : f(t) \in [-1,u]\} = [\frac{1}{2} - \frac{1}{\pi} \arcsin \frac{u+1}{2}, \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{u+1}{2}].$$

Finally we obtain

$$\phi(u) = \frac{d}{du} \int_{-1}^{u} \phi(y)dy = \frac{d}{du} \int_{\frac{1}{2}-\frac{1}{\pi}\arcsin\frac{u+1}{2}}^{\frac{1}{2}+\frac{1}{\pi}\arcsin\frac{u+1}{2}} dt = \frac{d}{du} \frac{2}{\pi} \arcsin \frac{u+1}{2} = \frac{1}{\pi\sqrt{1-u^2}}.$$

2.2 Exercise 3.2

2.2.1 Statement

The classical Bolza problem

$$\begin{array}{rcl} v^* & = & \inf_u \int_0^1 (x_t^2 + (u_t^2 - 1)^2)dt \\ & & \dot{x}_t = u_t, \; x_0 = 0 \\ & & x_t \in [-1,1], \; u_t \in [-1,1] \;\; \forall t \in [0,1] \end{array}$$

is relaxed to

$$\begin{array}{rcl} v_R^* & = & \inf_\omega \int_0^1 \int_U (x_t^2 + (u^2 - 1)^2)\, \omega_t(du)dt \\ & & \dot{x}_t = \int_U u\, \omega_t(du), \; x_0 = 0 \\ & & x_t \in [-1,1], \; \omega_t \in \mathscr{P}([-1,1]) \;\; \forall t \in [0,1] \end{array}$$

where $\mathscr{P}([-1,1])$ is the set of probability measures on $[-1,1]$. Prove that there is no relaxation gap: $v^* = v_R^*$ and that the relaxed infimum is attained at $\omega_t^* = \frac{1}{2}(\delta_{-1} + \delta_{+1})$.

2.2.2 Solution

A minimizing sequence for the classical Bolza problem is

$$t \in [0,1] \mapsto u_r = \text{sign}\; \cos(2r\pi t) \in U := [-1,1]$$

and the integral cost, equal to the square of the two norm of the integral of u_r on $[0,1]$, approaches from above the infimum $v^* = 0$. This is the optimal value since the integrand is non-negative.

We are seeking the relaxed control $t \in [0,1] \mapsto \omega_t^* \in \mathscr{P}(U)$ such that

$$\lim_{r\to\infty} \int_0^1 v(u_{rt})dt = \int_0^1 \int_{-1}^1 v(u)\omega_t^*(du)dt$$

for all test functions $v \in C(U)$. The control sequence $t \mapsto u_{rt}$ oscillates increasingly with r, but the amount of time for which u_{rt} is equal to -1 remains always equal to the amount of time for which u_{rt} is equal to -1. The integral can then be readily computed

$$\int_0^1 v(\text{sign } \sin(2r\pi t))dt = \frac{1}{2}(v(-1) + v(+1))$$

and it follows that $\omega_t^*(du) = \frac{1}{2}(\delta_{-1} + \delta_{+1})$. Inserting this value into the relaxed problem, we obtain $v_R^* = 0$ since the relaxed trajectory is now

$$x_t = \int_0^t u \ \omega_s^*(du)ds = 0$$

for all $t \in [0, 1]$.

2.3 Exercise 3.3

2.3.1 Statement

Prove that the measure LP

$$\begin{array}{rcll} p^*(t_0, x_0) & := & \min_{\mu,\mu_T} & \int l\mu + \int l_T \mu_T \\ & & \text{s.t.} & \frac{\partial \mu}{\partial t} + \text{div}(f\mu) + \mu_T = \delta_{t_0}\delta_{x_0} \\ & & & \mu \in C([t_0, T] \times X \times U)'_+, \ \ \mu_T \in C(\{T\} \times X_T)'_+ \end{array}$$

has a dual LP

$$\begin{array}{rcll} d^*(t_0, x_0) & := & \sup_v & v(t_0, x_0) \\ & & \text{s.t.} & l + \frac{\partial v}{\partial t} + \text{grad } v \cdot f \in C([t_0, T] \times X \times U)_+ \\ & & & l_T - v(T, .) \in C(\{T\} \times X_T)_+ \end{array}$$

on functions $v \in C^1([t_0, T] \times X)$ and that there is no duality gap: $p^* = d^*$.

2.3.2 Solution

Construct the Lagrangian

$$\ell(\mu, \mu_T, v) := \langle l, \mu \rangle + \langle l_T, \mu_T \rangle - \langle \frac{\partial \mu}{\partial t} + \text{div}(f\mu) + \mu_T - \delta_{t_0}\delta_{x_0}, v \rangle$$

where the dual variable $v \in C^1([t_0, T] \times X)$ corresponds to the Liouville equation. Rearrange the Lagrangian

$$\ell(\mu, \mu_T, v) = \langle v, \delta_{t_0}\delta_{x_0} \rangle + \langle l + \frac{\partial v}{\partial t} + \text{grad } v \cdot f, \mu \rangle + \langle l_T - v, \mu_T \rangle$$

such that the dual Lagrange function can be expressed as

$$d(v) := \inf_{\mu,\mu_T} \ell(\mu, \mu_T, v) = \langle v, \delta_{t_0}\delta_{x_0} \rangle = v(t_0, x_0)$$

provided $l + \frac{\partial v}{\partial t} + \mathrm{grad}\, v \cdot f \in C([t_0,T] \times X \times U)_+$ and $l_T - v \in C(\{T\} \times X_T)_+$. Maximization of the Lagrange function yields the expected dual LP.

To prove that there is no duality gap, we use [2, Theorem IV.7.2] and the fact that the cone $\{(\langle l, \mu\rangle + \langle l_T, \mu_T\rangle, \frac{\partial \mu}{\partial t} + \mathrm{div}(f\mu) + \mu_T) : \mu \in C([t_0,T] \times X \times U)'_+, \mu_T \in C(\{T\} \times X_T)'_+\}$ is nonempty and bounded in the metric inducing the weak-star topology on measures.

2.4 Exercise 3.4

2.4.1 Statement

In the dual LP

$$\begin{array}{ll} \sup_v & v(t_0, x_0) \\ \text{s.t.} & l + \frac{\partial v}{\partial t} + \mathrm{grad}\, v \cdot f \in C([t_0,T] \times X \times U)_+ \\ & l_T - v(T,.) \in C(\{T\} \times X_T)_+ \end{array}$$

by combining the dual inequalities evaluated on an admissible trajectory, prove that for any admissible v it holds $v^* \geq v$ on $[t_0, T] \times X$.

2.4.2 Solution

If v is dual admissible, then for an admissible trajectory starting at $x_{t_1} = x_1$, it holds

$$\int_{t_1}^{T} (l + \frac{\partial v}{\partial t} + \mathrm{grad}\, v \cdot f)(t, x_t, u_t)dt = \int_{t_1}^{T} l(x_t, u_t)dt + v(T, x_T) - v(t_1, x_1) \geq 0$$

since $l + \frac{\partial v}{\partial t} + \mathrm{grad}\, v \cdot f \geq 0$ on $[t_0, T] \times X \times U$. Moreover

$$\int_{t_1}^{T} l(x_t, u_t)dt + l_T(x_T) \geq \int_{t_1}^{T} l(x_t, u_t)dt + v(T, x_T)$$

since $l_T - v(T,.) \geq 0$ on X_T. Combining the two inequalities yields

$$\int_{t_1}^{T} l(x_t, u_t)dt + l_T(x_T) \geq v(t_1, x_1)$$

and the expected inequality follows by taking the infimum over admissible trajectories.

3 Questions and answers

A. What are the convexity assumptions for having no duality gap?

Q. There is no duality gap between the primal LP on measures on the dual LP on functions, as shown in Exercise 3.3. There is no relaxation gap between the POP with relaxed control and the primal LP on measures, as shown in [29]. There may be however a relaxation gap between the POP with classical controls and the POP with relaxed controls, and some convexity assumptions ensure that there is no relaxation gap, see the discussion in [14, Section 3.1] or [13, Appendix C].

A. Are the inf=min reached here ? why ? are they conditions ?

Q. The infimum can be attained if the optimization problem is formulated in a dual topological space, this is a key idea of the monograph [21]. In a dual topological space, we can use the weak star topology, which is the only infinite-dimensional topology for which the unit ball is compact (this is called the Banach-Alaoglu Theorem).

A. Is it possible to generalize the framework, that you just explained for general POCPs, to fractional OCPs which includes fractional differential equations?

Q. This is an interesting question. I do not know if there has been such extensions to fractional differential equations, i.e. differential equations with fractional derivatives. Another interesting research direction is optimal control of partial differential equations with sound convergence guarantees. See [22] for optimal control of linear PDEs modeled by Riesz spectral operators.

A. Is it possible to extract an approximate optimal control law from the solution of the moment-SOS problems? If so, could you please give an idea of the strategy?

Q. Thank you for asking. There are various strategies to extract an approximate polynomial control law from moments and SOS, see e.g. [11, 16]. Recently, Christoffel-Darboux kernels have been used to recover the graph of a function from its moments, with convergence guarantees [23]. I hope this will be covered during the series of POEMA lectures by Edouard Pauwels.

A. Could you please explain more about what you mean by “transporting the Lebesgue measure” on $[-1, 1]$ to obtain v^* ?

Q. See [14, Section 5] for more details. The Liouville equation is a linear transport equation classical in fluid mechanics, statistical physics and analysis of PDEs. It is also called the equation of conservation of mass, or the continuity equation, or the advection equation, see e.g. [28, Section 5.4] or [1] for extensions.

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